BANACH ALGEBRAS GENERATED BY A BOUNDED LINEAR OPERATOR

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§1. Banach algebras with generators.

In this paper we shall characterize the spectrum and the semi-spectrums of a bounded linear operator on a Banach space by a ring-theoretical method. And we shall determine all the type of Banach algebras generated by a fixed bounded linear operator in connection with its corresponding semi-spectrums.

Let $\mathbf{M}$ be a Banach algebra which contains the identity I. $\mathbf{M}$ is called *normalized* if $|I| = 1$ and $|AB| \leqq |A||B|$. Put

$$\|A\| = \sup_{|X| \leqq 1} |AX|,$$

then the norm $\|A\|$ is equivalent to the original norm $|A|$, and by this norm $\mathbf{M}$ is a normalized Banach algebra.

Hereafter we shall assume that $\mathbf{M}$ is normalized and commutative. Given an element A of $\mathbf{M}$ and given a complex rational function $r(z) = (\sum_i a_i z^i)/(\sum_j b_j z^j)$. If $\sum_j b_j A^j$ is inversible, the element $r(A) = (\sum_i a_i A^i)(\sum_j b_j A^j)^{-1}$ is called a *rational form*. The *spectrum* $\mathfrak{S}(\mathbf{M})$ of A is the set of all complex numbers z so that $A - zI$ fail to be inversible. The *resolvent set* of A is the complement of the spectrum in the complex Riemann sphere $\mathfrak{R}_0$.

If $\mathbf{M}$ contains a fixed element A whose rational forms are contained everywhere densely in $\mathbf{M}$, then A is called a *generator* of $\mathbf{M}$. The *spectrum* of the algebra $\mathbf{M}$ is the space of all continuous linear functionals f on $\mathbf{M}$ such that $f(I) = 1$ and $f(XY) = f(X)f(Y)$.

A fundamental relationship between spectrums of generators and the spectrum of a Banach algebra will be observed in the sequel.

Theorem 1. *Let $\mathbf{M}$ be a normalized Banach algebra with the identity I and a generator A. Then the spectrum $\mathfrak{S}$ of A is a bounded closed set in the complex number field. In order that a complex number z belong to $\mathfrak{S}$, it is necessary and sufficient that $|r(A)| \geqq |r(z)|$ hold for every rational form $r(A)$. Every rational form $r(A)$ satisfies*

$$\lim_{n \to \infty} |r^n(A)|^{\frac{1}{n}} = \sup_{z \in \mathfrak{S}} |r(z)|.$$

The spectrum $\mathfrak{G}$ of **M** *is homeomorphic to $\mathfrak{S}$ by the mapping*

$$x \in \mathfrak{G} \longrightarrow x(A) \in \mathfrak{S}.$$

Proof. The famous I. Gelfund's theorem states that:

If **M** is a commutative Banach algebra, the spectrum $\mathfrak{G}$ of **M** is a compact space. For every fixed $f \in$ **M**, $x \in \mathfrak{G} \to x(f)$ is a continuous mapping of $\mathfrak{G}$ in the complex field. And every $f \in$ **M** satisfies $\lim |f^n|^{1/n} = \sup_{x \in \mathfrak{G}} |x(f)|$.

Now our Banach algebra **M** in question contains the generator A. Then especially $x \in \mathfrak{G} \to x(A)$ is a continuous mapping of $\mathfrak{G}$ in the complex number field. So the range $\mathfrak{G}(A)$ of this mapping is a bounded closed set. We show that this mapping $x \to x(A)$ is a homeomorphism between $\mathfrak{G}$ and $\mathfrak{G}(A)$. Assume that u and v be two elements in $\mathfrak{G}$ which satisfy $u(A) = v(A) = t$. Then every rational form $r(A)$ satisfies $u(r(A)) = v(r(A)) = r(t)$. Since the set of all rational forms $r(A)$ is everywhere dense in **M**, u coincides with v. Hence $x \to x(A)$ is a one-to-one continuous mapping of the compact set $\mathfrak{G}$ on $\mathfrak{G}(A)$, i.e. a homeomorphism.

Next we show that $\mathfrak{G}(A)$ coincides with $\mathfrak{S}$. Let z be an element in $\mathfrak{S}$. Then $A - zI$ fails to be inversible, and the monomial ideal $(A - zI)$**M** of **M** is a proper ideal of **M**. By the Gelfunds theorem there exists at least one element u in $\mathfrak{G}$ which satisfies $u(X) = 0$ for every $X \in (A - zI)$**M**. Especially $u(A - zI) = 0$ and $z = u(A) \in \mathfrak{G}(A)$. Therefore $\mathfrak{G}(A)$ contains $\mathfrak{S}$.

Conversely let z be a complex number which does not belong to $\mathfrak{S}$. Then $A - zI$ is inversible, and

$$\sup_{u \in \mathfrak{G}(A)} |(u - z)^{-1}| = \sup_{x \in \mathfrak{G}} |x((A - zI)^{-1})| \leqq |(A - zI)^{-1}| = d.$$

That is,

$$\inf_{u \in \mathfrak{G}(A)} |(u - z)| \geqq d^{-1} > 0.$$

And $\mathfrak{G}(A)$ does not contain z. Hence $\mathfrak{G}(A)$ coincides with $\mathfrak{S}$.

Now our theorem is clear. In fact, $x \to x(A)$ is a homeomorphism between $\mathfrak{G}$ and $\mathfrak{S}$. Therefore we can determine $\mathfrak{G}$ as follows: for every $z \in \mathfrak{S}$ there exists a uniquely determined bounded linear functional z on **M** defined by $z(r(A)) = r(z)$ for every rational form $r(A)$. $\mathfrak{G}$ is the set of all such bounded linear functionals z. Thus $z \in \mathfrak{S}$ if and only if $|r(A)| \geqq |r(z)|$ for every rational form $r(A)$. And by the Gelfund's theorem,

$$\lim_{n\to\infty} |r^n(A)|^{1/n} = \sup_{x\in\mathfrak{G}} |x(r(A))| = \sup_{z\in\mathfrak{S}} |r(z)|.$$

q.e.d.

§2. The classification of congenetic subalgebras.

Let $\mathbf{M}$ be a normalized Banach algebra with the identity I and a fixed generator A. We shall now classify all the sub-algebras of $\mathbf{M}$ generated by the same A.

Theorem 2. *Let $\mathbf{M}$ be a normalized Banach algebra with the identity I and a fixed generator A. If $\mathbf{N}$ is a Banach sub-algebra of $\mathbf{M}$ which is also generated by A, then the spectrum $\mathfrak{S}(\mathbf{N})$ of A which corresponds with $\mathbf{N}$ contains the spectrum $\mathfrak{S}$ of A (which corresponds with $\mathbf{M}$). And $\mathfrak{S}(\mathbf{N})-\mathfrak{S}$ is an open set.*

Conversely let $\mathfrak{T}$ be a bounded closed set in the complex field containing $\mathfrak{S}$ such that $\mathfrak{T}-\mathfrak{S}$ is open, then there exists a uniquely determined Banach sub-algebra $\mathbf{N}$ of $\mathbf{M}$ such that its corresponding spectrum $\mathfrak{S}(\mathbf{N})$ of A coincides with $\mathfrak{T}$.

Proof. Let $\mathbf{N}$ be a Banach sub-algebra of $\mathbf{M}$ which is also generated by A. Then the resolvent set $\mathfrak{R}(\mathbf{N})$ of A which corresponds with $\mathbf{N}$ is contained in the resolvent set $\mathfrak{R}(\mathbf{M})$ of A which corresponds with $\mathbf{M}$. We show that $\mathfrak{R}(\mathbf{M})-\mathfrak{R}(\mathbf{N})$ is open.

Let $\{z_n\}$ be a sequence in $\mathfrak{R}(\mathbf{N})$ which converges to a number z in $\mathfrak{R}(\mathbf{M})$. Then $(A-z_nI)^{-1}$ belongs to $\mathbf{N}$, and $(A-zI)^{-1}$ belongs to $\mathbf{M}$. In a suitable neighbourhood of z, $(A-wI)^{-1}$ is developped to the power series

$$(A-wI)^{-1} = \sum_{n=0}^{\infty} (w-z)^n (A-zI)^{-(n+1)}.$$

Therefore $(A-z_nI)^{-1}$ converges uniformly to $(A-zI)^{-1}$. This means $(A-zI)^{-1}\in\mathbf{N}$, and $z\in\mathfrak{R}(\mathbf{N})$. Thus $\mathfrak{R}(\mathbf{N})$ is relatively closed in $\mathfrak{R}(\mathbf{M})$, so that $\mathfrak{R}(\mathbf{M})-\mathfrak{R}(\mathbf{N})$ is an open set. Hence $\mathfrak{S}(\mathbf{N})=\mathfrak{K}_0-\mathfrak{R}(\mathbf{N})$ is a bounded closed set which contains $\mathfrak{S}=\mathfrak{K}_0-\mathfrak{R}(\mathbf{M})$. And $\mathfrak{T}-\mathfrak{S}=\mathfrak{R}(\mathbf{M})-\mathfrak{R}(\mathbf{N})$ is open. The former half of Theorem 2 is thus proved.

To prove the latter half of the Theorem, we shall prepare the next Lemma.

Lemma 1. *Let $\mathbf{N}$ be a Banach sub-algebra generated by A. In order that a rational form $r(A)$ exist and belong to $\mathbf{N}$, it is necessary* *and sufficient that $r(z)$ have no pole on the spectrum $\mathfrak{S}(\mathbf{N})$ of A corresponding with* $\mathbf{N}$.

In fact, if $r(z)$ is a rational function which has no pole on $\mathfrak{T} = \mathfrak{S}(\mathbf{N})$, then $r(z) = p(z) \prod_{i=1}^{n} (z-a_i)^{-1}$, where $p(z)$ is a polynomial, and every a_i is not contained in $\mathfrak{T}$. Then $r(A) = p(A) \prod_{1}^{n} (A - a_i I)^{-1}$ exists and belongs to $\mathbf{N}$. Conversely if a rational form $r(A)$ belongs to $\mathbf{N}$, then $| r(A) | \geqq \sup_{z \in \mathfrak{T}} | r(z) |$. Therefore $r(z)$ is bounded on $\mathfrak{S}(\mathbf{N})$, and has no pole on it. This concludes the Lemma.

We now turn to the proof of our Theorem. Let $\mathfrak{T}$ be a bounded closed set of complex numbers which contains $\mathfrak{S}$ and satisfies that $\mathfrak{T} - \mathfrak{S}$ is open. We denote by $\mathbf{N}$ the smallest Banach sub-algebra of $\mathbf{M}$ which contains all rational forms $r(A)$ for which corresponding rational functions $r(z)$ have no pole on $\mathfrak{T}$. $\mathbf{N}$ is clearly generated by A, and contains the identity. We show that the spectrum $\mathfrak{S}(\mathbf{N})$ of A which corresponds with $\mathbf{N}$ coincides with $\mathfrak{T}$. Let z be any complex number which does not belong to $\mathfrak{T}$. Then $(A - zI)^{-1}$ exists and belongs to $\mathbf{N}$. Therefore z does not belong to $\mathfrak{S}(\mathbf{N})$, so $\mathfrak{T}$ contains $\mathfrak{S}(\mathbf{N})$.

Next we show the converse. Let $r(z)$ be a rational function which has no pole on $\mathfrak{T}$. $r(z)$ is regular and one-valued in the interior of $\mathfrak{T}$. Then by the well-known maximal modulous theorem for analitic functions, the maximal value of $| r(z) |$ in $\mathfrak{T}$ is taken on the boundary $\mathfrak{B}$ of $\mathfrak{T}$. Since $\mathfrak{B} = \mathfrak{T} - \text{int}\, \mathfrak{T}$ is contained in $\mathfrak{S}$ (because $\mathfrak{T} - \mathfrak{S}$ is open by the assumption), we have

$$\sup_{z \in \mathfrak{T}} | r(z) | = \sup_{z \in \mathfrak{B}} | r(z) | = \sup_{z \in \mathfrak{S}} | r(z) |.$$

Consider a fixed $w \in \mathfrak{T}$, then for every rational form $r(A)$ of which $r(z)$ has no pole on $\mathfrak{T}$, we have

$$| r(A) | \geqq \sup_{z \in \mathfrak{S}} | r(z) | = \sup_{z \in \mathfrak{T}} | r(z) | \geqq | r(w) |.$$

Those rational forms $r(A)$ are contained everywhere densely in $\mathbf{N}$. By Theorem 1 we conclude that w belongs to $\mathfrak{S}(\mathbf{N})$. Therefore $\mathfrak{S}(\mathbf{N})$ contains $\mathfrak{T}$, and coincides with $\mathfrak{T}$.

Finally we show that the correspondence between Banach algebras $\mathbf{N}$ generated by the same A and the related spectrums $\mathfrak{S}(\mathbf{N})$ is one-to-one. Let $\mathbf{N}$ and $\mathbf{L}$ be two Banach sub-algebras of $\mathbf{M}$ which are generated by A and which have the common related spectrums

$\mathfrak{T}=\mathfrak{S}(\mathbf{N})=\mathfrak{S}(\mathbf{L})$. We show that $\mathbf{L}$ and $\mathbf{N}$ coincide with each other. By Lemma 1 a rational form $r(A)$ belongs to $\mathbf{N}$ (and $\mathbf{L}$) if and only if the rational function $r(z)$ has no pole on $\mathfrak{T}$. Therefore $\mathbf{L}$ and $\mathbf{N}$ contain all the rational forms $r(A)$ commonly. Hence they must be coincident with each other. q.e.d.

§3. Spectrums and semi-spectrums of bounded linear operators.

We shall now apply our Theorem 1 and 2 to an analisis of the spectrums of general bounded linear operators on Banach spaces.

Let $\mathbf{B}$ be a Banach space and A be a bounded linear operator on it. The *spectrum* of A is the set of all complex numbers z so that $A-zI$ fail to be inversible. There exists the largest Banach algebra $\mathbf{R}(A)$ of bounded linear operators generated by A. $\mathbf{R}(A)$ is the uniform closure of the set of all rational forms $r(A)$. In order to avoid any mis-understand, a spectrum $\mathfrak{S}(\mathbf{N})$ of A corresponding to a Banach algebra $\mathbf{N}$ of bounded linear operators generated by A, is called a *semi-spectrum* of A. Then $\mathfrak{S}(\mathbf{N})$ is the set of all complex numbers z so that $(A-zI)^{-1}$ do not appear in $\mathbf{N}$. Clearly the semi-spectrum of A corresponding to $\mathbf{R}(A)$ is coincident with the usual spectrum $\mathfrak{S}$ of A. Then using Theorem 1 we have

Theorem 3. *Let A be a bounded linear operator on a Banach space B. In order that a complex number z belong to the spectrum of A, it is necessary and sufficient that $|r(A)|\geq|r(z)|$ hold for every rational form $r(A)$. The spectrum $\mathfrak{S}$ of A is a bounded closed set, and every rational form $r(A)$ satisfies*

$$\lim_{n\to\infty}|r^n(A)|^{1/n}=\sup_{z\in\mathfrak{S}}|r(z)|.$$

Every Banach algebra of bounded operators generated by A is a Banach sub-algebra of $\mathbf{R}(A)$. Then we can determine the type of all such Banach algebras using our Theorem 2.

Theorem 4. *Let $\mathfrak{S}$ be the spectrum of a bounded linear operator A. If $\mathbf{N}$ is a Banach algebra of bounded operators generated by A, then its corresponding semi-spectrum $\mathfrak{S}(\mathbf{N})$ is a bounded closed set containing $\mathfrak{S}$ such that $\mathfrak{S}(\mathbf{N})-\mathfrak{S}$ is open.*

Conversely if $\mathfrak{T}$ is a bounded closed set which contains $\mathfrak{S}$ such that $\mathfrak{T}-\mathfrak{S}$ is open, then there exists a uniquely determined Banach algebra $\mathbf{N}$ generated by A whose corresponding semi-spectrum coincides with $\mathfrak{T}$.

§4. Some remarks for Theorem 2.

Given a Banach algebra $\mathbf{M}$ generated by an element A. Let $\mathfrak{S}$ denote the spectrum of A, and $\mathfrak{R}$ denote the resolvent set $\mathfrak{K}_0 - \mathfrak{S}$. (Conveniently we add the infinity point of the Riemann sphere to the resolvent set $\mathfrak{R}$). $\mathfrak{R}$ is the sum of mutually disjoint connected components $\mathfrak{R} = \mathfrak{R}_\infty \cup (\bigcup_{i=1}^{\infty} \mathfrak{R}_i)$, where $\mathfrak{R}_\infty$ is the connected component of $\mathfrak{R}$ which contains the infinity.

Now if $\mathfrak{T}$ is a bounded closed set containing $\mathfrak{S}$ such that $\mathfrak{T} - \mathfrak{S}$ is open, then $\mathfrak{T} - \mathfrak{S}$ is a component of $\mathfrak{R}$ which does not contain $\mathfrak{R}_\infty$. That is, $\mathfrak{T} - \mathfrak{S}$ is a sum $\mathfrak{T} - \mathfrak{S} = \cup \mathfrak{R}_{n'}$, where $\mathfrak{R}_{n'}$ are connected components of $\mathfrak{R}$ which do not contain the infinity. Since every Banach sub-algebra $\mathbf{N}$ of $\mathbf{M}$ is determined by the corresponding semi-spectrum $\mathfrak{S}(\mathbf{N})$, $\mathbf{N}$ is completely determined by the system of connected components $\mathfrak{R}_{n'}$ which are contained in $\mathfrak{S}(\mathbf{N})$. Therefore

Theorem 5. *Let $\mathbf{M}$ be a normalized Banach algebra generated by an element A. If the resolvent set $\mathfrak{R}$ of A is decomposed into n pieces of connected components, then there exists exactly 2^{n-1} pieces of mutually different Banach sub-algebras of $\mathbf{M}$ generated by A.*

Let $\mathbf{P}$ denote the uniform closure of all rational integral forms $p(A)$. Then $\mathbf{P}$ is the smallest Banach sub-algebra of $\mathbf{M}$ generated by A. The complement of the corresponding semi-spectrum $\mathfrak{S}(\mathbf{P})$ coincides with the connected component of the resolvent set $\mathfrak{R}$ of A which contains the infinity. Especially when the resolvent set of A is connected, then $\mathbf{M}$ coincides with $\mathbf{P}$, and every rational from $r(A)$ is uniformly approximative by a sequence of rational integral forms $\{p_n(A)\}$.

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