ON CURVES IN KAEHLERIAN SPACES

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The main purpose of our present paper is to study local properties of holomorphically planar curves in Kählerian spaces. For the sake of Kählerian or more general Geometry on a complex space, it would be useful to reflect upon its foundations in the first section. In Section 2 we shall sketch the definitions and the results on Hermitian and Kählerian spaces. After considering geodesics in Section 3, we shall define, in Section 4, holomorphically planar curves by equations similar to those of geodesics. In Sections 5 and 6, we shall reconstruct Fubini spaces and know what holomorphically planar curves are in these spaces. On the other hand, in Section 7, geodesic complex curves will be also introduced as a generalization of real geodesics, and in Section 8 we shall prove theorems on the relation between these and holomorphically planar curves. In Section 9, we may introduce holomorphically projective correspondences playing the rôle for real projective ones in Riemannian spaces, although the non-existence of the latter in Hermitian spaces will be proved in the last section.

Throughout this paper latin indices h, i, j, k, l will run from 1 to $2n$ and greek indices $\kappa, \lambda, \mu, \nu, \omega, \rho$ from 1 to n. Barred index will indicate the value of the index increased or decreased by n units if the index is less than or greater than n, say

$$\bar{i} = i \pm n, \quad \bar{\lambda} = \lambda + n.$$

We shall sometimes use $z^{\bar{\lambda}}$ for $\bar{z}^{\lambda}$ and $x^{\bar{\lambda}}$ for y^{λ}.

§1. Complex spaces. In this paper we shall occupy ourselves on a complex analytic space V with the usual definitions, see, e.g., Bochner and Martin, [5]:

a) The underlying manifold of V is a connected point set of topological dimension $2n$ in which a topology is defined by open sets.

b) There exists a covering consisting of coordinate neighborhoods $\{U\}$, each being homeomorphic to a domain in a Euclidean $2n$-space. The homeomorphism assigns to each point p of U a set of parameters $(x_1, \cdots\cdots, x_n, y_1, \cdots\cdots, y_n)$, and the set of complex parameters $z^{\lambda} = x^{\lambda} + iy^{\lambda}$ is called a complex coordinate of the point.

c) If two coordinate neighborhoods intersect with non-empty set, then the transformation between the two coordinates has to be expressible by analytic functions

$$z'^{\lambda} = f^{\lambda}(z^1, \cdots\cdots, z^n). \tag{1.1}$$

These coordinate systems are called *allowable* ones and the transformations also *allowables*, the set of these allowable transformations being denoted by G.

To each point having coordinates z^{λ} in a neighborhood U of the complex space, we assign the set of the conjugate parameters $\bar{z}^{\lambda} = x^{\lambda} - iy^{\lambda}$, and denote the point assigned with these conjugate parameters by $\bar{p}$. It is compatible since the conjugate operation is independent of coordinate neighborhoods because of

$$\bar{z}'^{\lambda} = \bar{f}^{\lambda}(\bar{z}^1, \cdots\cdots, \bar{z}^n), \tag{1.2}$$

where the coefficients of the power series of $\bar{f}^{\lambda}$ are the conjugates of those of f^{λ} in (1.1).

We denote by $\bar{U}$ the set of points $\bar{p}$ corresponding to the points p of U. Then we obtain a complex space $\bar{V}$, said to be *conjugate* to the original V, having the following properties:

a') the underlying manifold is the same as of the original V,

b') the covering of coordinate neighborhoods consists of $\bar{U}$'s, and

c') the set $\bar{G}$ of allowable transformations consists of analytic transformations (1.2).

For convenience, in this section we shall denote points neighboring to a point $\bar{p}$ by $\bar{q}$ and their complex parameters in a neighborhood $\bar{U}$ by $\tilde{z}^{\lambda}$ instead of another letter to be used. An allowable transformation (1.2) of $\bar{G}$ is then expressed in form

$$\tilde{z}'^{\lambda} = \bar{f}^{\lambda}(\tilde{z}^1, \cdots\cdots, \tilde{z}^n) \tag{1.3}$$

in $\bar{U} \cap \bar{U}'$.

If we consider the product space $V \times \bar{V} = \{(p, \bar{q})\}$, then the space V may be imbedded on the diagonal subset of the product space by an inclusion map

$$\theta : V \longrightarrow V \times \bar{V} \tag{1.4}$$

defined by

$$\theta(p) = (p, \bar{p}). \tag{1.5}$$

The union W of products $U \times \bar{U}$ corresponding to neighborhoods U of V covers $\theta(V)$ in the product space, and it has as allowable transformations the transformations defined by (1.1) and (1.3) together, these forming the diagonal subset of the product set $G \times \bar{G}$.

All quantities of V appearing in sequel are supposed to be half-analytic [Schouten and Struik, 7], that is, that all components of quantities are in general complex valued and are analytic in $2n$ real parameters x^λ and y^λ. If in first we continuate analytically the real parameters x^λ and y^λ to the corresponding complex domains and next carry out a transformation

$$x^\lambda = \frac{1}{2}(z^\lambda + \bar{z}^\lambda), \qquad y^\lambda = \frac{1}{2i}(z^\lambda - \bar{z}^\lambda), \tag{1.6}$$

considered as a transformation $(x^\lambda, y^\lambda) \to (z^\lambda, \bar{z}^\lambda)$ between complex parameters, then, for a given half-analytic function $F^*(x, y)$, we have a unique function $F(z, \bar{z})$ analytic in $2n$ complex variables $z^\lambda, \bar{z}^\lambda$ in W such that

$$F(z, \bar{z}) = F^*(x, y). \tag{1.7}$$

Here and hereafter the phrase "analytic in W" means "analytic in a neighborhood of $\theta(U)$ for every envisaged U in V".

Conversely, if there is given a function $F(z, \bar{z})$ analytic in $z^\lambda, \bar{z}^\lambda$ in W, then the map θ^* induced by the inclusion map θ gives a function

$$F^*(x, y) = (\theta^* F)(x, y) = F(z, \bar{z}) \tag{1.8}$$

in V. Therefore *the map θ^* gives a one-one correspondence between analytic functions in W and half-analytic functions in V*, and is called the *realization.*

When there is given a half-analytic tensor or a connection in V, by the same means as for functions, we can construct a tensor or a connection analytic in W, and conversely, we can realize a tensor or a connection given in W into V.

For a half-analytic function $F^*(x, y)$ in V and its analytical continuation $F(z, \bar{z})$ in W, we have

$$\left.\frac{\partial F}{\partial z^\lambda}\right|_{\bar{z}=\bar{z}} = \frac{1}{2}\left(\frac{\partial F^*}{\partial x^\lambda} - \frac{\partial F^*}{\partial y^\lambda}\right) \text{ and } \left.\frac{\partial F}{\partial \bar{z}^\lambda}\right|_{\bar{z}=\bar{z}} = \frac{1}{2}\left(\frac{\partial F^*}{\partial x^\lambda} + i\frac{\partial F^*}{\partial y^\lambda}\right) \tag{1.9}$$

in V. To write briefly in symbol, the differentiation ∂^* on V is given by

$$\partial^* = \theta^* \partial, \tag{1.10}$$

∂ being the differentiation in W. When a connection is given in W or when a connection given on V is continuated analytically into W, from (1.10) together with property of the realization θ^* on tensors and connections, we may have a relation $D^* = \theta^* D$, D^* and D being the covariant differentiations on V and in W respectively.

Two half-analytic functions F and G in W are said to be *conjugate*, written in symbol $G = \bar{F}$, if the power series of one of them, in which all the coefficients are replaced by their conjugates and z^λ and $\bar{z}^\lambda$ are interchanged, is identical to the power series of the other. *If and only if F and G are conjugate in W to each other, $\theta^* F$ and $\theta^* G$ are conjugate in ordinary sense in V.*

A half-analytic function F in W is said to be *real-valued* if it is self-conjugate, i.e., $F = \bar{F}$, and to be *positive* if $\theta^* F$ is positive in V in addition to self-conjugateness. A half-analytic function F is said to be *holomorphic* if it is independent of $\bar{z}^\lambda$, i.e., $\dfrac{\partial F}{\partial \bar{z}^\lambda} = 0$.

Two tensors S and T or two affine connections Γ and Δ are said to be *conjugate*, written in symbol $T = \bar{S}$ or $\Delta = \bar{\Gamma}$, if each component with indices barred simultaneously of one is conjugate to the corresponding component of the other. We have clearly

$$\theta^* \bar{T} = \overline{\theta^* T},$$

where the bar in right hand side denotes the conjugate value in ordinary sense.

A tensor T (or an affine connection) is said to be *self-conjugate* if $\bar{T} = T$ in W, and *then and only then all the components of T are real-valued* because of $\overline{\theta^* T} = \theta^* \bar{T} = \theta^* T$ and of the biuniqueness of θ^*.

Self-conjugateness is closed with respect to arithmetical and differential operations, and consequently, to covariant differentiation in W.

In conformity to the consideration in this section, when a Geometry with half-analyticity is given on V, first we imbed V into the product space $V \times \bar{V}$ by the diagonal map, and extend analytically quantities in V into W, next we construct formally a Geometry in W, and finally we may return to the original Geometry on V by realization of the Geometry constructed in W.

To keep in mind on the things stated above and without repeating them we shall investigate Hermitian or Kählerian Geometry on V in the following sections.

The following lemma will be used frequently.

Lemma. *If the induced function θ^*F of a half-analytic function $F(z, \bar{z})$ defined in W vanishes identically on $\theta(V)$, then $F(z, \bar{z})$ vanishes identically in W.*

§2. Preliminaries on Hermitian and Kählerian spaces. (see Bochner, [1] - [4]). Now we introduce as a fundamental tensor on V a self-conjugate positive definite symmetric Hermitian tensor $g_{ij}(z, \bar{z})$ half-analytic on V, whose components are

$$(g_{ij}) = \begin{pmatrix} 0 & g_{\lambda\bar{\mu}} \\ g_{\bar{\lambda}\mu} & 0 \end{pmatrix},$$

where

$$g_{\lambda\bar{\mu}} = g_{\bar{\mu}\lambda} = \overline{g_{\bar{\lambda}\mu}}. \tag{2.1}$$

The fundamental form on V is defined by

$$ds^2 = g_{ij}dz^i dz^j = 2g_{\lambda\bar{\mu}}dz^\lambda d\bar{z}^\mu. \tag{2.2}$$

The geometry based on such a fundamental form is called a *Hermitian Geometry* on V.

Its affine connection defined by formulas as usual,

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{hj}}{\partial z^k} + \frac{\partial g_{hk}}{\partial z^j} - \frac{\partial g_{jk}}{\partial z^h}\right) \tag{2.3}$$

has components

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\bar{\kappa}}\left(\frac{\partial g_{\bar{\kappa}\mu}}{\partial z^\nu} + \frac{\partial g_{\bar{\kappa}\nu}}{\partial z^\mu}\right), \\ \Gamma^\lambda_{\mu\bar{\nu}} &= \frac{1}{2} g^{\lambda\bar{\kappa}}\left(\frac{\partial g_{\bar{\kappa}\mu}}{\partial \bar{z}^\nu} - \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\kappa}\right), \\ \Gamma^\lambda_{\bar{\mu}\bar{\nu}} &= 0, \end{aligned} \tag{2.4}$$

here and hereafter partially differential operators meaning the left hand sides of (1.9). When we introduce by usual way the covariant differentiation with respect to the connection Γ^i_{jk}, denoted by comma, equations

(2.5) $$g_{ij,k} = 0 \quad \text{and} \quad g^{ij}{}_{,k} = 0$$

hold for all combinations $i = \bar{\lambda}, \lambda$; $j = \mu, \bar{\mu}$; $k = \nu, \bar{\nu}$, and we may use them for pulling indices up and down.

If the fundamental tensor g_{ij} satisfies the metric condition of Kähler

(2.6) $$\frac{\partial g_{\lambda\bar{\mu}}}{\partial z^{\nu}} = \frac{\partial g_{\nu\bar{\mu}}}{\partial z^{\lambda}}.$$

then the space is called a *Kählerian* space. In such a space there exists locally such a function $\Phi(z, \bar{z})$ that

(2.7) $$g_{\lambda\bar{\mu}} = \frac{\partial^2 \Phi}{\partial z^{\lambda} \partial \bar{z}^{\mu}},$$

and the components $\Gamma^{\lambda}_{\mu\bar{\nu}}$ and their conjugates $\Gamma^{\bar{\lambda}}_{\bar{\mu}\nu}$ vanish. The components of Riemann-Christoffel curvature tensor $R^{i}{}_{jkl}$ of a Kählerian space all vanish except $R^{\lambda}{}_{\mu\bar{\nu}\omega}$ given by

(2.8) $$R^{\lambda}{}_{\mu\bar{\nu}\omega} = \frac{\partial \Gamma^{\lambda}_{\mu\omega}}{\partial \bar{z}^{\nu}}.$$

Its covariant components are given in form

(2.9) $$\begin{aligned} R_{\lambda\bar{\mu}\nu\bar{\omega}} &= \frac{\partial^2 g_{\lambda\bar{\mu}}}{\partial z^{\nu} \partial \bar{z}^{\omega}} - g^{\rho\bar{\sigma}} \frac{\partial g_{\lambda\bar{\sigma}}}{\partial z^{\nu}} \frac{\partial g_{\rho\bar{\mu}}}{\partial \bar{z}^{\omega}} \\ &= \frac{\partial^4 \Phi}{\partial z^{\lambda} \partial \bar{z}^{\mu} \partial z^{\nu} \partial \bar{z}^{\omega}} - g^{\rho\bar{\sigma}} \frac{\partial^3 \Phi}{\partial z^{\lambda} \partial \bar{z}^{\sigma} \partial z^{\nu}} \frac{\partial^3 \Phi}{\partial \bar{z}^{\mu} \partial z^{\rho} \partial \bar{z}^{\omega}} \end{aligned}$$

and satisfy relations

(2.10) $$\begin{aligned} R_{\lambda\bar{\mu}\nu\bar{\omega}} &= -R_{\lambda\bar{\mu}\bar{\omega}\nu} = -R_{\bar{\mu}\lambda\nu\bar{\omega}} = R_{\nu\bar{\omega}\lambda\bar{\mu}}, \\ R_{\lambda\bar{\mu}\nu\bar{\omega}} &= R_{\lambda\bar{\omega}\nu\bar{\mu}} = R_{\nu\bar{\mu}\lambda\bar{\omega}}. \end{aligned}$$

Moreover, for the Ricci tensor $R_{jk} = g^{il} R_{ijkl}$, we have components

(2.11) $$R_{\mu\nu} = R_{\bar{\mu}\bar{\nu}} = 0$$

and

(2.12) $$R_{\mu\bar{\nu}} = \frac{\partial^2 \log G}{\partial z^{\mu} \partial \bar{z}^{\nu}},$$

where G is the determinant

(2.13) $$G = \det (g_{\lambda\bar{\mu}})$$

and is positive real-valued. The scalar curvature is

$$R = g^{jk} R_{jk} = 2g^{\lambda\bar{\mu}} R_{\lambda\bar{\mu}}. \tag{2.14}$$

§3. **Geodesics.** We consider a curve in a Hermitian space V defined by parametric representations in a real parameter

$$z^\lambda = z^\lambda(t). \tag{3.1}$$

Its arc-length is given by

$$s = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} F\left(z, \bar{z}, \frac{dz}{dt}, \frac{d\bar{z}}{dt}\right) dt, \tag{3.2}$$

where we have put

$$F = \sqrt{2g_{\lambda\bar{\mu}}(z, \bar{z}) \frac{dz^\lambda}{dt} \frac{d\bar{z}^\mu}{dt}}, \tag{3.3}$$

which is positive real valued because of self-conjugateness.

Let us seek for curves which make their arc-lengths extremal. Such curves are called *geodesics* in V. By calculus of variations,

$$\begin{aligned}
\delta \int_{t_0}^{t_1} ds &= \int_{t_0}^{t_1} \delta F \, dt \\
&= \int_{t_0}^{t_1} \frac{1}{F} \left(\delta g_{\lambda\bar{\mu}} \frac{dz^\lambda}{dt} \frac{d\bar{z}^\mu}{dt} + g_{\lambda\bar{\mu}} \frac{\delta\, dz^\lambda}{dt} \frac{d\bar{z}^\mu}{dt} + g_{\lambda\bar{\mu}} \frac{dz^\lambda}{dt} \frac{\delta\, d\bar{z}^\mu}{dt} \right) dt \\
&= \int_{t_0}^{t_1} \frac{1}{F} \Bigg[\Bigg(\frac{\partial g_{\nu\bar{\mu}}}{\partial z^\lambda} \frac{dz^\nu}{dt} \frac{d\bar{z}^\mu}{dt} - \frac{\partial g_{\lambda\bar{\mu}}}{\partial z^\nu} \frac{dz^\nu}{dt} \frac{d\bar{z}^\mu}{dt} - \frac{\partial g_{\lambda\bar{\mu}}}{\partial \bar{z}^\nu} \frac{d\bar{z}^\nu}{dt} \frac{d\bar{z}^\mu}{dt} \\
&\qquad - g_{\lambda\bar{\mu}} \frac{d^2 \bar{z}^\mu}{dt^2} + g_{\lambda\bar{\mu}} \frac{d\bar{z}^\mu}{dt} \frac{d \log F}{dt} \Bigg) \delta z^\lambda \\
&\qquad + \left(\frac{\partial g_{\nu\bar{\mu}}}{\partial \bar{z}^\lambda} \frac{dz^\nu}{dt} \frac{d\bar{z}^\mu}{dt} - \frac{\partial g_{\bar{\lambda}\mu}}{\partial z^\nu} \frac{dz^\nu}{dt} \frac{dz^\mu}{dt} - \frac{\partial g_{\bar{\lambda}\mu}}{\partial \bar{z}^\nu} \frac{d\bar{z}^\nu}{dt} \frac{dz^\mu}{dt} \right. \\
&\qquad \left. - g_{\bar{\lambda}\mu} \frac{d^2 z^\mu}{dt^2} + g_{\bar{\lambda}\mu} \frac{dz^\mu}{dt} \frac{d \log F}{dt} \right) \delta \bar{z}^\lambda \Bigg] dt.
\end{aligned}$$

In order that the arc-length becomes extremal, it is necessary and sufficient that the expression in brackets vanishes, and, by Lemma, it is equivalent to the vanishing of the coefficients of δz^λ and $\delta \bar{z}^\lambda$. Hence, from the coefficients of $\delta \bar{z}^\lambda$, we obtain

$$\begin{aligned}
& g_{\bar{\lambda}\mu} \frac{d^2 z^\mu}{dt^2} + \frac{1}{2} \left(\frac{\partial g_{\bar{\lambda}\mu}}{\partial z^\nu} + \frac{\partial g_{\bar{\lambda}\nu}}{\partial z^\mu} \right) \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} + \left(\frac{\partial g_{\bar{\lambda}\mu}}{\partial \bar{z}^\nu} - \frac{\partial g_{\bar{\nu}\mu}}{\partial \bar{z}^\lambda} \right) \frac{dz^\mu}{dt} \frac{d\bar{z}^\nu}{dt} \\
&= \rho(t) g_{\bar{\lambda}\mu} \frac{dz^\mu}{dt},
\end{aligned}$$

and, $\rho(t) = d(\log F)/dt$ being real-valued, we observe that the coefficients of δz^λ are the conjugates of the coefficients of $\delta\bar{z}^\lambda$. Pulling up the index $\bar{\lambda} \to \lambda$, we obtain in final as equations of geodesics

$$\text{(3.4)} \qquad \frac{d^2 z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} + 2\Gamma^\lambda_{\mu\bar{\nu}} \frac{dz^\mu}{dt} \frac{d\bar{z}^\nu}{dt} = \rho(t) \frac{dz^\lambda}{dt}.$$

These equations show that the geodesics are autoparallel with respect to the connection Γ^i_{jk} defined by (2.3). If we take the arc-length s as parameter, then $\rho(s)$ is equal to zero.

In the following, unless otherwise stated, we shall confine ourselves to Kählerian spaces, and then the equations (3.4) of geodesics are reduced to

$$\text{(3.5)} \qquad \frac{D^2 z^\lambda}{dt^2} \equiv \frac{d^2 z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} = \rho(t) \frac{dz^\lambda}{dt},$$

and, when the arc-length s is taken as parameter,

$$\text{(3.6)} \qquad \frac{D^2 z^\lambda}{ds^2} \equiv \frac{d^2 z^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{ds} \frac{dz^\nu}{ds} = 0.$$

§4. **Holomorphically planar curves.** Next we consider curves defined by equations, similar to (3.5),

$$\text{(4.1)} \qquad \frac{D^2 z^\lambda}{dt^2} \equiv \frac{d^2 z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} = \rho(t) \frac{dz^\lambda}{dt},$$

in which $\rho(t)$ is not real-valued, but complex-valued in general. Although we take the arc-length s as parameter, $\rho(s)$ does not vanish for these curves, and we have

$$\text{(4.2)} \qquad \frac{D\mathbf{e}_1}{ds} = \rho(s)\,\mathbf{e}_1,$$

where, for simplicity, $\mathbf{e}_1$ denotes the unit tangent vector $\frac{dz^\lambda}{ds}$ in vector symbol. Since the tangent vector $\mathbf{e}_1$ is unitary, we have

$$\rho(s) + \bar{\rho}(s) = 0.$$

Hence $\rho(s)$ is pure imaginary, and we put $\rho(s) = i\sigma(s)$, $\sigma(s)$ being a real-valued function. Moreover, if we put $\mathbf{e}_2 = i\mathbf{e}_1$, then the vector $\mathbf{e}_2$ is also unitary and unitary orthogonal to the vector $\mathbf{e}_1$, and the equation (4.2) is put in form

$$\frac{D\mathbf{e}_1}{ds} = \sigma(s)\,\mathbf{e}_2\,, \tag{4.3}$$

and we have

$$\frac{D\mathbf{e}_2}{ds} = -\sigma(s)\,\mathbf{e}_1\,. \tag{4.4}$$

These equations (4.3) and (4.4) are the first two of Frenet-Serret formulas of the present curves in V, and the remainders of the formulas do not appear. Conversely, if a real-valued function $\sigma(s)$ is given, then a curve satisfying (4.3) and (4.4) is uniquely determined within situation, since

$$\frac{D}{ds}(\mathbf{e}_2 - i\,\mathbf{e}_1) = -i\sigma(s)\,(\mathbf{e}_2 - i\,\mathbf{e}_1)$$

vanishes by the initial condition $\mathbf{e}_2 = i\mathbf{e}_1$.

The two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form a holomorphic section and (4.3) and (4.4) show that these curves are planar in this holomorphic section. Conversely, it is evident that a curve osculating the holomorphic section partaining to its tangent vector has (4.3) and (4.4) as Frenet-Serret formulas. We say these curves to be *holomorphically planar*.

The above arguments are also applicable to curves in more general Hermitian spaces defined by

$$\frac{d^2z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} + 2\Gamma^\lambda_{\mu\bar{\nu}}\frac{dz^\mu}{dt}\frac{d\bar{z}^\nu}{dt} = \rho(t)\frac{dz^\lambda}{dt}\,, \tag{4.5}$$

$\rho(t)$ being complex valued, which are also called holomorphically planar curves in Hermitian space.

§5. **Fubini spaces.** In preparation for the next section, we reconstruct here a Fubini space in one way appropriate to our purpose.

In a complex projective space P_n we take a fundamental domain F whose points have homogeneous coordinates $(\xi) = (\xi_0, \xi_1, \cdots\cdots, \xi_n)$ such that

$$Q(\xi, \bar{\xi}) = \xi_0\bar{\xi}_0 + \frac{k}{2}\sum_{\lambda=1}^{n}\xi_\lambda\bar{\xi}_\lambda > 0 \tag{5.1}$$

corresponding to a complex elliptic space for $k > 0$ and to a complex hyperbolic space for $k < 0$. The hyperquadric in P_n defined by equations

$$Q(\xi, \bar{\xi}) = 0 \tag{5.2}$$

is called the absolute of the space F.

The complex line joining two analytic points (η) and (ζ) in F is parametrically represented such as

$$(\xi) = (\eta) + \sigma(\zeta), \tag{5.3}$$

where σ is a complex parameter, $\sigma = re^{i\theta}$. The origin $\sigma = 0$ of Gaussian σ-palne corresponds to the point (η), the infinity $\sigma = \infty$ to the point (ζ), and the straight line for each value of θ to a Möbius circle passing (η) and (ζ). The chain of the complex line (5.3), i.e., the intersection of the line with the absolute, is the set of points which are parametrized by solutions σ of equation

$$Q(\xi, \bar{\xi}) = Q(\eta, \bar{\eta}) + \bar{\sigma}Q(\eta, \bar{\zeta}) + \sigma Q(\bar{\eta}, \zeta) + \sigma\bar{\sigma}Q(\zeta, \bar{\zeta}) = 0. \tag{5.4}$$

If we put

$$Q(\eta, \bar{\eta}) = a, \quad Q(\eta, \bar{\zeta}) = \beta = be^{i\varphi}, \quad Q(\zeta, \bar{\zeta}) = c, \tag{5.5}$$

then (5.4) is reduced to

$$a + 2rb \cos(\varphi - \theta) + r^2 c = 0. \tag{5.6}$$

This equation with a fixed value of θ gives an involution on the corresponding Möbius circle.

Among the Möbius circles passing through the two points (η) and (ζ) we choose a special one characterized by $\varphi = \theta$, this condition being independent of the analytic representation of homogeneous coordinates of the two points. In fact, the special circle is characterized by the property that σ/β is real, and if the two points have another homogeneous coordinates $(\eta_1) = \lambda(\eta)$ and $(\zeta_1) = \mu(\zeta)$, then the complex line

$$(\xi) = \frac{(\eta_1)}{\lambda} + \sigma\frac{(\zeta_1)}{\mu}$$

is parametrized by $\sigma_1 = \dfrac{\lambda}{\mu}\sigma$, and then $\beta_1 = Q(\eta_1, \bar{\zeta}_1) = \lambda\bar{\mu}\beta$. Hence we see

$$\frac{\sigma_1}{\beta_1} = \frac{1}{\mu\bar{\mu}}\frac{\sigma}{\beta}$$

to be real. If we put $(\eta_1) = (\eta)$ and $(\zeta_1) = \beta(\zeta)$, then $\sigma_1 = \dfrac{\sigma}{\beta}$ is real, say put r, and the circle is parametrized by the real parameter r such as

$$(\xi) = (\eta_1) + r(\zeta_1). \tag{5.7}$$

In other words, when we choose the homogeneous coordinates (η) and (ζ) such as $Q(\eta, \bar{\zeta})$ is real, the circle corresponds to the real axis of σ-plane. Consequently, for any two points on the circle, the circle joining the two points of the characteristic property just stated is identical to the former. We call it the *geodesic* joining the points (η) and (ζ).

Along geodesic, (5.6) is reduced to

$$a + 2rb + r^2c = 0 \tag{5.8}$$

and it has real conjugate roots $\rho_+ = r_+$ and $\rho_- = r_-$ for hyperbolic case and conjugate imaginary roots $\rho_+ = re^{i\varphi}$ and $\rho_- = re^{-i\varphi}$ for elliptic case. Then a signed distance between (η) and (ζ) is defined by

$$\text{dis}\,(\eta, \zeta) = \frac{1}{\sqrt{-k}} \log\,(\eta, \zeta, \xi_+, \xi_-) = \frac{1}{\sqrt{-k}} \log \frac{\rho_+}{\rho_-}, \tag{5.9}$$

where we make ξ_+ and ξ_- correspond to $\sigma_+ = \rho_+ e^{i\theta}$ and $\sigma_- = \rho_- e^{i\theta}$ respectively and $(\eta, \zeta, \xi_+, \xi_-)$ indicates the harmonic ratio of the four points.

For any three points (η), (ζ), (κ) on a geodesic, the relation

$$\text{dis}\,(\eta, \zeta) + \text{dis}\,(\zeta, \kappa) = \text{dis}\,(\eta, \kappa)$$

holds.

The distance (5.9) is given by explicit formula

$$\text{dis}\,(\eta, \zeta) = \frac{2}{\sqrt{-k}} \log \frac{[Q(\eta, \bar{\zeta})Q(\bar{\eta}, \zeta)]^{\frac{1}{2}} - [Q(\eta, \bar{\zeta})Q(\bar{\eta}, \zeta) - Q(\eta, \bar{\eta})Q(\zeta, \bar{\zeta})]^{\frac{1}{2}}}{[Q(\eta, \bar{\eta})Q(\zeta, \bar{\zeta})]^{\frac{1}{2}}}.$$

If a point (ζ) is infinitesinally near by a point (η), i.e., $(\zeta) = (\eta + d\eta)$, then, after some infinitesimal calculations, we can obtain the differential form of the distance

$$\begin{aligned} ds^2 &= [\text{dis}\,(\eta, \eta + d\eta)]^2 \\ &= \frac{4}{k} \frac{1}{Q(\eta, \bar{\eta})^2} [Q(\eta, \bar{\eta})Q(d\eta, d\bar{\eta}) - Q(\eta, d\bar{\eta})Q(\bar{\eta}, d\eta)], \end{aligned}$$

at which we can also arrive in different ways on infinitesimal analysis.

In normalized homogeneous coordinates

$$Q(\eta, \bar{\eta}) = \eta_0\bar{\eta}_0 + \frac{k}{2}\sum \eta_\lambda\bar{\eta}_\lambda = 1$$

the formula of distance has simpler form

$$\begin{aligned} ds^2 &= \frac{4}{k}\{Q(d\eta, d\bar{\eta}) - Q(\eta, d\bar{\eta})Q(\bar{\eta}, d\eta)\} \\ &= \frac{4}{k}\Big\{\Big(d\eta_0 d\bar{\eta}_0 + \frac{k}{2}\sum d\eta_\lambda d\bar{\eta}_\lambda\Big) \\ &\quad - \Big(\eta_0 d\bar{\eta} + \frac{k}{2}\sum \eta_\lambda d\bar{\eta}_\lambda\Big)\Big(\bar{\eta}_0 d\eta_0 + \frac{k}{2}\sum \bar{\eta}_\lambda d\eta_\lambda\Big)\Big\}. \end{aligned} \tag{5.10}$$

If we introduce non-homogeneous coordinates $z_\lambda = \frac{\eta_\lambda}{\eta_0}$, then we can verify that, the fundamental form has the expression

$$\frac{ds^2}{2} = \frac{\sum |dz_\lambda|^2 + \frac{k}{2}(\sum |z_\lambda|^2 \sum |dz_\mu|^2 - |\sum \bar{z}_\lambda dz_\lambda|^2)}{\Big(1 + \frac{k}{2}\sum |z_\lambda|^2\Big)^2} \tag{5.11}$$

which is a so-called Fubini metric. In consequence, we know that the fundamental domain of an elliptic or a hyperbolic complex geometry is locally equivalent to a Fubini space with $k > 0$ or $k < 0$. Fubini space is a special Kählerian space.

The fundamental tensor of a Fubini space has covariant components

$$g_{\lambda\bar{\mu}} = \frac{\delta_{\lambda\mu}}{S} - \frac{k}{2}\frac{\bar{z}_\lambda z_\mu}{S^2} \tag{5.12}$$

and contravariant components

$$g^{\lambda\bar{\mu}} = S\Big(\delta_{\lambda\mu} + \frac{k}{2} z_\lambda \bar{z}_\mu\Big), \tag{5.13}$$

where we have put

$$S = 1 + \frac{k}{2}\sum \bar{z}_\lambda z_\lambda . \tag{5.14}$$

Its Christoffel symbols are

$$\Gamma^\lambda_{\mu\nu} = -\frac{k}{2S}(\delta^\lambda_\mu \bar{z}_\nu + \delta^\lambda_\nu \bar{z}_\mu) \tag{5.15}$$

and the curvature tensor satisfies the equations

$$R_{\lambda\bar{\mu}\nu\bar{\omega}} = -\frac{k}{2}(g_{\lambda\bar{\mu}}g_{\nu\bar{\omega}} + g_{\lambda\bar{\omega}}g_{\nu\bar{\mu}}). \tag{5.16}$$

Therefore a Fubini space is locally equivalent to a space of constant holomorphic curvature, see [2, Theorem 6].

§6. Holomorphically planar curves in Fubini space. In a Fubini space the equations (4.1) of holomorphic planar curves are reduced, by (5.15), to

$$\frac{d^2z^\lambda}{dt^2} - \frac{k}{S}\sum \bar{z}_\nu \frac{dz^\nu}{dt}\frac{dz^\lambda}{dt} = \rho(t)\frac{dz^\lambda}{dt}. \tag{6.1}$$

To find what holomorphically planar curves are in a Fubini space, let us consider curves in complex analytic lines of a complex projective space with a Fubini metric. Any curve lying in a complex line represented by (5.3) is represented by parametric equation

$$(\xi(t)) = (\eta) + \sigma(t)(\zeta), \tag{6.2}$$

where $\sigma(t)$ is a complex valued function of one real parameter t. In non homogeneous coordinates, the curve is represented by

$$z_\lambda = \frac{\eta_\lambda + \sigma(t)\zeta_\lambda}{\eta_0 + \sigma(t)\zeta_0}. \tag{6.3}$$

Differentiating (6.3) successively, we have

$$\frac{dz^\lambda}{dt} = \frac{(\eta_0\zeta_\lambda - \eta_\lambda\zeta_0)}{(\eta_0 + \sigma(t)\zeta_0)^2}\frac{d\sigma}{dt},$$

$$\frac{d^2z^\lambda}{dt^2} = \frac{dz^\lambda}{dt}\frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}} - \frac{2\zeta_0\dfrac{d\sigma}{dt}}{\eta_0 + \sigma(t)\zeta_0}\frac{dz^\lambda}{dt},$$

from which, taking account of (5.16), there is obtained the differential equations of the curve (6.3)

$$\frac{d^2z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} = \tau(t)\frac{dz^\lambda}{dt}, \tag{6.4}$$

where

$$\tau(t) = \frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}} - \left[\frac{2\zeta_0}{(\eta_0 + \sigma(t)\zeta_0)} + \frac{k}{S}\frac{(\bar{\eta}_\lambda + \bar{\sigma}(t)\bar{\zeta}_\lambda)(\eta_0\zeta_\lambda - \eta_\lambda\zeta_0)}{(\bar{\eta}_0 + \bar{\sigma}(t)\bar{\zeta}_0)(\eta_0 + \sigma(t)\zeta_0)^2}\right]\frac{d\sigma}{dt}. \tag{6.5}$$

In order to show that the integral curves of (6.1) with a given $\rho(t)$ lie always in complex lines of projective space, it suffices to prove the existence of the solution of differential equations (6.5) in $\sigma(t)$, where $\tau(t)$ is equated to $\rho(t)$ given arbitrarily. To do this we may assume that the points (η), (ξ) have homogeneous coordinates

$$\eta_0 = 1,\ \eta_1 = 0,\ \cdots\cdots\cdots\cdots\cdots\cdots,\ \eta_n = 0,$$
$$\zeta_0 = 0,\ \zeta_1 = 1,\ \zeta_2 = 0,\ \cdots\cdots\cdots\cdots,\ \zeta_n = 0,$$

since the integral curve of (6.1) is determined within motion in Fubini space. For hyperbolic case, although the point (ζ) lies outside our fundamental domain, it is enough to restrict ourselves in the part of the line inside the absolute. Then the current point of the curve (6.3) is represented by

$$z_1 = \sigma(t), \quad z_2 = \cdots\cdots = z_n = 0,$$

and (6.5) is reducible to

$$\frac{\dfrac{d^2\sigma}{dt^2}}{\dfrac{d\sigma}{dt}} - \frac{k\bar{\sigma}}{1+\dfrac{k}{2}\sigma\bar{\sigma}}\frac{d\sigma}{dt} = \rho(t). \tag{6.6}$$

Separating this equation into real and imaginary parts, we obtain a system of ordinary differential equations of order two in two unknown functions, which are the real and imaginary parts of $\sigma(t)$. In virtue of the uniqueness of solutions of such a system, the integral curves of (6.1) lie in complex lines of a projective space.

§7. **Geodesic complex curves.** Let us now consider a complex curve defined by analytic functions of one complex parameter τ

$$z^\lambda = z^\lambda(\tau). \tag{7.1}$$

A complex curve is an analytic subspace of topological dimension two. There is defined the induced metric whose fundamental tensor has one covariant component

$$g_{1\bar{1}}(\tau,\bar{\tau}) \equiv g(\tau,\bar{\tau}) = g_{\lambda\bar{\mu}}\frac{dz^\lambda}{d\tau}\overline{\frac{dz^\mu}{d\tau}} \tag{7.2}$$

and one contravariant component

$$g^{1\bar{1}}(\tau,\bar{\tau}) = \frac{1}{g(\tau,\bar{\tau})}. \tag{7.3}$$

Its Christoffel symbol is only

$$\Gamma_{1}{}^{1}{}_{1} \equiv \Gamma = g^{1\bar{1}} \frac{\partial g_{\bar{1}1}}{\partial \tau} = \frac{\partial \log g}{\partial \tau}. \tag{7.4}$$

An analytic change of parameter of the curve

$$\tau' = \tau'(\tau) \tag{7.5}$$

causes the above quantities to be transformed as follows:

$$g = g' \frac{d\tau'}{d\tau} \frac{\overline{d\tau'}}{d\tau} \tag{7.6}$$

and

$$\Gamma = \Gamma' \frac{d\tau'}{d\tau} + \frac{\dfrac{d^2\tau'}{d\tau^2}}{\dfrac{d\tau'}{d\tau}}. \tag{7.7}$$

We put

$$\frac{d\tau'}{d\tau} = pe^{i\theta}. \tag{7.8}$$

The vector ε^λ defined by

$$\varepsilon^\lambda = (2g)^{-\frac{1}{2}} \frac{dz^\lambda}{d\tau} \tag{7.9}$$

is unitary and, under change of parameter (7.5), it accepts the change

$$\varepsilon^\lambda = e^{i\theta} \varepsilon'^\lambda, \tag{7.10}$$

that is, it rotates in the holomorphic section containg itself. We call it the *unitary tangent vector of the complex curve.*

We put

$$h^\lambda = \frac{d^2 z^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} - \frac{\partial \log g}{\partial \tau} \frac{dz^\lambda}{d\tau} \tag{7.11}$$

and define a vector κ^λ by

$$\kappa^\lambda = \frac{1}{2g} h^\lambda. \tag{7.12}$$

Under (7.5), the vector h^λ is changed by

$$h^\lambda = h'^\lambda \left(\frac{d\tau'}{d\tau} \right)^2, \tag{7.13}$$

and, in virtue of (7.8), the vector κ^λ by

$$\kappa^\lambda = e^{2i\theta}\kappa'^\lambda, \tag{7.14}$$

the latter meaning that the vector κ^λ rotates in the holomorphic section containing itself.

Taking account of (7.6), (7.11) and (7.12), we can easily verify that

$$g_{\lambda\bar{\mu}}\kappa^\lambda\bar{\varepsilon}^\mu = 0, \tag{7.15}$$

that is, the vector κ^λ is unitary orthogonal to the tangent vector ε^λ, in other words, the holomorphic section attendant to the former is unitary orthogonal to that attendant to the latter.

We call the vector κ^λ *the vector of the first curvature of complex curve* [Schouten and Struik, 7], and a complex curve with vanishing vector of the first curvature a *geodesic complex* curve. From (7.11) and (7.12) the differential equations of geodesic complex curve are

$$\frac{d^2z^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}(z(\tau), \bar{z}(\bar{\tau}))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = \frac{\partial \log g(\tau, \bar{\tau})}{\partial\tau}\frac{dz^\lambda}{d\tau}. \tag{7.16}$$

§8. Theorems on geodesic complex curves. Any curve in a geodesic complex curve (7.1) is given by (7.1) in which τ is a complex valued function $\tau(t)$ of a real parameter t and satisfies the equations

$$\frac{d^2z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}(z(\tau(t)), \bar{z}(\bar{\tau}(t)))\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} = \rho_1(t)\frac{dz^\lambda}{dt}, \tag{8.1}$$

$\rho_1(t)$ being a complex-valued proportional factor. From the above equations, we can obtain at once the following

Theorem 1. *Any curve on a geodesic complex curve is holomorphically planar, and conversely.*

Besides geodesic complex curves, we may have *extensions* of real geodesics, uniquely determined by analytic continuation of real parameter t into a neighborhood of the real axis in the Gaussian τ-plane. The extension of a geodesic (3.5) is given by

$$\frac{d^2z^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}(z(\tau), \bar{z}(\tau))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = \rho(\tau)\frac{dz^\lambda}{d\tau} \tag{8.2}$$

$\rho(\tau)$ being real-valued.

Theorem 2. *In order that the system of extensions of real geodesics and that of geodesic complex curves are identical to each other, it* *is necessary and sufficient that the space is a Fubini one.*

Proof. Subtracting (7.16) from (8.2), we have

$$\{\Gamma^{\lambda}_{\mu\nu}(z(\tau), \bar{z}(\tau)) - \Gamma^{\lambda}_{\mu\nu}(z(\tau), \bar{z}(\bar{\tau}))\} w^{\mu} w^{\nu} = \rho_2(\tau, \tau) w^{\lambda}, \tag{8.3}$$

$\rho_2(\tau, \bar{\tau})$ being a complex proportional factor and w^{λ} denoting $\dfrac{dz^{\lambda}}{d\tau}$ for brievity. Since, in the neighborhood of t-axis, by neglecting terms of higher order in $(\tau - \bar{\tau})$,

$$\bar{z}^{\lambda}(\tau) = \overline{z^{\lambda}(\bar{\tau})} = \overline{z^{\lambda}(\tau)} + \overline{z^{\lambda}(\bar{\tau})} - \overline{z^{\lambda}(\tau)} = \overline{z^{\lambda}(\tau)} + \overline{w}^{\lambda}(\bar{\tau})(\tau - \bar{\tau})$$

hold, we have, from (8.3)

$$(\tau - \bar{\tau}) \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial \bar{z}^{\omega}} w^{\mu} w^{\nu} \overline{w}^{\omega} = \rho_2(\tau, \bar{\tau}) w^{\lambda}$$

or, by (2.8),

$$R^{\lambda}_{\cdot\mu\nu\bar{\omega}} w^{\mu} w^{\nu} \overline{w}^{\omega} = \rho_3(\tau, \bar{\tau}) w^{\lambda}, \tag{8.4}$$

ρ_3 being a proportional factor. In order that (8.4) hold for any vector w^{λ}, we must have, by Lemma in §1,

$$R^{\lambda}_{\cdot\mu\nu\bar{\omega}}\delta^{\kappa}_{\rho} + R^{\lambda}_{\cdot\nu\rho\bar{\omega}}\delta^{\kappa}_{\mu} + R^{\lambda}_{\cdot\rho\mu\bar{\omega}}\delta^{\kappa}_{\nu} = R^{\kappa}_{\cdot\mu\nu\bar{\omega}}\delta^{\lambda}_{\rho} + R^{\kappa}_{\cdot\nu\rho\bar{\omega}}\delta^{\lambda}_{\mu} + R^{\kappa}_{\cdot\rho\mu\bar{\omega}}\delta^{\lambda}_{\nu},$$

from which it can be easily verify that

$$R_{\lambda\bar{\mu}\nu\bar{\omega}} = \frac{R}{2n(n+1)}(g_{\lambda\bar{\mu}}g_{\nu\bar{\omega}} + g_{\lambda\bar{\omega}}g_{\nu\bar{\mu}}). \tag{8.5}$$

Thus the space is a Fubini space with $k = -R/n(n+1)$. The sufficiency is easily verified.

§9. Holomorphically projective correspondence. The differential equations (4.5) of holomorphically planar curves in a Hermitian space are equivalent to

$$\begin{aligned}&\left(\frac{d^2z^{\lambda}}{dt^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dz^{\mu}}{dt}\frac{dz^{\nu}}{dt} + 2\Gamma^{\lambda}_{\mu\bar{\nu}}\frac{dz^{\mu}}{dt}\frac{d\bar{z}^{\nu}}{dt}\right)\frac{dz^{\kappa}}{dt} \\ &= \left(\frac{d^2z^{\kappa}}{dt^2} + \Gamma^{\kappa}_{\mu\nu}\frac{dz^{\mu}}{dt}\frac{dz^{\nu}}{dt} + 2\Gamma^{\kappa}_{\mu\bar{\nu}}\frac{dz^{\mu}}{dt}\frac{d\bar{z}^{\nu}}{dt}\right)\frac{dz^{\lambda}}{dt}.\end{aligned} \tag{9.1}$$

Now we consider two Hermitian spaces with fundamental tensors $g_{\lambda\bar{\mu}}$ and $h_{\lambda\bar{\mu}}$ and denote their Christoffel symbols by Γ and Δ respectively. If the two spaces have all holomorphically planar curves in common, then the equations (9.1) with Γ replaced by Δ are also satisfied, and, by subtracting (9.1) from these, we have identities

(9.2) $$(A^{\lambda}_{\mu\nu}w^{\mu}w^{\nu} + 2B^{\lambda}_{\mu\bar{\nu}}w^{\mu}\bar{w}^{\nu})w^{\kappa} = (A^{\kappa}_{\mu\nu}w^{\mu}w^{\nu} + 2B^{\kappa}_{\mu\bar{\nu}}w^{\mu}\bar{w}^{\nu})w^{\lambda},$$

where we have put

$$A^{\lambda}_{\mu\nu} = \Delta^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu}, \quad B^{\lambda}_{\mu\bar{\nu}} = \Delta^{\lambda}_{\mu\bar{\nu}} - \Gamma^{\lambda}_{\mu\bar{\nu}} \quad \text{and} \quad w^{\lambda} = \frac{dz^{\lambda}}{dt}.$$

From the above identities, by Lemma in §1, we have equations

(9.3) $$A^{\lambda}_{\mu\nu}\delta^{\kappa}_{\omega} + A^{\lambda}_{\nu\omega}\delta^{\kappa}_{\mu} + A^{\lambda}_{\omega\mu}\delta^{\kappa}_{\nu} = A^{\kappa}_{\mu\nu}\delta^{\lambda}_{\omega} + A^{\kappa}_{\nu\omega}\delta^{\lambda}_{\mu} + A^{\kappa}_{\omega\mu}\delta^{\lambda}_{\nu}$$

and

(9.4) $$B^{\lambda}_{\mu\bar{\nu}}\delta^{\kappa}_{\omega} + B^{\lambda}_{\omega\bar{\nu}}\delta^{\kappa}_{\mu} = B^{\kappa}_{\mu\bar{\nu}}\delta^{\lambda}_{\omega} + B^{\kappa}_{\omega\bar{\nu}}\delta^{\lambda}_{\mu}.$$

By contracting κ and ω in both equations, we have

(9.5) $$\begin{aligned} A^{\lambda}_{\mu\nu} &= \delta^{\lambda}_{\mu}\varphi_{\nu} + \delta^{\lambda}_{\nu}\varphi_{\mu}, \\ B^{\lambda}_{\mu\bar{\nu}} &= \delta^{\lambda}_{\mu}\psi_{\bar{\nu}}, \end{aligned}$$

where we have put

$$\varphi_{\nu} = \frac{1}{n+1}A^{\lambda}_{\lambda\nu}, \quad \psi_{\nu} = \frac{1}{n}B^{\bar{\lambda}}_{\bar{\lambda}\nu}.$$

Therefore the Christoffel symbols Γ and Δ are related by formulas

(9.6) $$\begin{aligned} \Delta^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu}\varphi_{\nu} + \delta^{\lambda}_{\nu}\varphi_{\mu} \\ \Delta^{\lambda}_{\mu\bar{\nu}} &= \Gamma^{\lambda}_{\mu\bar{\nu}} + \delta^{\lambda}_{\mu}\psi_{\bar{\nu}}. \end{aligned}$$

We shall call such a correspondence a *holomorphically projective* one, which was introduced and called "Bahntreue Transformation" by Schouten and Struik [7], however, their introduction seemed rather formally.

After these, we can state the following

Theorem 3. *If a Hermitian space is holomorphically projective to a Kählerian space, then the former is also Kählerian.*

Proof. If a metric $g_{\lambda\bar{\mu}}$ is Kählerian, then $\Gamma^{\lambda}_{\mu\bar{\nu}}$ vanish and we have from the conjugates of the seconds of (2.4) and of (9.6)

$$\Delta^{\bar{\lambda}}_{\bar{\mu}\nu} = \frac{1}{2}h^{\bar{\lambda}\kappa}\left(\frac{\partial h_{\kappa\bar{\mu}}}{\partial z^{\nu}} - \frac{\partial h_{\bar{\mu}\nu}}{\partial z^{\kappa}}\right) = \delta^{\bar{\lambda}}_{\bar{\mu}}\psi_{\nu}$$

or

$$\frac{\partial h_{\lambda\bar{\mu}}}{\partial z^{\nu}} - \frac{\partial h_{\nu\bar{\mu}}}{\partial z^{\lambda}} = h_{\lambda\bar{\mu}}\psi_{\nu}\,. \tag{9.7}$$

Since the left hand sides are alternative in λ and ν, we have

$$h_{\lambda\bar{\mu}}\psi_{\nu} + h_{\nu\bar{\mu}}\psi_{\lambda} = 0,$$

from which, by contracting $h^{\lambda\bar{\mu}}$, we see ψ_{ν} vanishing. Thus the tensor $h_{\lambda\bar{\mu}}$ satisfies the condition of Kähler (2.6). Then $\Delta^{\lambda}_{\mu\bar{\nu}}$ are equal to zero and the non-vanishing symbols are only

$$\Delta^{\lambda}_{\mu\nu} = h^{\lambda\bar{\kappa}}\frac{\partial h_{\bar{\kappa}\mu}}{\partial z^{\nu}}\,. \tag{9.8}$$

In the following we return to Kählerian spaces. If we contract λ and μ in (9.6), we have equations

$$\frac{\partial \log H}{\partial z^{\nu}} = \frac{\partial \log G}{\partial z^{\nu}} + (n+1)\varphi_{\nu}\,, \tag{9.9}$$

H and G being the determinants $|\,h_{\lambda\bar{\mu}}\,|$ and $|\,g_{\lambda\bar{\mu}}\,|$ respectively, both real-valued and positive, and consequently φ_{ν} is gradient.

Expressing the condition that the covariant derivatives of $h_{\lambda\bar{\mu}}$ with respect to the symbols $\Delta^{\lambda}_{\mu\nu}$ are equal to zero and replacing the symbols $\Delta^{\lambda}_{\mu\nu}$ by (9.6), we get equations

$$h_{\lambda\bar{\mu},\nu} = h_{\lambda\bar{\mu}}\varphi_{\nu} + h_{\nu\bar{\mu}}\varphi_{\lambda}\,, \tag{9.10}$$

where comma denotes the covariant differentiation with respect to the original $\Gamma^{\lambda}_{\mu\nu}$. In virtue of Ricci identities and the fact that the components $R^{\cdot}{}_{\cdot\nu\omega}$ vanish, the integrability conditions of (9.10) in $h_{\lambda\bar{\mu}}$ are reducible to, by substitution of (9.10) themselves,

$$0 = h_{\bar{\lambda}\nu}(\varphi_{\mu,\omega} - \varphi_{\mu}\varphi_{\omega}) - h_{\bar{\lambda}\omega}(\varphi_{\mu,\nu} - \varphi_{\mu}\varphi_{\nu}) \tag{9.11}$$

and

$$h_{\lambda\bar{\kappa}}R^{\bar{\kappa}}{}_{\cdot\bar{\mu}\nu\bar{\omega}} + h_{\kappa\bar{\mu}}R^{\kappa}{}_{\cdot\lambda\nu\bar{\omega}} = h_{\nu\bar{\mu}}\varphi_{\lambda,\bar{\omega}} - h_{\lambda\bar{\omega}}\varphi_{\bar{\omega},\nu}\,. \tag{9.12}$$

The equations (9.11) are equivalent to, for $n > 1$,

$$\varphi_{\mu,\nu} = \varphi_{\mu}\varphi_{\nu}\,. \tag{9.13}$$

If we denote by $S_{\lambda\bar{\mu}\nu\bar{\omega}}$ the Riemann-Christoffel tensor for $h_{\lambda\bar{\mu}}$, then they are related to the original ones by formulas

$$\begin{aligned} S^{\bar{\lambda}}{}_{\cdot\bar{\mu}\nu\bar{\omega}} &= \frac{\partial \Delta^{\bar{\lambda}}_{\bar{\mu}\bar{\omega}}}{\partial z^{\nu}} = R^{\bar{\lambda}}{}_{\cdot\bar{\mu}\nu\bar{\omega}} + \delta^{\bar{\lambda}}_{\bar{\mu}}\varphi_{\bar{\omega},\nu} + \delta^{\bar{\lambda}}_{\bar{\omega}}\varphi_{\bar{\mu},\nu}\,, \\ S^{\lambda}{}_{\cdot\mu\nu\bar{\omega}} &= -\frac{\partial \Delta^{\lambda}_{\mu\nu}}{\partial \bar{z}^{\omega}} = R^{\lambda}{}_{\cdot\mu\nu\bar{\omega}} - \delta^{\lambda}_{\mu}\varphi_{\nu,\bar{\omega}} - \delta^{\lambda}_{\nu}\varphi_{\mu,\bar{\omega}}\,. \end{aligned} \tag{9.14}$$

From these it follows that the equations (9.12) are equivalent to the identities

$$S_{\lambda\bar{\mu}\nu\bar{\omega}} + S_{\bar{\mu}\lambda\nu\bar{\omega}} = 0.$$

Differentiating (9.9) and taking account of (2.12), we have

$$S_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} + (n+1)\varphi_{\lambda,\bar{\nu}}. \tag{9.15}$$

Substituting these into (9.14), it can be seen that a tensor

$$P^{\lambda}_{\cdot\mu\bar{\nu}\omega} = R^{\lambda}_{\cdot\mu\bar{\nu}\omega} - \frac{1}{n+1}(\delta^{\lambda}_{\mu}R_{\bar{\nu}\omega} + \delta^{\lambda}_{\omega}R_{\bar{\nu}\mu}) \tag{9.16}$$

is invariant under holomorphically projective correspondence (9.6). We call it *holomorphically projective curvature tensor,* which is the same that was introduced by Bochner [4, p. 85].

A Kählerian space is said to be *holomorphically projectively flat* if its holomorphically projective curvature tensor $P^{\lambda}_{\mu\bar{\nu}\omega}$ vanishes. Then we have

$$R_{\bar{\lambda}\mu\bar{\nu}\omega} = \frac{1}{n+1}(g_{\bar{\lambda}\mu}R_{\bar{\nu}\omega} + g_{\bar{\lambda}\omega}R_{\mu\bar{\nu}}) \tag{9.17}$$

and, by contracting $g^{\mu\bar{\nu}}$,

$$2nR_{\bar{\lambda}\omega} = g_{\bar{\lambda}\omega}R.$$

From these equations we can state the following

Theorem 4. *A holomorphically projectively flat space is equivalent to a Fubini space.*

§10. Holomorphically projective correspondence between Fubini spaces. For a Fubini space (5.16) hold and then the equations (9.12) are reducible to

$$h_{\lambda\bar{\omega}}A_{\nu\bar{\mu}} - h_{\nu\bar{\mu}}A_{\lambda\bar{\omega}} = 0 \tag{10.1}$$

and (9.14) to

$$S_{\lambda\bar{\mu}\nu\bar{\omega}} = -h_{\lambda\bar{\mu}}A_{\nu\bar{\omega}} - h_{\lambda\bar{\omega}}A_{\nu\bar{\mu}}, \tag{10.2}$$

where we have put

$$A_{\mu\bar{\nu}} = \frac{k}{2}g_{\mu\bar{\nu}} - \varphi_{\mu,\bar{\nu}}. \tag{10.3}$$

From (10.1) we get relations

$$A_{\lambda\bar{\mu}} = \frac{l}{2} h_{\lambda\bar{\mu}} . \tag{10.4}$$

l being a real proportional factor. Then (10.2) become

$$S_{\lambda\bar{\mu}\nu\bar{\omega}} = -\frac{l}{2}(h_{\lambda\bar{\mu}} h_{\nu\bar{\omega}} + h_{\lambda\bar{\omega}} h_{\nu\bar{\mu}}),$$

and, as soon verified, l is constant. Hence we have a generalization of Beltrami's theorem:

Theorem 5. *The only spaces corresponding holomorphically projectively to a Fubini space are Fubini spaces.*

This theorem is also proved more geometrically as follows. Let R be a Fubini space, S a Kählerian space and φ a holomorphically projective correspondence $S \to R$. If C is a real geodesic in S and ζ is its analytic extension, then $\varphi(C)$ is a holomorphically planar curve. As R is a Fubini space, the extension η of $\varphi(C)$ is also geodesic complex curve containing $\varphi(C)$, $\varphi^{-1}(\eta)$ is the extension of C and is also the geodesic complex curve containing C. Consequently, $\varphi^{-1}(\eta)$ is identical with ζ, and the theorem follows from Theorem 2.

Theorem 6. *Any two Fubini spaces correspond holomorphically projectively to each other, locally.*

Proof. Since Fubini space is locally equivalent to an elliptic or a hyperbolic domain of a complex projective space, two Fubini spaces are imbedded locally into a projective space. A complex projective collineation mapping the image of one into the image of the other carries complex straight lines to complex straight lines, and it is this collineation that is seeked for.

However, in grobal case, it shows different looks, and the following theorem is obtained.

Theorem 7. *A compact Kählerian space V cannot correspond holomorphically projectively to a space with Ricci curvature null if its scalar curvature is non-negative* (*or non-positive*) *and somewhere positive* (*or negative*) *strictly.*

In fact, under our assumption we have

$$\varphi_{,\lambda,\bar{\mu}} = \frac{-1}{n+1} R_{\lambda\bar{\mu}}$$

and consequently

$$(n+1)\Delta\varphi = 2(n+1)g^{\lambda\bar{\mu}}\varphi_{,\lambda,\bar{\mu}} = -R.$$

If we integrate this over the space, we have

$$(n+1)\int_V \Delta\varphi dv = -\int_V R dv > 0 \quad (\text{or} < 0),$$

which is contrary to the fact that $\int \Delta\varphi dv = 0$ for any scalar φ, see [1].

§11. Non-existence of real projective correspondence. We can state a theorem due to Bochner [2, Theorem 2] in somewhat generalized form.

Theorem 8. *If a Hermitian space corresponds to a Kählerian space real-projectively, that is, such as all geodesics are held in common, then the former is also Kählerian and is identical to the latter.*

Noting for $\rho(t)$ to be real valued in (3.4), we have not only the identities (9.1) but also

$$\left(\frac{d^2z^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt} + 2\Gamma^\lambda_{\mu\bar{\nu}}\frac{dz^\mu}{dt}\frac{d\bar{z}^\nu}{dt}\right)\frac{d\bar{z}^\kappa}{dt}$$
$$= \left(\frac{d^2\bar{z}^\kappa}{dt^2} + \Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}\frac{d\bar{z}^\mu}{dt}\frac{d\bar{z}^\nu}{dt} + 2\Gamma^{\bar{\lambda}}_{\bar{\mu}\nu}\frac{d\bar{z}^\mu}{dt}\frac{dz^\nu}{dt}\right)\frac{d\bar{z}^\lambda}{dt}.$$

By the same way to get (9.3) and (9.4), besides these we have

$$A^\lambda_{\mu\nu} = \delta^\lambda_\mu\psi_\nu + \delta^\lambda_\nu\psi_\mu .$$

Since our present correspondence is a special holomorphically projective one, ψ_ν vanish as in the proof of Theorem 3. Hence $A^\lambda_{\mu\nu}$ vanish also, and we have $\Delta^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$.

REFERENCES

[1] S. BOCHNER, Vector fields and Ricci curvature, Bull. Amer. Math. Soc., 52 (1946), pp. 776-797.
[2] ———, Curvature in Hermitian metric, Bull. Amer. Math. Soc., 53 (1947), pp. 179-195.
[3] ———, Curvature and Betti numbers, Annals of Math., 49 (1948), pp. 379-390.
[4] ———, Curvature and Betti numbers, II, Annals of Math., 50 (1949), pp. 77-93.
[5] ——— and T. W. MARTIN, Several Complex Variables, Princeton Univ. Press, 1948.
[6] L. P. EISENHART, Riemannian Geometry, second edition, Princeton Univ. Press, 1949.
[7] J. A. SCHOUTEN and D. J. STRUIK, Einführung in die neueren Methoden der Differentialgeometrie, I and II, P. Noordhoff-N. V. Groningen-Batavia, 1938.

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