ON THE REPRESENTATIONS OF THE GENERALIZED SYMMETRIC GROUP

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Introduction. All permutations of the mn symbols commutative with

$$(1_1\ 2_1\ \cdots\cdots\ m_1)\ (1_2\ 2_2\ \cdots\cdots\ m_2)\ \cdots\cdots\ (1_n\ 2_n\ \cdots\cdots\ m_n)$$

constitute a group of order $n!\,m^n$. Let us denote this group by $S(n, m)$. Obviously $S(1, m)$ is the cyclic group with generator $Q = (1\ 2\ \cdots\cdots\ m)$. Since $S(n, 1)$ is the symmetric group S_n on n symbols, $S(n, m)$ will be called the *generalized symmetric group* [10]. $S(n, 2)$ is the hyper-octahedral group of A. Young. The group $S(n, m)$ was treated from other point of view by H. S. M. Coxeter [2]. We set $Q_i = (1_i\ 2_i\ \cdots\cdots\ m_i)$. The n cycles Q_i generate an invariant subgroup $\mathfrak{Q}$ of order m^n of $S(n, m)$. The totality of permutations

$$W^* = \begin{pmatrix} 1_1\ 2_1\ \cdots\cdots\ m_1\ 1_2\ 2_2\ \cdots\cdots\ m_2\ \cdots\cdots\ 1_n\ 2_n\ \cdots\cdots\ m_n \\ 1_{i_1}\ 2_{i_1}\ \cdots\cdots\ m_{i_1}\ 1_{i_2}\ 2_{i_2}\ \cdots\cdots\ m_{i_2}\ \cdots\cdots\ 1_{i_n}\ 2_{i_n}\ \cdots\cdots\ m_{i_n} \end{pmatrix}$$
$$= \begin{pmatrix} 1_1\ 1_2\ \cdots\cdots\ 1_n \\ 1_{i_1}\ 1_{i_2}\ \cdots\cdots\ 1_{i_n} \end{pmatrix} \begin{pmatrix} 2_1\ 2_2\ \cdots\cdots\ 2_n \\ 2_{i_1}\ 2_{i_2}\ \cdots\cdots\ 2_{i_n} \end{pmatrix} \cdots\cdots \begin{pmatrix} m_1\ m_2\ \cdots\cdots\ m_n \\ m_{i_1}\ m_{i_2}\ \cdots\cdots\ m_{i_n} \end{pmatrix}$$

which transform the n cycles Q_i into each other, constitutes a subgroup S_n^* of $S(n, m)$. S_n^* is isomorphic to S_n by the mapping

$$W = \begin{pmatrix} 1\ 2\ \cdots\cdots\ n \\ i_1\ i_2\ \cdots\cdots\ i_n \end{pmatrix} \longrightarrow W^*.$$

We see easily that

$$S(n, m) = S_n^*\mathfrak{Q}, \qquad S_n^* \cap \mathfrak{Q} = 1,$$

so that $S(n, m)/\mathfrak{Q} \cong S_n$. Every element P of $S(n, m)$ is expressed uniquely in the form $P = W^*Q$, where $W^* \in S_n^*$ and

$$Q = Q_1^{l_1}\ Q_2^{l_2}\ \cdots\cdots\ Q_n^{l_n} \qquad (0 \leqq l_i \leqq m-1).$$

We have also

$$(W^*)^{-1}QW^* = Q_{i_1}^{l_1}\ Q_{i_2}^{l_2}\ \cdots\cdots\ Q_{i_n}^{l_n}.$$

In the present paper we shall first determine the irreducible representations of $S(n, m)$ [10, Theorem 2]. For this purpose, we state in §1 some preliminary results for the induced representations of a group of finite order. As an application, the irreducible representations of $S(n, m)$ will be determined in §2. In §3 some results in [11] and [12] are generalized for $S(n, m)$. In particular, a generalization of the Murnaghan-Nakayama recursion formula plays an important role in the following section. Let p be a prime number. As was shown in [10], there exists the close relationship between the theory of the representations of $S(b, p)$ and that of the modular representations of S_n for p. In §4 we shall prove the theorems in [10] which were stated without proofs.

1. Preliminaries. Let $\mathfrak{G}$ be a group of finite order. We consider the representations of $\mathfrak{G}$ in an algebraically closed field of characteristic 0. Let $\mathfrak{H}$ be an invariant subgroup of $\mathfrak{G}$ and let $\chi_1, \chi_2, \cdots\cdots, \chi_n$; $\zeta_1, \zeta_2, \cdots\cdots, \zeta_m$ be the distinct irreducible characters of $\mathfrak{G}$ and $\mathfrak{H}$ respectively. As is well known, n is equal to the number of conjugate classes of $\mathfrak{G}$. The characters ζ of $\mathfrak{H}$ are distributed in classes of characters which are associated with regard to $\mathfrak{G}$; two characters ζ_μ and ζ_ν being *associated* if

$$\zeta_\nu(H) = \zeta_\mu(G^{-1}HG), \tag{1.1}$$

where H is a variable element of $\mathfrak{H}$ and G is a fixed element of $\mathfrak{G}$. The totality of elements $G \in \mathfrak{G}$ which satisfy

$$\zeta_\mu(H) = \zeta_\mu(G^{-1}HG) \qquad (\text{for} \quad H \in \mathfrak{H}) \tag{1.2}$$

constitutes a subgroup $\mathfrak{G}_\mu$ of $\mathfrak{G}$. Obviously $\mathfrak{H} \subseteq \mathfrak{G}_\mu$. $\mathfrak{G}_\mu$ is called the subgroup of $\mathfrak{G}$ corresponding to ζ_μ. Let $\zeta_1, \zeta_2, \cdots\cdots, \zeta_k$ be the characters of $\mathfrak{H}$ such that they all lie in different associated classes and every character ζ is associated with one of them. Let $(\mathfrak{G} : \mathfrak{G}_\mu) = s_\mu$ and

$$\mathfrak{G} = \mathfrak{G}_\mu T_1 + \mathfrak{G}_\mu T_2 + \cdots\cdots + \mathfrak{G}_\mu T_{s_\mu}, \qquad T_1 = 1.$$

Then the number of characters ζ associated with ζ_μ is s_μ. If we denote these characters by $\zeta_\mu = \zeta_\mu^{(1)}, \zeta_\mu^{(2)}, \cdots\cdots, \zeta_\mu^{(s_\mu)}$, we may set

$$\zeta_\mu^{(i)}(H) = \zeta_\mu(T_i^{-1}HT_i). \tag{1.3}$$

We set

$$\theta_\mu(H) = \sum_i \zeta_\mu^{(i)}(H) = \sum_i \zeta_\mu(T_i^{-1}HT_i). \tag{1.4}$$

Every character χ_i, considered as a character of $\mathfrak{H}$, is expressed as

$$\chi_i(H) = a_i\theta_\mu(H) \qquad (\text{for } H \in \mathfrak{H}) \tag{1.5}$$

with a suitable θ_μ. Here a_i is a positive integer. We shall say that χ_i is the character of $\mathfrak{G}$ determined by ζ_μ. Denote by $\chi_\mu^{(1)}, \chi_\mu^{(2)}, \cdots\cdots, \chi_\mu^{(t_\mu)}$ the irreducible characters of $\mathfrak{G}$ determined by ζ_μ. We then have

$$\sum_{\mu=1}^{k} t_\mu = n. \tag{1.6}$$

We consider a subgroup $\mathfrak{G}'$ of $\mathfrak{G}$. Let $(\mathfrak{G} : \mathfrak{G}') = r$ and

$$\mathfrak{G} = \mathfrak{G}'S_1 + \mathfrak{G}'S_2 + \cdots\cdots + \mathfrak{G}'S_r, \qquad S_1 = 1.$$

Let $G' \to D(G')$ be a representation of $\mathfrak{G}'$. We set $D(S_i^{-1}GS_j) = 0$ if $S_i^{-1}GS_j$ is not contained in $\mathfrak{G}'$. Then

$$G \longrightarrow D^*(G) = (D(S_i^{-1}GS_j))_{ij}, \qquad (\text{for } G \in \mathfrak{G}) \tag{1.7}$$

forms a representation D^* of $\mathfrak{G}$ and is called the representation of $\mathfrak{G}$ induced by the representation D of $\mathfrak{G}$. If ξ is the character of D, we denote by $\tilde{\xi}$ the character of D^*. We define $\xi(S_i^{-1}GS_i) = 0$, if $S_i^{-1}GS_i$ is not contained in $\mathfrak{G}'$. By (1.7) we then have

$$\tilde{\xi}(G) = \sum_{i=1}^{r} \xi(S_i^{-1}GS_i). \tag{1.8}$$

Let $\mathfrak{H}$ be an invariant subgroup of $\mathfrak{G}$ as before. The irreducible character ζ_μ of $\mathfrak{H}$ is not associated with any other ζ *with regard to* $\mathfrak{G}_\mu$. Applying Frobenius' reciprocity theorem on induced characters, we obtain the following

Theorem 1. *Let ζ_μ be any irreducible character of an invariant subgroup $\mathfrak{H}$ of $\mathfrak{G}$. Denote by $\chi_\mu^{(1)}, \chi_\mu^{(2)}, \cdots\cdots, \chi_\mu^{(t_\mu)}$ the irreducible characters of $\mathfrak{G}$ determined by ζ_μ and by $\xi_\mu^{(1)}, \xi_\mu^{(2)}, \cdots\cdots, \xi_\mu^{(h_\mu)}$ those of $\mathfrak{G}_\mu$. Then $t_\mu = h_\mu$ and $\tilde{\xi}_\mu^{(i)} = \chi_\mu^{(i)}$, if the notation is suitably chosen.*

2. The irreducible representations of $S(n, m)$. Any element Q of $\mathfrak{Q}$ is expressed uniquely in the form $Q = Q_1^{l_1} Q_2^{l_2} \cdots\cdots Q_n^{l_n}$ $(0 \leq l_i \leq m-1)$. Q is called an element of type $(n_0, n_1, \cdots\cdots, n_{m-1})$, if the number of l_i such that $l_i = k$ is n_k.

Lemma 1. *Two elements Q and Q' of $\mathfrak{Q}$ are conjugate in $S(n, m)$ if and only if they are of same type.*

In this section we assume that Q is an element of type $(n_0, n_1, \cdots\cdots, n_{m-1})$ such that $l_1 = \cdots\cdots = l_{n_0} = 0,\ l_{n_0+1} = \cdots\cdots = l_{n_0+n_1} = 1$, and so on.

Since the invariant subgroup $\mathfrak{Q}$ is a commutative group, every irreducible representation of $\mathfrak{Q}$ is of degree one. Denote by ω a primitive m-th root of unity. Then

$$Q_i \longrightarrow \omega^{\alpha_i} \qquad (0 \leqq \alpha_i \leqq m-1),\quad i = 1, 2, \cdots\cdots, n$$

forms an irreducible representation of $\mathfrak{Q}$. We denote by $\zeta^{(\alpha_i)}$ the character of the representation defined above. The character $\zeta^{(\alpha_i)}$ is called the character of type $(n_0, n_1, \cdots\cdots, n_{m-1})$, if the number of α_i such that $\alpha_i = k$ is n_k. Two characters $\zeta^{(\alpha_i)}$ and $\zeta^{(\beta_i)}$ are associated with regard to $S(n, m)$ if and only if they are of same type. In what follows we assume that $\zeta^{(\alpha_i)}$ is a character of type $(n_0, n_1, \cdots\cdots, n_{m-1})$ such that $\alpha_1 = \cdots\cdots = \alpha_{n_0} = 0,\ \alpha_{n_0+1} = \cdots\cdots = \alpha_{n_0+n_1} = 1$, and so on.

Lemma 2. *Let $\mathfrak{G}^{(\alpha_i)}$ be the subgroup of $S(n, m)$ corresponding to the character $\zeta^{(\alpha_i)}$. Then $\mathfrak{G}^{(\alpha_i)}$ is the normalizer $\mathfrak{N}(Q)$ of Q in $S(n, m)$.*

We have

$$\mathfrak{G}^{(\alpha_i)} = S_{(n_i)}^{*}\,\mathfrak{Q}, \qquad S_{(n_i)}^{*} \cap \mathfrak{Q} = 1, \tag{2.1}$$

where $S_{(n_i)}^{*}$ is the subgroup of S_n^{*} and is the direct product of $S_{n_i}^{*}$:

$$S_{(n_i)}^{*} = S_{n_0}^{*} \times S_{n_1}^{*} \times \cdots\cdots \times S_{n_{m-1}}^{*}.$$

Hence

$$(S(n, m) : \mathfrak{G}^{(\alpha_i)}) = (S_n^{*} : S_{(n_i)}^{*}) = (S_n : S_{(n_i)}). \tag{2.2}$$

This implies that the number of irreducible characters ζ of $\mathfrak{Q}$ associated with $\zeta^{(\alpha_i)}$ with regard to $S(n, m)$ is $\dfrac{n!}{n_0!\ n_1!\ \cdots\cdots\ n_{m-1}!}$. Let

$$S_n = S_{(n_i)}P_1 + S_{(n_i)}P_2 + \cdots\cdots + S_{(n_i)}P_r \tag{2.3}$$

be the coset decomposition of S_n with respect to $S_{(n_i)}$. *Then*

$$S(n, m) = \mathfrak{G}^{(\alpha_i)}P_1^{*} + \mathfrak{G}^{(\alpha_i)}P_2^{*} + \cdots\cdots + \mathfrak{G}^{(\alpha_i)}P_r^{*}, \tag{2.4}$$

where P_i^{*} is the element of S_n^{*} corresponding to P_i of S_n.

Let $U^* \rightarrow D(U^*)$, $U^* \in S_{(n_i)}^*$, be an irreducible representation of degree f of $S_{(n_i)}^*$. Then $G = U^*Q \rightarrow \zeta^{(\alpha_i)}(Q)D(U^*)$ is an irreducible representation of $\mathfrak{G}^{(\alpha_i)}$ determined by $\zeta^{(\alpha_i)}$. Conversely, if $G \rightarrow D'(G)$ is an irreducible representation of $\mathfrak{G}^{(\alpha_i)}$ determined by $\zeta^{(\alpha_i)}$, then $U^* \rightarrow D'(U^*)$ is an irreducible representation of $S_{(n_i)}^*$. This implies that the number of irreducible representations of $\mathfrak{G}^{(\alpha_i)}$ determined by $\zeta^{(\alpha_i)}$ is equal to the number of irreducible representations of $S_{(n_i)} = S_{n_0} \times S_{n_1} \times \cdots\cdots S_{n_{m-1}}$.

We shall denote by $[\alpha]$ the irreducible representation of S_n corresponding to a diagram $[\alpha]$ of n nodes, and by χ_α its character. The degree $\chi_\alpha(1)$ of $[\alpha]$ will be denoted by f_α. Any irreducible representation of $S_{(n_i)}$ is given by the Kronecker product representation

$$[\alpha_0] \times [\alpha_1] \times \cdots\cdots \times [\alpha_{m-1}], \tag{2.5}$$

where $[\alpha_i]$ is an irreducible representation of S_{n_i}.

Let us denote by $\xi^{(\alpha_i)}$ the character of the irreducible representation (2.5). As was shown previously, any irreducible character of $\mathfrak{G}^{(\alpha_i)}$ determined by $\zeta^{(\alpha_i)}$ is given by $\xi^{(\alpha_i)} \times \zeta^{(\alpha_i)}$. Theorem 1 shows that the character of $S(n, m)$ induced by $\xi^{(\alpha_i)} \times \zeta^{(\alpha_i)}$ is irreducible. Hence the irreducible characters of $S(n, m)$ determined by $\zeta^{(\alpha_i)}$ are in (1- 1) correspondence with star diagrams

$$[\alpha]_m^* = [\alpha_0]\cdot[\alpha_1]\cdot \cdots\cdots \cdot[\alpha_{m-1}]$$

of n nodes such that the i-th component $[\alpha_i]$ is a diagram of n_i nodes.

We shall denote by $(\alpha)^*$ the irreducible representation of $S(n, m)$ corresponding to $[\alpha]_m^*$, and by ϑ_{α^*} its character. We see by (2.3) and (2.4) that

$$\vartheta_{\alpha^*}(W^*) = \sum_{j=1}^{r} \xi^{(\alpha_i)}(P_j^{-1} WP_j) \qquad \text{for} \quad W^* \in S_n^*, \tag{2.6}$$

where we set $\xi^{(\alpha_i)}(P_j^{-1} WP_j) = 0$ if $P_j^{-1} WP_j$ is not contained in $S_{(n_i)}$, and

$$\vartheta_{\alpha^*}(Q) = f_{\alpha_0} f_{\alpha_1} \cdots\cdots f_{\alpha_{m-1}} \sum_{j=1}^{r} \zeta^{(\alpha_i)}((P_j^*)^{-1} QP_j^*) \qquad \text{for } Q \in \mathfrak{Q}. \tag{2.7}$$

In particular, if W^* in S_n^* is not contained in $S_{(n_i)}^*$, then

$$\vartheta_{\alpha^*}(W^*) = 0. \tag{2.8}$$

Let $k(n)$ be the number of partitions of n. The number of distinct irreducible representations of $S_{(n_i)}$ is $k(n_0)\,k(n_1)\cdots\cdots k(n_{m-1})$. Hence, by (1.6) and Theorem 1, the number of irreducible representations of $S(n, m)$ is given by

$$(2.9)\qquad l(n, m) = \sum_{n_0, n_1, \cdots\cdots, n_{m-1}} k(n_0)\,k(n_1)\cdots\cdots k(n_{m-1}),$$
$$(\textstyle\sum n_i = n, \quad 0 \leqq n_i \leqq n).$$

As in [12], we shall denote by $[\alpha]_m{}^*$ the reducible representation of S_n induced by the irreducible Kronecker product representation $[\alpha_0]\times[\alpha_1]\times\cdots\cdots\times[\alpha_{m-1}]$ of $S_{(n_i)}$. The representation $[\alpha]_m{}^*$ is called the skew representation corresponding to the star diagram $[\alpha]_m{}^*$. We shall denote by χ_{α^*} the character of $[\alpha]_m{}^*$ and by f_{α^*} its degree. (2.6) implies

$$(2.10)\qquad \vartheta_{\alpha^*}(W^*) = \chi_{\alpha^*}(W).$$

In particular, the degree of $(\alpha)^*$ is equal to

$$(2.11)\qquad f_{\alpha^*} = \frac{n!}{n_0!\,n_1!\cdots\cdots n_{m-1}!} f_{\alpha_0} f_{\alpha_1}\cdots\cdots f_{\alpha_{m-1}}.$$

Thus we have proved the following

Theorem 2. *The irreducible representations of $S(n, m)$ are in (1-1) correspondence with star diagrams $[\alpha]_m{}^*$ of n nodes.*

Let H_α be the hook product [4; 4a] of a diagram $[\alpha]$ of n nodes. The degree f_α of $[\alpha]$ is given by $n!/H_\alpha$. We shall define the hook product H_{α^*} of a star diagram $[\alpha]_m{}^*$ by

$$(2.12)\qquad H_{\alpha^*} = H_{\alpha_0}\cdot H_{\alpha_1}\cdot\cdots\cdots\cdot H_{\alpha_{m-1}}.$$

Theorem 3. *Let $(\alpha)^*$ be an irreducible representation of $S(n, m)$ corresponding to $[\alpha]_m{}^*$. The degree of $(\alpha)^*$ is given by $n!/H_{\alpha^*}$.*

Proof. Our assertion follows immediately from $f_{\alpha_i} = n_i!/H_{\alpha_i}$ and (2.11).

Let P be any element of S_n with b_1 1-cycles, b_2 2-cycles, $\cdots\cdots$, b_k k-cycles. The normalizer $\mathfrak{N}(P)$ of P in S_n is the direct product of $S(b_i, i)$:

$$\mathfrak{N}(P) = S(b_1, 1)\times S(b_2, 2)\times\cdots\cdots\times S(b_k, k).$$

Hence we can easily determine the irreducible representations of $\mathfrak{N}(P)$.

Let A_n^* be the subgroup of S_n^* corresponding to the alternating group A_n of S_n. Evidently $A_n^*\mathfrak{Q}$ is an invariant subgroup of $S(n, m)$. This will be denoted by $A(n, m)$ and will be called the *generalized alternating group.* We shall determine the irreducible representations of $A(n, m)$. If the rows and columns of a diagram $[\alpha]$ are interchanged, the resulting diagram $[\bar{\alpha}]$ is said to be conjugate to $[\alpha]$. If $[\alpha]=[\bar{\alpha}]$, then $[\alpha]$ is called self-conjugate. For a star diagram, we shall say that $[\bar{\alpha}]^*=[\bar{\alpha}_0]\cdot[\bar{\alpha}_1]\cdot\cdots\cdots\cdot[\bar{\alpha}_{m-1}]$ is conjugate to $[\alpha]^*$. A star diagram $[\alpha]^*$ is called self-conjugate, if $[\alpha]^*=[\bar{\alpha}]^*$.

Theorem 4. *Let $(\alpha)^*$ be an irreducible representation of $S(n, m)$ corresponding to a star diagram $[\alpha]^*$. If $[\alpha]^*$ is self-conjugate, then $(\alpha)^*$ breaks up into two irreducible conjugate parts of equal degree as a representation of $A(n, m)$. If $[\alpha]^*$ is not self-conjugate, then $(\alpha)^*$ remains irreducible as a representation of $A(n, m)$. Moreover two representations $(\alpha)^*$ and $(\bar{\alpha})^*$ of $A(n, m)$ are equivalent.*

We shall study the modular representations of $S(n, m)$ in a forthcoming paper.

3. A generalization of the Murnaghan-Nakayama recursion formula. We first consider the conjugate classes of $S(n, m)$. We see easily that if two elements W^* and U^* of S_n^* are conjugate in $S(n, m)$, then they are conjugate in S_n^*. Generally we have

Lemma 3. *If two elements W^*Q and U^*Q' are conjugate in $S(n, m)$, then W^* and U^* are conjugate in S_n^*.*

Let C^* be an element of S_n^* corresponding to a b-cycle $C=(i_1 i_2 \cdots\cdots i_b)$ of S_n:

$$C^* = (1_{i_1} 1_{i_2} \cdots\cdots 1_{i_b})(2_{i_1} 2_{i_2} \cdots\cdots 2_{i_b}) \cdots\cdots (m_{i_1} m_{i_2} \cdots\cdots m_{i_b}). \tag{3.1}$$

$C^*Q_{i_\alpha}{}^l$ $(1\leqq l\leqq m-1,\ 1\leqq \alpha\leqq b)$ is the cycle of length mb. We shall say that $C^*Q_{i_\alpha}{}^l$ is a permutation of type (b, l) and denote it by $P(b, l)$. Of course, $P(b, 0)=C^*$. If $i\neq j$, then $P(b, i)$ and $P(b, j)$ are not conjugate in $S(n, m)$. We consider a permutation P of $S(n, m)$ such that

$$P = P(a_1^{(0)}, 0)\, P(a_2^{(0)}, 0) \cdots\cdots P(a_t^{(m-1)}, m-1),$$

where no two of $P(a_\mu^{(k)}, k)$ have common symbols. For a fixed i, we may assume that $a_1^{(i)}\geqq a_2^{(i)}\geqq\cdots\cdots\geqq a_{r_i}^{(i)}\geqq 0$. We set

$$a_1^{(i)}+a_2^{(i)}+\cdots\cdots+a_{r_i}^{(i)} = b_i.$$

Then

$$b_0 + b_1 + \cdots\cdots + b_{m-1} = n \qquad (0 \leqq b_i \leqq n).$$

We set $[\alpha_i] = [a_1^{(i)}, a_2^{(i)}, \cdots\cdots, a_{r_i}^{(i)}]$ and associate P with a star diagram $[\alpha]_m^* = [\alpha_0]\cdot[\alpha_1]\cdot\cdots\cdots\cdot[\alpha_{m-1}]$ of n nodes. We then have

Lemma 4. *Let S and T be two elements of $S(n, m)$ corresponding to the star diagrams $[\alpha]_m^*$ and $[\beta]_m^*$ of n nodes respectively. S and T are conjugate in $S(n, m)$ if and only if $[\alpha]_m^* = [\beta]_m^*$.*

Since there exists an element of $S(n, m)$ corresponding to an arbitrary star diagram of n nodes, Lemma 4 implies that there exist at least the $l(n, m)$ elements which are not mutually conjugate in $S(n, m)$. On the other hand, Theorem 2 shows that the number of conjugate classes of $S(n, m)$ is $l(n, m)$. Thus, if we denote by P_{α^*} the element of $S(n, m)$ corresponding to $[\alpha]_m^*$, then the $l(n, m)$ elements P_{α^*} form a complete system of representatives for the conjugate classes of $S(n, m)$. Hence we have obtained the following

Theorem 5. *The conjugate classes of $S(n, m)$ are in (1- 1) correspondence with star diagrams $[\alpha]_m^*$ of n nodes.*

We shall summarlize some results of G. de B. Robinson [**11**; **12**] on the skew representations of the symmetric group which are significant hereafter. Let $[\alpha]-[\beta]$ be a skew diagram [**11**] of l nodes. $[\alpha]-[\beta]$ determines a reducible representation of S_l. This is called a skew representation of S_l and is denoted by $[\alpha]-[\beta]$. We shall denote by χ_α^β the character of $[\alpha]-[\beta]$. The irreducible representation $[\alpha]$ of S_n is reducible considered as a representation of a subgroup $S_k \times S_l$. Let $[\alpha] = \sum g_{\alpha\beta\gamma}[\beta] \times [\gamma]$. *Then* $[\alpha]-[\beta] = \sum_\gamma g_{\alpha\beta\gamma}[\gamma]$, so that

$$[\alpha] = \sum_\beta [\beta] \times ([\alpha]-[\beta]). \tag{3.2}$$

Hence we have for $S = S^{(1)}S^{(2)} \in S_k \times S_l$

$$\chi_\alpha(S) = \sum_\beta \chi_\beta(S^{(1)})\chi_\alpha^\beta(S^{(2)}). \tag{3.3}$$

If C is a cycle of length l in S_l, then

$$\chi_\alpha^\beta(C) = (-1)^r \text{ or } 0, \tag{3.4}$$

according as $[\alpha]-[\beta]$ is a skew hook equivalent to the right hook $H_r = [n-r, 1^r]$ or not. We can prove, as in [**11**], the Murnaghan-Nakayama recursion formula [5; 7] by (3.3) and (3.4).

We shall prove, by the analogous method, a generalization of the Murnaghan-Nakayama recursion formula for $S(n, m)$. Let $(\alpha)^*$ be an irreducible representation of $S(n, m)$ corresponding to a star diagram $[\alpha]_m^*$. Let $[\alpha_i]-[\beta_i]$ be a skew diagram of l_i nodes. A diagram which has $[\alpha_i]-[\beta_i]$ as its i-th component will be called a skew star diagram and will be denoted by $[\alpha]^*-[\beta]^*$:

$$[\alpha]^*-[\beta]^* = [\alpha_0]-[\beta_0]\cdot[\alpha_1]-[\beta_1]\cdot\cdots\cdots\cdot[\alpha_{m-1}]-[\beta_{m-1}].$$

We set $\sum l_i = l$. Then $[\alpha]^*-[\beta]^*$ corresponds to a reducible representation of $S(l, m)$, which will be denoted by $(\alpha)^*-(\beta)^*$, where $(\beta)^*$ denotes the irreducible representation of $S(n-l, m)$ corresponding to $[\beta]^* = [\beta_0]\cdot[\beta_1]\cdot\cdots\cdots\cdot[\beta_{m-1}]$. The representation $(\alpha)^*$ is reducible considered as a representation of a subgroup $S(n-l, m) \times S(l, m)$. Let

$$(3.5) \qquad (\alpha)^* = \sum h_{\alpha\beta\gamma}(\beta)^* \times (\gamma)^*$$

as a representation of $S(n-l, m) \times S(l, m)$.

Theorem 6. *Let* $[\alpha_i]-[\beta_i] = \sum g_{\alpha_i\beta_i\gamma_i}[\gamma_i]$. *Then*

$$(\alpha)^* - (\beta)^* = \sum h_{\alpha\beta\gamma}(\gamma)^*,$$

where $h_{\alpha\beta\gamma} = \prod_i g_{\alpha_i\beta_i\gamma_i}$ *and* $(\gamma)^*$ *is an irreducible representation of* $S(l, m)$ *corresponding to* $[\gamma]^* = [\gamma_0]\cdot[\gamma_1]\cdot\cdots\cdots\cdot[\gamma_{m-1}]$.

If $[\alpha_i] = [\beta_i]$, we must set $g_{\alpha_i\beta_i\gamma_i} = 1$ in Theorem 6. We obtain by Theorem 6 and (3.5)

$$(3.6) \qquad (\alpha)^* = \sum_{\beta^*} (\beta)^* \times ((\alpha)^* - (\beta)^*).$$

We shall denote by $\vartheta_{\alpha^*}{}^{\beta^*}$ the character of $(\alpha)^*-(\beta)^*$. By (3.6) we have for $T = T^{(1)}T^{(2)} \in S(n-l, m) \times S(l, m)$

$$(3.7) \qquad \vartheta_{\alpha^*}(T) = \sum \vartheta_{\beta^*}(T^{(1)})\,\vartheta_{\alpha^*}{}^{\beta^*}(T^{(2)}).$$

In particular, if $T^{(2)} = U^*$ is an element of the subgroup S_l^* of $S(l, m)$, then

$$(3.8) \qquad \vartheta_{\alpha^*}{}^{\beta^*}(U^*) = \sum h_{\alpha\beta\gamma}\chi_{\gamma^*}(U),$$

where U is an element of S_l corresponding to U^* of S_l^*. Let C^* be an element of type $(l, 0)$, that is, an element of S_l^* corresponding to an l-cycle C of S_l. We shall determine the value of $\chi_{\gamma^*}(C)$. Let $l_i < l$ for every i. Since C is not contained in a subgroup $S_{l_0} \times S_{l_1}$

$\times \cdots\cdots \times S_{l_{m-1}}$ of S_l, we have $\chi_{\gamma^*}(C) = 0$ by (2.8). Next we consider the case when one of l_i, say l_0, is equal to l and $l_i = 0$ $(0 < i)$. We see by (3.4) that $\chi_{\gamma^*}(C) = \chi_{\alpha_0}{}^{\beta_0}(C) = (-1)^r$ or 0, according as $[\alpha_0] - [\beta_0]$ is a skew hook equivalent to the right hook $H_r = [l-r, 1^r]$ or not. In this case we have $g_{\alpha_i \beta_i \gamma_i} = 1$ for every $i > 0$. Hence we can conclude that

$$(3.9) \qquad \vartheta_{\alpha^*}{}^{\beta^*}(C^*) = (-1)^r \quad \text{or} \quad 0,$$

according as $[\alpha]^* - [\beta]^*$ is a skew hook of some component $[\alpha_i]$ equivalent to the right hook $H_r = [l-r, 1^r]$ or not. (3.7), combined with (3.9), yields a generalization of the Murnaghan-Nakayama recursion formula for $S(n, m)$.

Theorem 7. *Let $H_1, H_2, \cdots\cdots$ be the totality of hooks of length l in the star diagram $T^* = [\alpha]^*$, and let $\vartheta^*(T^*)$ be the character of $(\alpha)^*$ of $S(n, m)$ corresponding to T^*. Then*

$$\vartheta^*(T^*;\ P) = \sum_i (-1)^{r_i} \vartheta^*(T^* - H_i;\ \bar{P}),$$

where P is any permutation of $S(n, m)$ which contains a permutation C^* of S_n^* corresponding to a cycle C of length l and $\bar{P}$ is the permutation of $S(n-l, m)$ obtained by removing C^* from P. If T^* has no hook of length l, then $\vartheta^*(T^*;\ P) = 0$.

As a special case of Theorem 7, we obtain

Corollary. *Let $H_1, H_2, \cdots\cdots$ be the totality of hooks of length l in the star diagram $T^* = [\alpha]^*$, and let $\chi^*(T^*)$ be the character of the skew representation $[\alpha]^*$ of S_n. Then*

$$\chi^*(T^*;\ P) = \sum_i (-1)^{r_i} \chi^*(T^* - H_i;\ \bar{P}),$$

where P is any permutation of S_n which contains a cycle C of length l and $\bar{P}$ is the permutation on $n-l$ symbols obtained by removing C from P. If T^* has no hook of length l, then $\chi^*(T^*;\ P) = 0$.

In what follows we shall denote by $[\alpha]^*$ the irreducible representation of $S(n, m)$ corresponding to a star diagram $[\alpha]^*$ in place of $(\alpha)^*$ and by χ_{α^*} its character.

4. The decomposition numbers of S_n. Let p be a fixed prime number. If b p-hooks are removable from $[\alpha]$ of n nodes, we shall say that $[\alpha]$ is of weight b and residue $[\alpha^{(0)}]$ of $n-bp$ nodes is called the p-core of $[\alpha]$. The p-hook structure of $[\alpha]$ is completely represented by the star diagram $[\alpha]_p^* = [\alpha_0]\cdot[\alpha_1]\cdot\cdots\cdots\cdot[\alpha_{p-1}]$ of b nodes [12; also 8, 13]. Namely, each node of $[\alpha]_p^*$ represents a p-hook of $[\alpha]$ and each r-hook of $[\alpha]_p^*$ represents an rp-hook of $[\alpha]$. Let $H = [g-r, 1^r]$ be a g-hook of $[\alpha]$. $(-1)^r$ is called the parity of H and is denoted by $\sigma(H)$. Let us consider a cp-hook $H = [cp-r, 1^r]$ of $[\alpha]$ and suppose that its representative in $[\alpha]_p^*$ is $H^* = [c-s, 1^s]$. If we denote by H_i the i-th of the c component p-hooks of H, then we have [11]

$$\sigma(H) = \sigma(H^*) \prod_i \sigma(H_i). \tag{4.1}$$

Let $[\beta]$ be a diagram obtained by removing succesively b_1p-hook H_1, b_2p-hook H_2, $\cdots\cdots$, b_sp-hook H_s from $[\alpha]$. We set $\sigma'(\alpha, \beta) = \prod_i \sigma(H_i)$. Suppose that the representatives of H_i in $[\alpha]_p^*$ are H_i^*. We set $\sigma^*(\alpha^*, \beta^*) = \prod_i \sigma(H_i^*)$. Let $b = \sum_i b_i$. Since $[\beta]$ is obtained by removing succesively b p-hooks from $[\alpha]$, we shall denote by $\sigma(\alpha, \beta)$ the product of parities of these b p-hooks. Then it follows from (4.1) that

$$\sigma'(\alpha, \beta) = \sigma^*(\alpha^*, \beta^*)\sigma(\alpha, \beta). \tag{4.2}$$

Let $P \in S_n$ be the product of a_1p-cycle Q_1, a_2p-cycle Q_2, $\cdots\cdots$, a_sp-cycle Q_s, where $a_1 \geq a_2 \geq \cdots\cdots \geq a_s \geq 1$. P is called an element of type $(a_1, a_2, \cdots\cdots, a_s)$ and of weight $a = \sum_i a_i$ [10]. *We shall associate P with the diagram $[\mu] = [a_1, a_2, \cdots\cdots, a_s]$ and P will be denoted by P_μ.* The number of elements of weight a such that they all lie in different conjugate classes of S_n is $k(a)$, where $k(a)$ denotes, as before, the number of diagrams of a nodes. We set $n = n' + tp$ $(0 \leq n' < p)$ and $\sum_{a=0}^{t} k(a) = r$. We then have r elements P_μ of S_n, where $[\mu]$ ranges over r diagrams of a nodes $(0 \leq a \leq t)$. Every conjugate class contains an element of the form VP_μ, where $[\mu]$ is uniquely determined by the class and where V is a p-regular element of S_{n-ap}, if $[\mu]$ is a diagram of a nodes. In what follows we shall denote by n_μ the number of nodes of $[\mu]$. Let $[\alpha^{(0)}]$ be a p-core with m nodes and $n = m + bp$, and let B be the p-block of S_n with p-core $[\alpha^{(0)}]$. We denote by $\chi_\beta^{(a)}$ the character of the irreducible representation $[\beta]$ of S_{n-ap}. Let P_μ be an element of type $[\mu] = [a_1, a_2, \cdots\cdots, a_s]$. Applying the Murnaghan-Nakayama recursion formula iterated s times to $[\alpha] \subset B$, we obtain

$$(4.3)\qquad \chi_\alpha(VP_\mu) = \begin{cases} \sum_\beta \sigma'(\alpha,\beta)\, h^{(\mu)}(\alpha,\beta)\, \chi_\beta^{(n_\mu)}(V), & [\beta]\subset B^{(n_\mu)} \\ & (\text{for } n_\mu \leqq b), \\ 0 & (\text{for } b < n_\mu), \end{cases}$$

where the $h^{(\mu)}(\alpha, \beta)$ are rational integers $\geqq 0$, and $B^{(n_\mu)}$ denotes the block of $S_{n-n_\mu p}$ with p-core $[\alpha^{(0)}]$. Let $\varphi_\lambda^{(n_\mu)}$ be the character of $S_{n-n_\mu p}$ in the modular irreducible representation λ. We then have

$$(4.4)\qquad \chi_\beta^{(n_\mu)}(V) = \sum_\lambda d_{\beta\lambda}^{(n_\mu)} \varphi_\lambda^{(n_\mu)}(V) \qquad (V \text{ in } S_{n-n_\mu p},\ p\text{-regular}),$$

where the $d_{\beta\lambda}^{(n_\mu)}$ are the decomposition numbers of $S_{n-n_\mu p}$. Hence (4.3), combined with (4.4), yields

$$(4.5)\qquad \chi_\alpha(VP_\mu) = \sum_\lambda u_{\alpha\lambda}^{(\mu)} \varphi_\lambda^{(n_\mu)}(V),$$

where

$$(4.6)\qquad u_{\alpha\lambda}^{(\mu)} = \sum_\beta \sigma'(\alpha,\beta)\, h^{(\mu)}(\alpha,\beta)\, d_{\beta\lambda}^{(n_\mu)}.$$

The $u_{\alpha\lambda}^{(\mu)}$ will be called the *u-numbers* of S_n. Let $D=(d_{\alpha\lambda})$ be the decomposition matrix of S_n. For $P_0=1$, we have

$$(4.7)\qquad u_{\alpha\lambda}^{(0)} = d_{\alpha\lambda}.$$

In [10] we have proved the orthogonality relations for the u-numbers $u_{\alpha\lambda}^{(\mu)}$:

$$(4.8)\qquad \sum_\alpha u_{\alpha\lambda}^{(\mu)} u_{\alpha\kappa}^{(\nu)} = 0 \qquad [\alpha]\subset B, \qquad \text{if } [\mu]\neq[\nu].$$

$$(4.9)\qquad \sum_\alpha u_{\alpha\lambda}^{(\mu)} u_{\alpha\kappa}^{(\mu)} = c_{\lambda\kappa}^{(n_\mu)} \prod_i (k_i!\,(ip)^{k_i}) \qquad [\alpha]\subset B,$$

where the $c_{\lambda\kappa}^{(n_\mu)}$ denote the Cartan invariants of $S_{n-n_\mu p}$ and $[\mu]=(1^{k_1}, 2^{k_2}, \cdots\cdots, m^{k_m})$. In particular, by (4.7) and (4.8)

$$(4.10)\qquad \sum_\alpha d_{\alpha\lambda} u_{\alpha\kappa}^{(\mu)} = 0 \qquad [\alpha]\subset B, \qquad \text{if } [\mu]\neq[0].$$

Let P_{α^*} be, as before, a complete system of representatives for the conjugate classes of $S(b, p)$. P_{α^*} is contained in S_b^* if and only if the first component $[\alpha_0]$ of $[\alpha]_p^*$ is a diagram of b nodes and $[\alpha_i]=[0]$ for $0<i$. On the other hand, P_{α^*} is contained in $\mathfrak{Q}$ if and only if $[\alpha_i]=[1^{b_i}]$ or $[0]$ for every i. *We associate P_{α^*} with a diagram $[\mu]$, if $[\alpha_0]=[\mu]$.* The number of P_{α^*} associated with a fixed $[\mu]$ is $l^*(b-n_\mu)$. Here $l^*(a)$ is defined by

$$l^*(a) = \sum_{b_1, b_2, \cdots\cdots, b_{p-1}} k(b_1) k(b_2) \cdots\cdots k(b_{p-1}), \quad (\textstyle\sum b_i = a, \quad 0 \leq b_i \leq a). \tag{4.11}$$

We have proved [9; also 6, 3, 10] that the number of modular irreducible representations in a p-block of weight a is $l^*(a)$. Let P_{α^*} be any element of $S(b, p)$ associated with $[\mu]$. Then P_{α^*} is expressed in the form $T_i^{(n_\mu)} R_\mu^* = R_\mu^* T_i^{(n_\mu)}$, where R_μ^* is an element of $S_{n_\mu}^*$ corresponding to $[\mu]\cdot[0]\cdot \cdots\cdots \cdot[0]$, considered as an element of $S(n, p)$, and $T_i^{(n_\mu)}$ is an element corresponding to $[0]\cdot[\alpha_1]\cdot \cdots\cdots \cdot[\alpha_{p-1}]$, considered as an element of $S(b - n_\mu, p)$. Hence the $l(b, p)$ elements

$$T_i^{(n_\mu)} R_\mu^* \qquad (i = 1, 2, \cdots\cdots, l^*(b - n_\mu))$$

form a complete system of representatives for the conjugate classes of $S(b, p)$, if $[\mu]$ ranges over all diagrams of a nodes $(0 \leq a \leq b)$. In particular, the $T_i^{(0)}$ $(i = 1, 2, \cdots\cdots, l^*(b))$ are the elements of $S(b, p)$ corresponding to $[\alpha]^*$ such that $[\alpha_0] = [0]$.

We consider a diagram $[\alpha]$ with p-core $[\alpha^{(0)}]$ belonging to a p-block B of weight b. *Let* $[\alpha]^*$ *be the irreducible representation of* $S(b, p)$ *corresponding to the star diagram* $[\alpha]^*$ *of* $[\alpha]$ *and let* $[\mu] = [a_1, a_2, \cdots\cdots, a_s]$. Applying the Murnaghan-Nakayama recursion formula (Theorem 7) iterated s times to $[\alpha]^*$, we obtain

$$\chi_{\alpha^*}(T_i^{(n_\mu)} R_\mu^*) = \sum_{\beta^*} \sigma^*(\alpha^*, \beta^*) h^{(\mu)}(\alpha^*, \beta^*) \chi_{\beta^*}{}^{(n_\mu)}(T_i^{(n_\mu)}), \tag{4.12}$$

where $[\beta]^*$ ranges over all star diagrams of $S(b - n_\mu, p)$. Moreover we see that $h^{(\mu)}(\alpha^*, \beta^*)$ is equal to $h^{(\mu)}(\alpha, \beta)$ in (4.3):

$$h^{(\mu)}(\alpha^*, \beta^*) = h^{(\mu)}(\alpha, \beta). \tag{4.13}$$

For any R_μ^* of S_b^* corresponding to $[\mu]\cdot[0]\cdot \cdots\cdots \cdot[0]$, we have

$$\chi_{\alpha^*}(R_\mu^*) = \sigma^*(\alpha^*, 0) h^{(\mu)}(\alpha^*, 0) = \sigma^*(\alpha^*, 0) h^{(\mu)}(\alpha, \alpha^{(0)}).$$

Let VP_μ be an element of S_n such that $[\mu]$ is a diagram of b nodes and V is any p-regular element on the fixed symbols of P_μ. We have by (4.2) and (4.3)

$$\begin{aligned} \chi_\alpha(VP_\mu) &= \sigma'(\alpha, \alpha^{(0)}) h^{(\mu)}(\alpha, \alpha^{(0)}) \chi_{\alpha^{(0)}}(V) \\ &= \sigma^*(\alpha^*, 0) \sigma(\alpha, \alpha^{(0)}) h^{(\mu)}(\alpha, \alpha^{(0)}) \chi_{\alpha^{(0)}}(V) \\ &= \sigma_\alpha \chi_{\alpha^*}(R_\mu^*) \chi_{\alpha^{(0)}}(V), \end{aligned}$$

where $\sigma_\alpha = \sigma(\alpha, \alpha^{(0)})$. This result was first obtained by R. M. Thrall and G. de B. Robinson [14]. Since $[\alpha^{(0)}]$ is the p-core, $\chi_{\alpha^{(0)}}$ is irreducible as a modular character of S_{n-bp}. If we set $\chi_{\alpha^{(0)}} = \varphi_\lambda^{(b)}$, we have

$$u_{\alpha\lambda}^{(\mu)} = \sigma_\alpha \chi_{\alpha^*}(R_\mu^*) \qquad \text{(for } [\mu] \text{ of } b \text{ nodes)}. \tag{4.14}$$

(4.14) combined with (4.10), yields

$$\sum \sigma_\alpha d_{\alpha\lambda} \chi_{\alpha^*}(R_\mu^*) = 0 \qquad \text{(for } [\mu] \text{ of } b \text{ nodes)}, \tag{4.15}$$

where $[\alpha]$ ranges over all diagrams in a p-block B of weight b. Generally, by (4.8) and (4.13), we have [10, Theorem 3] for any $[\mu]$ of b nodes and $[\nu]$ of a nodes with $a \neq b$

$$\sum \chi_\alpha(VP_\nu)\chi_{\alpha^*}(R_\mu^*) = 0 \qquad [\alpha] \subset B. \tag{4.16}$$

We shall consider the special case when $b = 1$. Since $S(1, p)$ is the cyclic group of order p with generator $Q = (1\ 2 \cdots\cdots p)$, the number of irreducible characters of $S(1, p)$ is p. Let ω be a primitive p-th root of unity. The irreducible character χ_{α^*} of the representation $Q \to \omega^i$ $(0 \leqq i \leqq p-1)$ corresponds to the star diagram $[\alpha]^*$ of one node with i-th component $[\alpha_i] = [1]$. Also Q^i corresponds to the same star diagram. Let $(d_{\alpha\lambda})$ be the decomposition matrix of a p-block B of weight 1. As was shown previously, $(d_{\alpha\lambda})$ is a matrix of type $(p, p-1)$. Hence each column of $(\sigma_\alpha d_{\alpha\lambda})$ can be written as a linear combination of the columns of $(\chi_{\alpha^*}(Q^l))$:

$$\sigma_\alpha d_{\alpha\lambda} = \sum_{l=0}^{p-1} m_{l\lambda} \chi_{\alpha^*}(Q^l) \qquad [\alpha] \subset B.$$

By the orthogonality relations for group characters of $S(1, p)$, we have

$$m_{l\lambda} = \frac{1}{p} \sum_\alpha \sigma_\alpha d_{\alpha\lambda} \chi_{\alpha^*}(Q^{-l}).$$

According to (4.14), we obtain

$$\sum_\alpha \sigma_\alpha d_{\alpha\lambda} \chi_{\alpha^*}(1) = \sum_\alpha \sigma_\alpha d_{\alpha\lambda} = 0,$$

whence $m_{0\lambda} = 0$. This implies that

$$(\sigma_\alpha d_{\alpha\lambda}) = (\chi_{\alpha^*}(Q^l)) M_1 \qquad l = 1, 2, \cdots\cdots, p-1. \tag{4.17}$$

Here $M_1 = (m_{l\lambda})$ with l $(1 \leqq l \leqq p-1)$ as row index and λ as column index. We see easily that M_1 is non-singular.

Now we shall prove the following theorem [10, Theorem 5].

Theorem 8. *Let $D=(d_{\alpha\lambda})$ be the decomposition matrix of a p-block B of weight b. Let $T_i^{(0)}$ $(i=1,2,\cdots\cdots,l^*(b))$ be the elements of $S(b,p)$ associated with $[\mu]=[0]$. There exists a non-singular matrix M_b of degree $l^*(b)$ which satisfy*

$$(\sigma_\alpha d_{\alpha\lambda}) = (\chi_\alpha*(T_i^{(0)}))M_b.$$

Proof. D is a matrix of type $(l(b), l^*(b))$. (Since p is a fixed prime number, we shall denote $l(b,p)$ simply by $l(b)$.) It follows from (4.12) and (4.13) that

$$(4.18)\qquad (\chi_\alpha*(T_i^{(n_\mu)}R_\mu^*)) = (\sigma^*(\alpha^*,\beta^*)h^{(\mu)}(\alpha,\beta))(\chi_\beta*^{(n_\mu)}(T_i^{(n_\mu)}))$$

for a fixed diagram $[\mu]\neq[0]$. As was shown before, the theorem is true for $b=1$. We shall assume it to be true for all p-blocks of weight less than $b>1$. By our inductive assumption, we have

$$(4.19)\qquad (\sigma_\beta d_{\beta\lambda}^{(n_\mu)}) = (\chi_\beta*(T_i^{(n_\mu)}))M_{b-n_\mu}.$$

Observe that $T_i^{(n_\mu)}$ corresponds to the star diagram of $b-n_\mu$ nodes with the first component [0], considered as the element of $S(b-n_\mu,p)$. We have by (4.2)

$$\sigma_\alpha = \sigma_\beta\sigma(\alpha,\beta) = \sigma_\beta\sigma'(\alpha,\beta)\sigma^*(\alpha^*,\beta^*),$$

where we set $\sigma_\beta=\sigma(\beta,\alpha^{(0)})$. Hence it follows from (4.18), (4.19) and (4.6) that

$$\begin{aligned}(4.20)\qquad (\chi_\alpha*(T_i^{(n_\mu)}R_\mu^*)) &= (\sigma^*(\alpha^*,\beta^*)h^{(\mu)}(\alpha,\beta))(\sigma_\beta d_{\beta\lambda}^{(n_\mu)})M_{b-n_\mu}^{-1}\\ &= (\sum_\beta\sigma_\alpha\sigma'(\alpha,\beta)h^{(\mu)}(\alpha,\beta)d_{\beta\lambda}^{(n_\mu)})M_{b-n_\mu}^{-1}\\ &= (\sigma_\alpha u_{\alpha\lambda}^{(\mu)})M_{b-n_\mu}^{-1}.\end{aligned}$$

This, combined with (4.10), yields

$$(4.21)\qquad \sum_\alpha\sigma_\alpha d_{\alpha\lambda}\chi_\alpha*(T_i^{(n_\mu)}R_\mu^*)) = 0 \qquad [\alpha]\subset B,$$

for any $[\mu]\neq[0]$. By the orthogonality relations for group characters of $S(b,p)$, each column of $(\sigma_\alpha d_{\alpha\lambda})$ can be written as a linear combination of the columns of $(\chi_\alpha*(T_i^{(0)}))$ $(i=1,2,\cdots\cdots,l^*(b))$. Thus we have

$$(\sigma_\alpha d_{\alpha\lambda}) = (\chi_\alpha*(T_i^{(0)}))M_b,$$

where M_b is non-singular.

(4.21) yields

$$\sum_{\alpha} \sigma_\alpha \chi_\alpha(V) \chi_\alpha{}^*(T_l^{(n_\mu)} R_\mu{}^*)) = 0 \qquad [\alpha]\subset B \tag{4.22}$$

for any p-regular element V of S_n and any $[\mu] \neq [0]$ [10, Theorem 4].
Generally we have by (4.8) and (4.20)

$$\sum_{\alpha} \sigma_\alpha \chi_\alpha(VP_\nu) \chi_\alpha{}^*(T_l^{(n_\mu)} R_\mu{}^*) = 0 \qquad [\alpha]\subset B, \qquad \text{if } [\nu] \neq [\mu]. \tag{4.23}$$

As an application of Theorem 8, we shall prove the following theorem [10, Corollary to Theorem 5].

Theorem 9. *Let* $(d_{\alpha\lambda})$ *and* $(\bar{d}_{\alpha'\lambda'})$ *be the decomposition matrices of* p*-blocks* B *and* $\bar{B}$ *of same weight* b *respectively, and let* $[\alpha]$ *and* $[\alpha']$ *have the same star diagram* $[\alpha]^*$. *Then*

$$(\sigma_{\alpha'} \bar{d}_{\alpha'\lambda'}) = (\sigma_\alpha d_{\alpha\lambda})(w_{\lambda\lambda'}),$$

where the $w_{\lambda\lambda'}$ *are rational integers and* $|w_{\lambda\lambda'}| = \pm 1$.

Proof. We have by Theorem 8

$$(\sigma_{\lambda'} \bar{d}_{\lambda'\lambda'}) = (\chi_\alpha{}^*(T_l^{(0)}))\bar{M}_b.$$

Hence

$$({}_{\alpha'} \bar{d}_{\alpha'\lambda'}) = (\sigma_\alpha d_{\alpha\lambda}) M_b^{-1} \bar{M}_b. \tag{4.24}$$

If we set $M_b^{-1}\bar{M}_b = W_b = (w_{\lambda\lambda'})$, then we see by Theorem 14 [1] that each column of $(w_{\lambda\lambda'})$ can be written as a linear combination $\sum_{\alpha'} s_{\alpha'}(\sigma_{\alpha'} \bar{d}_{\alpha'\lambda'})$, where the $s_{\alpha'}$ are rational integers which do not depend on λ. This shows that the $w_{\lambda\lambda'}$ are rational integers. Then, applying again Theorem 14 [1] to $(\sigma_{\alpha'} \bar{d}_{\alpha'\lambda'})$, we can conclude that $|W_b| = \pm 1$.

It follows from (4.24) that

$$(\bar{c}_{\kappa'\lambda'}) = W_b'(c_{\kappa\lambda}) W_b, \tag{4.25}$$

where W_b' denotes the transpose of W_b and where $(c_{\kappa\lambda})$, $(\bar{c}_{\kappa'\lambda'})$ are the matrices of Cartan invariants corresponding to B and $\bar{B}$ respectively. (4.25), combined with $|W_b| = \pm 1$, yields the following theorem [10, Theorem 6].

Theorem 10. *Two matrices of Cartan invariants corresponding to the* p*-blocks of same weight have the same elementary divisors.*

Let $U = (u_{\alpha\lambda}{}^{(\mu)})$ be the matrix of u-numbers corresponding to a

p-block B of weight b [10]. U is a square matrix of degree $l(b)$ and is non-singular. We have by (4.20)

$$(4.26) \qquad (\sigma_\alpha u_{\alpha\lambda}^{(\mu)}) = (\chi_{\alpha^*}(T_i^{(n_\mu)} R_\mu^*))M,$$

where

$$M = \begin{pmatrix} M_b & & 0 \\ & M_{b-1} & \\ & \cdots\cdots\cdots & \\ 0 & & M_0 \end{pmatrix}, \qquad M_0 = I,$$

if the rows and columns are arranged suitably.

Theorem 11. *Let $(u_{\alpha\lambda}^{(\mu)})$ and $(\bar{u}_{\alpha'\lambda'}^{(\mu)})$ be the matrices of u-numbers corresponding to the p-blocks B and $\bar{B}$ of same weight respectively, and let $[\alpha]$ and $[\alpha']$ have the same star diagram $[\alpha]^*$. Then $(\sigma_\alpha u_{\alpha\lambda}^{(\mu)})$ and $(\sigma_{\alpha'} \bar{u}_{\alpha'\lambda'}^{(\mu)})$ have the same elementary divisors.*

Proof. We have by (4.26)

$$(\sigma_{\alpha'} \bar{u}_{\alpha'\lambda'}^{(\mu)}) = (\sigma_\alpha u_{\alpha\lambda}^{(\mu)}) W,$$

where

$$W = \begin{pmatrix} W_b & & 0 \\ & W_{b-1} & \\ & \cdots\cdots\cdots & \\ 0 & & W_0 \end{pmatrix}.$$

Since $|W| = \pm 1$, our assertion follows immediately.

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