THEORY OF CONNECTIONS AND A THEOREM OF E. CARTAN ON HOLONOMY GROUPS I

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E. Cartan [1][1)] proved locally a fundamental theorem on holonomy groups of spaces with generalized connections as follows:

Theorem. *Let H be the holonomy group of a space with a connection of structure group G, then the space is equivalent to a space with a connection of structure group H.*

The proof of E. Cartan holds good for the space whose underlying manifold is an n-cell. In this paper, we shall investigate the theorem in the large by means of fibre bundles. For fibre bundles, we shall utilize the notations in [2]. In §§2-5, we will give an elementary explanation on the relation between the concept of infinitesimal connections in fibre bundles introduced by C. Ehresmann [3] and the classical one of E. Cartan [1].

§1. We consider a fibre bundle $\mathfrak{B} = \{B, p, X, Y, G\}$. For the purpose of differential geometry the following assumptions will be made:

1) The bundle space B, the base space X, the fibre Y are connected, differentiable[2)] manifolds;

2) the group G of the bundle is a Lie group which acts differentiably and effectively on Y;

3) the projection p of B onto X is differentiable.

We assume *that a differentiable family of tangent subspaces to B which are transversal to the fibres is given.* For any curve $\mathscr{C}$ of class C^r $(r \geqq 2)$ in X from x_0 to x_1 and any point $b_0 \in p^{-1}(x_0)$, we have an uniquely determined curve ζ in B from b_0 to a point $b_1 \in p^{-1}(x_1)$ such that $p(\zeta) = \mathscr{C}$ and at any point $b \in \zeta$, ζ is tangent to the tangent subspace at b of the family. Then, corresponding b_1 to b_0, we get a homeomorphism

$$\rho(\mathscr{C}) : p^{-1}(x_1) = Y_{x_1} \longrightarrow p^{-1}(x_0) = Y_{x_0}.$$

Furthermore, we assume *that $\rho(\mathscr{C})$ is a bundle mapping.* Then, according to C. Ehresmann [3], we will say an *infinitesimal connection*

1) Numbers enclosed in brackets refer to the bibliography.

2) In the following, we suppose that all the manifolds B, X, Y, etc. are of class C^r $(r \geqq 2)$ and the differentiabilities of mappings are of suitable orders respectively.

Γ is given in $\mathfrak{B}$. Then the group G is called the *structure group* of the connection.

Let us put

$$\Omega_{x_0, x_1} = \text{the set of curves of class } D^r \text{ [1] in } X \text{ from } x_0 \text{ to } x_1$$

and

$$\Omega = \bigcup_{x_0, x_1 \in X} \Omega_{x_0, x_1}.$$

The above-mentioned $\rho(\mathscr{C})$ can be also defined for any curve of class D^r by combining the homeomorphisms corresponding to subarcs of class C^r. Then, by the definition, we have

$$(1) \qquad \rho(\mathscr{C}_1\mathscr{C}_2) = \rho(\mathscr{C}_1)\rho(\mathscr{C}_2), \qquad \mathscr{C}_1 \in \Omega_{x_0, x_1}, \qquad \mathscr{C}_2 \in \Omega_{x_1, x_2}.$$

Let $\Omega_x = \Omega_{x,x}$, $\chi_x = \rho \mid \Omega_x$, then by (1) the transformation $\chi_x : \Omega_x \rightarrow \chi_x(\Omega_x) = \Phi_x$ is a homomorphism of the group Ω_x of closed paths at x and a group of bundle mappings of Y_x on itself. Let ξ be any admissible map at $x \in X$, then $H_x = \overline{\xi^{-1}\Phi_x\xi}$ is a subgroup of G.[2] We call H_x the *holonomy group* at x of the bundle $\mathfrak{B}$ with the infinitesimal connection Γ.

Let be given another fibre bundle $\mathfrak{B}' = \{B', p', X, Y, G\}$ with an infinitesimal connection Γ' as $\mathfrak{B}$. Let $\rho', \chi_x', \Phi_x', H_x'$ be the maps and the groups defined for $\mathfrak{B}'$ as analogous to $\rho, \chi_x, \Phi_x, H_x$.

If for a point $x \in X$, we can take two admissible mappings $\xi : Y \rightarrow Y_x$, $\xi' : Y \rightarrow Y_x'$ such that $\xi^{-1}\chi_x(\mathscr{C})\xi = \xi^{-1}\chi_x'(\mathscr{C})\xi'$ for any $\mathscr{C} \in \Omega_x$, which we denote simply by $\xi^{-1}\chi_x\xi = \xi'^{-1}\chi_x'\xi'$, we denote this by $\chi_x \approx \chi_x'$.

We shall prove the following lemma.

Lemma 1. *Fibre bundles $\mathfrak{B}, \mathfrak{B}'$ with infinitesimal connections, the same base space, fibre and group are equivalent in G (G-equivalent) as fibre bundles if $\chi_{x_0} \approx \chi_{x_0}'$ at a point $x_0 \in X$.*

Proof. By the assumption of this theorem, let us put

$$(2) \qquad \xi^{-1}\chi_{x_0}\xi = \xi'^{-1}\chi_{x_0}'\xi',$$

where ξ, ξ' are admissible mappings of $\mathfrak{B}, \mathfrak{B}'$ at x_0.

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1) A curve in X is said to be of class D^r, $r > 0$, if it is defined by a continuous mapping of a closed interval into X, and if the interval can be divided into a finite set of subintervals on the closure of each of which the mapping is of class C^r.

2) $\xi^{-1}\Phi\xi$ is an abstract subgroup of G and may not be a closed subgroup of G.

For any point $x \in X$, let $\mathscr{C}$ be a curve of $\Omega_{x_0, x}$ and define h_x: $Y_x \to Y'_x$ by

$$h_x = \rho'(\mathscr{C}^{-1})\xi'\xi^{-1}\rho(\mathscr{C}). \tag{3}$$

If $\mathscr{C}_1$ is another curve of $\Omega_{x_0, x}$ and $h_{1,x}$ is the corresponding mapping, then we have by (1), (2)

$$\begin{aligned} h_x^{-1}h_{1,x} &= [\rho(\mathscr{C}^{-1})\xi\xi'^{-1}\rho'(\mathscr{C})][\rho'(\mathscr{C}_1^{-1})\xi'\xi^{-1}\rho(\mathscr{C}_1)] \\ &= [\rho(\mathscr{C}^{-1})\xi][\xi'^{-1}\rho'(\mathscr{C}\mathscr{C}_1^{-1})\xi'][\xi^{-1}\rho(\mathscr{C}_1)] \\ &= [\rho(\mathscr{C}^{-1})\xi][\xi'^{-1}\chi'_{x_0}(\mathscr{C}\mathscr{C}_1^{-1})\xi'][\xi^{-1}\rho(\mathscr{C}_1)] \\ &= [\rho(\mathscr{C}^{-1})\xi][\xi^{-1}\chi_{x_0}(\mathscr{C}\mathscr{C}_1^{-1})\xi][\xi^{-1}\rho(\mathscr{C}_1)] \\ &= \rho(\mathscr{C}^{-1})\rho(\mathscr{C}\mathscr{C}_1^{-1})\rho(\mathscr{C}_1) = 1, \end{aligned}$$

that is $h_x = h_{1,x}$.

Then, we define an one-to-one transformation $h: B \to B'$ by $h \mid Y_x = h_x$. For a fixed point $x_1 \in X$, let U be a coordinate neighborhood of x_1 which is simply covered by a differentiable family of curves issuing from x_1. For $x \in U$, let $\mathscr{C}_x$ be the curve from x_1 to x of the family. Then, since Γ is differentiable, $\rho(\mathscr{C}_x)(b)$, $b \in p^{-1}(x)$, is a differentiable mapping of $p^{-1}(U)$ onto Y_{x_1}, and $\rho(\mathscr{C}_x^{-1})(b)$, $b \in Y_{x_1}$, is a differentiable homeomorphism of $Y_{x_1} \times U$ onto $p^{-1}(U)$. $\rho'(\mathscr{C}_x)$ has the same property as $\rho(\mathscr{C}_x)$. Let $\mathscr{C}_1$ be a curve of Ω_{x_0, x_1}, then we have

$$h_x = \rho'(\mathscr{C}_x^{-1}\mathscr{C}_1^{-1})\xi'\xi^{-1}\rho(\mathscr{C}_1\mathscr{C}_x) = \rho'(\mathscr{C}_x^{-1})h_{x_1}\rho(\mathscr{C}_x).$$

This relation shows that h is continuous at x_1, furthermore, h is a differentiable homeomorphism.

Let $\{U_\alpha\}$ be a system of admissible coordinate neighborhoods as above which is a covering of X and

$$\begin{aligned} \phi_\alpha &: U_\alpha \times Y \longrightarrow p^{-1}(U_\alpha), \\ \phi'_\alpha &: U_\alpha \times Y \longrightarrow p'^{-1}(U_\alpha) \end{aligned}$$

be the coordinate functions of $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. Define

$$\begin{aligned} p_\alpha &: p^{-1}(U_\alpha) \longrightarrow Y, \\ p'_\alpha &: p'^{-1}(U_\alpha) \longrightarrow Y \end{aligned}$$

by $p_\alpha \mid Y_x = \phi_{\alpha,x}^{-1}$, $p'_\alpha \mid Y'_x = \phi'^{-1}_{\alpha,x}$. If $U_\alpha \cap U_\beta \neq \phi$, let

$$g_{\alpha\beta},\ g'_{\alpha\beta}\ :\ U_\alpha \cap U_\beta \longrightarrow G,$$
$$g_{\alpha\beta}(x) = \phi_{\alpha,x}{}^{-1}\phi_{\beta,x},\quad g'_{\alpha\beta}(x) = \phi'_{\alpha,x}{}^{-1}\phi'_{\beta,x}$$

be the coordinate transformations of $\mathfrak{B}, \mathfrak{B}'$ respectively.

These mappings have the property as

$$(4)\qquad \begin{aligned} g_{\alpha\beta}(x)g_{\beta\gamma}(x) &= g_{\alpha\gamma}(x), \\ g'_{\alpha\beta}(x)g'_{\beta\gamma}(x) &= g'_{\alpha\gamma}(x), \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

If the point $x_1 \in U_\alpha \cap U_\gamma$, $\mathscr{C}_x \subset U_\alpha \cap U_\gamma$. then we have

$$\begin{aligned} \bar{g}_{\gamma\alpha}(x) &= p'_\gamma h_x \phi_{\alpha,x} \\ &= p'_\gamma \rho'(\mathscr{C}_x{}^{-1}) h_{x_1} \rho(\mathscr{C}_x)\phi_{\alpha,x} \\ &= [p'_\gamma \rho'(\mathscr{C}_x{}^{-1})\phi'_{\gamma,x_1}][p'_\gamma h_{x_1}\phi_{\alpha,x_1}][p_\alpha \rho(\mathscr{C}_x)\phi_{\alpha,x}] \\ &= [p'_\gamma \rho'(\mathscr{C}_x{}^{-1})\phi'_{\gamma,x_1}]\bar{g}_{\gamma\alpha}(x_1)[p_\alpha \rho(\mathscr{C}_x)\phi_{\alpha,x}]. \end{aligned}$$

The first and third factors enclosed in square brackets of the last side of the above equations are differentiable on x. Hence, the map $\bar{g}_{\gamma\alpha} : U_\alpha \cap U_\gamma \to G$ is differentiable in a neighborhood of x_1 in $U_\alpha \cap U_\gamma$. By the definition of $\bar{g}_{\gamma\alpha}$, it has the property as

$$(5)\qquad \bar{g}_{\delta\beta}(x) = g'_{\delta\gamma}(x)\bar{g}_{\gamma\alpha}(x)g_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta.$$

Therefore, h is a differentiable bundle mapping. $\mathfrak{B}, \mathfrak{B}'$ are equivalent in G.

§2. Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a fibre bundle with an infinitesimal connection Γ as in §1, then we can give an infinitesimal connection $\bar{\Gamma}$ for the associated principal bundle[1] $\bar{\mathfrak{B}} = \{\bar{B}, \bar{p}, X, G, G\}$ of $\mathfrak{B}$ such that for any point $x_0, x_1 \in X$ and any curve $\mathscr{C} \in \Omega_{x_0, x_1}$

$$(6)\qquad \bar{\rho}(\mathscr{C})(\xi_{x_1}) = \rho(\mathscr{C})\xi_{x_1}, \qquad \xi_{x_1} \in G_x,$$

since $\rho(\mathscr{C})$ is a bundle mapping. Denoting the right translation corresponding to $g \in G$ by $\mathrm{r}(g)$, we get from (6)

$$\begin{aligned} (\bar{\rho}(\mathscr{C})\mathrm{r}(g))(\xi_{x_1}) &= \bar{\rho}(\mathscr{C})(\xi_{x_1}g) \\ &= \rho(\mathscr{C})(\xi_{x_1}g) \\ &= (\rho(\mathscr{C})\xi_{x_1})g \\ &= \mathrm{r}(g)(\bar{\rho}(\mathscr{C})(\xi_{x_1})), \end{aligned}$$

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1) See [2], §8.

hence

$$\bar{\rho}(\mathscr{C})\mathrm{r}(g) = \mathrm{r}(g)\bar{\rho}(\mathscr{C}). \tag{7}$$

This shows that $\bar{\Gamma}$ *is invariant under right translations.*

Conversely, if *we have a differentiable family of tangent subspaces to* $\bar{B}$ *which are transversal to the fibres and are invariant under right translations, there exists an infinitesimal connection* Γ *in* $\mathfrak{B}$ *such that* (6) *holds good.*

By virtue of the above argument, in the following, we may consider only principal fibre bundles.

Let $\mathfrak{B} = \{B, p, X, G, G\}$ be a differentiable principal fibre bundle as in §1 and let Γ be a differentiable family of tangent subspaces $\Gamma_b \subset T_b(B)$, $b \in B$[1], which are transversal to the fibres $G_{p(b)}$ and are invariant under right translations, that is

$$\begin{cases} p_*(\Gamma_b) = T_{p(b)}(X), \\ \mathrm{r}(g)_*\Gamma_b = \Gamma_{\mathrm{r}(g)(b)}, \end{cases} \qquad b \in B, \quad g \in G \tag{8}$$

where p_*, $\mathrm{r}(g)_*$ denote the differential mappings of p, $\mathrm{r}(g)$[2].

The decomposition of $T_b(B)$ into the direct sum

$$T_b(B) = \Gamma_b + T_b(G_{p(b)})$$

define the projection $\mu_b : T_b(B) \to T_b(G_{p(b)})$. Let μ be the mapping $T(B) \to T(B)$ by $\mu(\mathfrak{v}) = \mu_b(\mathfrak{v})$ for any $\mathfrak{v} \in T_b(B)$. Let ι_x be the imbedding mapping of G_x into B, then, by the definition of μ_b, we get

$$\mu\iota_{x*} = \iota_{x*}. \tag{9}$$

For any $\mathfrak{v} \in T_b(B)$, $g \in G$, by (8) and the relation

$$\begin{aligned} \mathrm{r}(g)_*(\mathfrak{v}) &= \mathrm{r}(g)_*(\mathfrak{v} - \mu_b(\mathfrak{v}) + \mu_b(\mathfrak{v})) \\ &= \mathrm{r}(g)_*(\mathfrak{v} - \mu_b(\mathfrak{v})) + \mathrm{r}(g)_*\mu_b(\mathfrak{v}) \end{aligned}$$

we get

$$\mathrm{r}(g)_*\mu_b = \mu_{\mathrm{r}(g)(b)}\mathrm{r}(g)_* \tag{10}$$

or

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1) For a differentiable manifold X, we denote the tangent space at $x \in X$ by $T_x(X)$ and the bundle space of the tangent bundle of X by $T(X)$.

2) Let X, Y be any differentiable manifolds and let f be a differentiable mapping $X \to Y$. Then we denote by $f_* : T(X) \to T(X)$ the differential mapping of f. If $f : X \to Y$, $h : Y \to Z$, then $(fh)_* = f_*h_*$. See [4] or [5].

$$\mathfrak{r}(g)_*\mu = \mu\mathfrak{r}(g)_* .$$

We denote by the same notation b the mapping of G onto G_x that $b(e) = b$ and define a linear transformation $\pi_b : T_b(B) \to T_e(G)$ by

$$(11) \qquad \pi_b = (b_*)^{-1}\mu_b$$

where e denotes the identity element of G. Thus, we obtain a set of linear differential forms on B with values in the Lie algebra $L(G) \approx T_e(G)$ (as vector space).[1]

Since $\mathfrak{r}(g)(b) \equiv bg = bl(g)$, $b\mathfrak{r}(g) = \mathfrak{r}(g)b$, where $l(g) : G \to G$ denotes the left translation corresponding to g, for any $\mathfrak{b} \in T_b(B)$, we get by (10), (11)

$$\begin{aligned}
\mathfrak{r}(g)^*\pi(\mathfrak{b}) &= \pi(\mathfrak{r}(g)_*\mathfrak{b}) = \pi_{bg}(\mathfrak{r}(g)_*\mathfrak{b})^{2)} \\
&= ((bg)_*)^{-1}\mu_{bg}\mathfrak{r}(g)_*\mathfrak{b} \\
&= ((bg)_*)^{-1}\mathfrak{r}(g)_*\mu_b\mathfrak{b} \\
&= ((bl(g))_*)^{-1}\mathfrak{r}(g)_*\mu_b\mathfrak{b} \\
&= l(g^{-1})_*\mathfrak{r}(g)_*\mathfrak{r}(g)_*^{-1}b_*^{-1}\mathfrak{r}(g)_*\mu_b\mathfrak{b} \\
&= l(g^{-1})_*\mathfrak{r}(g)_*b_*^{-1}\mu_b\mathfrak{b} \\
&= l(g^{-1})_*\mathfrak{r}(g)_*\pi(\mathfrak{b}).
\end{aligned}$$

Putting $\mathrm{ad}(g) = (l(g)\mathfrak{r}(g^{-1}))_*$ which is the differential mapping of the adjoint mapping $A(g) : G \to G$ by $A(g)(y) = gyg^{-1}$, $y \in G$, the above relation is written as

$$(12) \qquad \mathfrak{r}(g)^*\pi = \mathrm{ad}(g^{-1})\pi.$$

For $\mathfrak{b} \in T_g(G)$, $b \in p^{-1}(x)$, we have

$$\begin{aligned}
(\iota_x b)^*\pi(\mathfrak{b}) &= \pi((\iota_x b)_*\mathfrak{b}) \\
&= (bg)_*^{-1}\mu_{bg}(\iota_{x*}b_*\mathfrak{b}) \\
&= l(g^{-1})_*b_*^{-1}b_*\mathfrak{b} = l(g^{-1})_*(\mathfrak{b}).
\end{aligned}$$

If we define

$$(13) \qquad (\iota_x b)^*\pi = \omega,$$

1) We denote by $T^*(X, L(G))$ the bundle space of the fibre bundle over X whose fibre at $x \in X$ is $\mathscr{L}(T_x(X) ; L(G))$. Let f be a differentiable mapping $X \to Y$, then we denote by $f^* : T^*(Y, L(G)) \to T^*(X, L(G))$ the dual mapping of f_*. It $f : X \to Y$, $h : Y \to Z$, then $(hf)^* = f^*h^*$.

2) By the natural isomorphism $l(g)_* : T_e(G) \to T_g(G)$, $T_e(G) \approx T_g(G)$.

the above relation is written as

$$\omega(\mathfrak{v}) = l(g^{-1})_* \mathfrak{v}, \qquad \mathfrak{v} \in T_g(G), \quad g \in G.$$

From this relation, we obtain

$$(14) \qquad \begin{cases} l(g)^* \omega = \omega, & g \in G, \\ \omega(\mathfrak{v}) = \mathfrak{v}, & \mathfrak{v} \in T_e(G). \end{cases}$$

This shows that the $L(G)$-valued linear differential form ω on G is independent of $b \in B$.

Conversely, *we can define a differentiable family of tangent subspaces satisfying* (8) *from a* $L(G)$*-valued linear differential from* π *on* B *satisfying* (12), (13).

§3. Now, let ι_α be the imbedding mapping $p^{-1}(U_\alpha) \to B$ and define a mapping $\rho_\alpha : U_\alpha \to U_\alpha \times G$ by

$$\rho_\alpha(x) = x \times e, \qquad x \in U_\alpha.$$

Define a $L(G)$-valued linear differential form θ_α on U_α by

$$(15) \qquad \theta_\alpha = (\iota_\alpha \phi_\alpha \rho_\alpha)^* \pi.$$

Since $b = \mathrm{r}(p_\alpha(b)) \iota_\alpha \phi_\alpha \rho_\alpha p(b)$, $b \in p^{-1}(U_\alpha)$, for any $\mathfrak{v} \in T_b(B)$, we have

$$\mathfrak{v} = (\mathrm{r}(g) \iota_\alpha \phi_\alpha \rho_\alpha p)_* \mathfrak{v} + (\iota_x \phi_\alpha(x, e))_* p_{\alpha *} \mathfrak{v}, \qquad x = p(b), \ g = p_\alpha(b).$$

Hence, we get by (12), (13), (14), (15)

$$\begin{aligned} \pi_b &= p^* \rho_\alpha^* \phi_\alpha^* \iota_\alpha^* \mathrm{r}(g)^* \pi_b + p_\alpha^* \phi(x, e)^* \iota_x^* \pi_b \\ &= p^* (\iota_\alpha \phi_\alpha \rho_\alpha)^* (\mathrm{ad}(g^{-1}) \pi_b) + p_\alpha^* (\phi_\alpha(x, g) l(g^{-1}))^* \iota_x^* \pi_b \\ &= \mathrm{ad}(g^{-1}) p^* \theta_{\alpha, x} + p_\alpha^* l(g^{-1})^* \omega_e \\ &= \mathrm{ad}(g^{-1}) p^* \theta_{\alpha, x} + p_\alpha^* \omega_g, \end{aligned}$$

that is

$$(16) \qquad \pi_b = \mathrm{ad}(g^{-1}) p^* \theta_{\alpha, x} + p_\alpha^* \omega_g, \qquad p(b) = x, \ p_\alpha(b) = g.$$

If $b \in p^{-1}(U_\alpha \cap U_\beta)$, then $p_\beta(b) = g_{\beta\alpha}(p(b)) p_\alpha(b)$. Hence, at b, we have the relation

$$\begin{aligned} p_{\beta *} &= l(g_{\beta\alpha}(p(b)))_* p_{\alpha *} + \mathrm{r}(p_\alpha(b))_* g_{\beta\alpha *} p_*, \\ p_\beta^* \omega &= p_\alpha^* l(g_{\beta\alpha}(p(b)))^* \omega + p^* g_{\beta\alpha}^* \mathrm{r}(p_\alpha(b))^* \omega \\ &= p_\alpha^* \omega + p^* g_{\beta\alpha}^* (\mathrm{ad}(p_\alpha(b)^{-1}) \omega). \end{aligned}$$

By the relations above and the equation

$$\mathrm{ad}(p_\alpha(b)^{-1})p^*\theta_{\alpha,x} + p_\alpha{}^*\omega_{p_\alpha(b)} = \mathrm{ad}(p_\beta(b)^{-1})p^*\theta_{\beta,x} + p_\beta{}^*\omega_{p_\beta(b)},$$

we get

$$p^*\theta_{\alpha,x} = p^*\{\mathrm{ad}(g_{\beta\alpha}(x)^{-1})\theta_{\beta,x} + g_{\beta\alpha}{}^*\omega_{g_{\beta\alpha}(x)}\},$$

from which we get

$$\theta_{\alpha,x} = \mathrm{ad}(g_{\beta\alpha}(x)^{-1})\theta_{\beta,x} + g_{\beta\alpha}{}^*\omega_{g_{\beta\alpha}(x)}, \tag{17}$$

or simply

$$\theta_\alpha = \mathrm{ad}(g_{\beta\alpha}{}^{-1})\theta_\beta + g_{\beta\alpha}{}^*\omega, \tag{17'}$$

since p is onto.

Conversely, *on each* U_α, *let be given a system of* $L(G)$*-valued linear differential forms* θ_α *satisfying* (17), *then we can obtain a* $L(G)$*-valued linear differential form* π *satisfying* (12), (13) *by* (16).

Thus we see that an infinitesimal connection Γ as in §1 is given in $\mathfrak{B}$ is equivalent to that on each coordinate neighborhood U_α, a $L(G)$-valued linear differential form satisfying (17′) is given. The components of θ_α are the parameters of the connection in the classical sense and (17′) is the transformation equation of the parameters for coordinate transformations.

§4. In U_1, let be given a differentiable family of curves $\mathscr{C}(x_1, x) \in \Omega_{x_1,x}$ which covers simply over U_1 except x_1. Then, $\rho(\mathscr{C}(x_1, x)) : G_x \to G_{x_1}$ define a differentiable mapping

$$F : p^{-1}(U_1) \to G_{x_1} \qquad \text{by} \quad F(b) = \rho(\mathscr{C}(x_1, p(b)))(b).$$

Since $F \mid G_x$ is a bundle mapping, we can define a differentiable mapping $\eta : U_1 \to G$ by

$$p_1 F\phi_1(x, g) = \eta(x)g = f(x, g). \tag{18}$$

Let $\tau_1 : U_1 \times G \to U_1$, $\tau_2 : U_1 \times G \to G$ be the natural projections, then for any $\mathfrak{v} \in T_{x_1,e}(U_1 \times G)$, we get by (14), (16)

$$\begin{aligned} f_*\mathfrak{v} &= (\eta\tau_1)_*\mathfrak{v} + \tau_{2*}\mathfrak{v}, \\ (p_1F\phi_1)_*\mathfrak{v} &= p_{1*}\mu_{b_1}\phi_{1*}\mathfrak{v} = b_{1*}{}^{-1}\mu_{b_1}\phi_{1*}\mathfrak{v} \\ &= \pi_{b_1}(\phi_{1*}\mathfrak{v}) \end{aligned}$$

$$= p^* \theta_1(\phi_{1*}\mathfrak{v}) + p_1^* \omega(\phi_{1*}\mathfrak{v})$$
$$= \theta_1(p_* \phi_{1*}\mathfrak{v}) + \omega(p_{1*}\phi_{1*}\mathfrak{v})$$
$$= \theta_1(\tau_{1*}\mathfrak{v}) + \omega(\tau_{2*}\mathfrak{v})$$
$$= \theta_1(\tau_{1*}\mathfrak{v}) + \tau_{2*}\mathfrak{v}, \qquad b_1 = \phi_1(x_1, e),$$

since $\eta(x_1) = e$, $p_1(b_1) = e$, $\tau_1 = p\phi_1$, $\tau_2 = p_1\phi_1$. Hence, from (18) and the above relation we obtain

$$\eta_*(\tau_{1*}\mathfrak{v}) = \theta_1(\tau_{1*}\mathfrak{v})$$

or

(19) $$\eta^* \omega_e = \theta_{1, x_1}.$$

This equation will imply the following result which is in connection with the development of a curve in X on a tangent space to X at a point of the curve, in the classical differential geometry.

For any curve $\mathscr{C}$ of class C^r from x_0 to $x_1 : x = \psi(t)$, $0 \leqslant t \leqslant 1$, let $\mathscr{C}_\lambda \subset U_{\alpha_\lambda}$, $\lambda = 1, 2, \cdots\cdots, m$, be the subarc of $\mathscr{C}$ corresponding to the interval $t_{\lambda-1} \leqslant t \leqslant t_\lambda$, $0 = t_0 < t_1 < \cdots\cdots < t_m = 1$. Then, we can determine mappings

$$\eta_\lambda : [t_{\lambda-1}, t_\lambda] \longrightarrow G,$$

so that

(20) $$\eta_\lambda^* \omega = \psi_\lambda^* \theta_{\alpha_\lambda}, \qquad \eta_\lambda(t_{\lambda-1}) = \eta_{\lambda-1}(t_{\lambda-1}) g_{\alpha_{\lambda-1}\alpha_\lambda}(\psi(t_{\lambda-1}))$$

where $\psi_\lambda = \psi \mid [t_{\lambda-1}, t_\lambda]$. This is to integrate some system of ordinary differential equations in each coordinate neighborhood under certain conditions. If we extend each solution $\eta_\lambda(t)$ for $[t_{\lambda-1}, t_\lambda]$ to both sides of the interval, then in $U_{\alpha_{\lambda-1}} \cap U_{\alpha_\lambda}$, by means of (17′) we have

$$\eta_\lambda(t) = \eta_{\lambda-1}(t) g_{\alpha_{\lambda-1}\alpha_\lambda}(\psi(t)).$$

We define an element of G by

(21) $$k_{\alpha_0\alpha_m}(\mathscr{C}) = \eta_1(0)^{-1}\eta_m(1),$$

and for any curve $\mathscr{C} \in \Omega_{x_0, x_1}$, we define likewise $k_{\alpha_0\alpha_m}(\mathscr{C})$. Since ω is left-invariant, $k_{\alpha_0\alpha_m}(\mathscr{C})$ is independent of the choice of the initial point $\eta_1(0)$. Furthermore, we get easily the relation

(22) $$k_{\alpha_1\alpha_3}(\mathscr{C}_1\mathscr{C}_2) = k_{\alpha_1\alpha_2}(\mathscr{C}_1)k_{\alpha_2\alpha_3}(\mathscr{C}_2),$$
$$\mathscr{C}_1 \in \Omega_{x_1,x_2},\ \mathscr{C}_2 \in \Omega_{x_2,x_3},\ x_1 \in U_{\alpha_1},\ x_2 \in U_{\alpha_2},\ x_3 \in U_{\alpha_3}.$$

By means of (19), between ρ and k, there exists the following relation

(23) $$\rho(\mathscr{C})\cdot\phi_{\alpha_2,x_2} = \phi_{\alpha_1,x_1}\cdot k_{\alpha_1\alpha_2}(\mathscr{C}),$$
$$\mathscr{C} \in \Omega_{x_1,x_2},\ x_1 \in U_1,\ x_2 \in U_2.$$

§5. Now, in each coordinate neighborhood U_α, we take a differentiable mapping $f_\alpha : U_\alpha \to G$ and define a $L(G)$-valued linear differential form by

(24) $$\hat{\theta}_\alpha = \mathrm{ad}(f_\alpha)\theta_\alpha + (f_\alpha^{-1})^*\omega,$$

then we get

$$\hat{\theta}_\beta = \mathrm{ad}(f_\beta g_{\alpha\beta}^{-1} f_\alpha^{-1})\hat{\theta}_\alpha + (f_\alpha g_{\alpha\beta} f_\beta^{-1})^*\omega, \qquad x \in U_\alpha \cap U_\beta,$$

where we put $f_\alpha^{-1}(x) = (f_\alpha(x))^{-1}$. If we take, in each neighborhood U_α, a coordinate function

(25) $$\hat{\phi}_{\alpha,x} = \phi_{\alpha,x} f_\alpha(x)^{-1},$$

then we get the coordinate transformation of the bundle

(26) $$\hat{g}_{\alpha\beta}(x) = \hat{\phi}_{\alpha,x}^{-1}\hat{\phi}_{\beta,x} = f_\alpha(x) g_{\alpha\beta}(x) f_\beta(x)^{-1}, \qquad x \in U_\alpha \cap U_\beta.$$

Then, the fibre bundle $\hat{\mathfrak{B}} = \{B, p, X, Y, G, \hat{\phi}_\alpha\}$ with the infinitesimal connection $\{\hat{\theta}_\alpha\}$ is G-equivalent to the fibre bundle $\mathfrak{B} = \{B, p, X, Y, G, \phi_\alpha\}$ with the infinitesimal connection $\{\theta_\alpha\}$, that is, $\{\hat{\theta}_\alpha\}$ is obtained from $\{\theta_\alpha\}$ by transformations of frames. In both $\mathfrak{B}$ and $\hat{\mathfrak{B}}$, B has the same family of tangent subspaces to B which are transversal to the fibres. For $\hat{k}$ in $\hat{\mathfrak{B}}$ and k in $\mathfrak{B}$, from (23), (25) we get easily the relation

(27) $$\hat{k}_{\alpha\beta} = f_\alpha k_{\alpha\beta} f_\beta^{-1}.$$

Now, we take a coordinate neighborhood U such that if $U \ni x = (x^1, \cdots\cdots, x^n)$, then $U \ni (tx^1, \cdots\cdots, tx^n)$, $0 \leqslant t \leqslant 1$. Let θ be the $L(G)$-valued linear differential form in U. Let o be the origin of the coordinate system and $\widehat{ox}$ be the image of the segment joining o and x in the coordinates. Define a mapping $f: U \to G$ by

$$k(\widehat{ox}) = k_{UU}(\widehat{ox}) = f(x). \tag{28}$$

The mapping f is differentiable. For any point $x \in U$, we define the mapping $a_x : 0 \leqslant t \leqslant 1 \to U$ by $a_x(t) = (tx^i)$. Then, we have by (20), (28), (24)

$$a_x^* f^* \omega = a_x^* \theta,$$
$$a_x^* \theta = a_x^*(\mathrm{ad}(f^{-1})\hat{\theta} + f^* \omega).$$

Hence we obtain

$$a_x^* \hat{\theta} = 0. \tag{29}$$

Now, let X be an n-cell. $U = X$ be an coordinate neighborhood as above. Then, we get from (29)

$$\hat{k}(\widehat{ox}) = e.$$

Hence, by (23), (27), for any $\mathscr{C} \in \Omega_{x, x_1}$ we have

$$\hat{\phi}_{U,0}{}^{-1}\chi_0(\widehat{ox}\,\mathscr{C}\,\widehat{ox}_1{}^{-1})\hat{\phi}_{U,0} = \hat{k}(\widehat{ox}\,\mathscr{C}\,\widehat{ox}^{-1}) = \hat{k}(\mathscr{C})$$
$$= k(\widehat{ox}\,\mathscr{C}\,\widehat{ox}^{-1}) \in H_0$$

since $f(o) = e$. From this and (19), $\hat{\theta}$ is a $L(H_0)$-valued linear differential form. In other words, *if X is an n-cell, we can take a $L(H_0)$-valued linear differential form $\hat{\theta}$ from the $L(G)$-valued linear differential form θ by a suitable transformation of coordinate functions* (that is, by a suitable choice of frames).

§6. **Lemma 2.** *Let X, Y, G be differentiable manifolds, a Lie group as stated in Section 1. For a point $x_0 \in X$, let be given a transformation $\chi_0 : \Omega_{x_0} \to G$ with the properties as follows:*

i) $\chi_0(\mathscr{C}_1\mathscr{C}_2) = \chi_0(\mathscr{C}_1)\chi_0(\mathscr{C}_2)$, $\mathscr{C}_1, \mathscr{C}_2 \in \Omega_{x_0}$;

ii) $\chi_0(\mathscr{D}_1\mathscr{D}_2) = \chi_0(\mathscr{D}_1\mathscr{D}\mathscr{D}^{-1}\mathscr{D}_2)$, $\mathscr{D}_1, \mathscr{D}, \mathscr{D}_2 \in \Omega$, $\mathscr{D}_1\mathscr{D}\mathscr{D}^{-1}\mathscr{D}_2 \in \Omega_{x_0}$;

iii) χ_0 *is differentiable.*

Then there exists a fibre bundle $\mathfrak{B} = \{B, p, X, Y, G\}$ with an infinitesimal connection Γ such that $\chi_{x_0} \approx \chi_0$.

In the lemma, the differentiablity of χ_0 is in the sense as follows.

For any points $x_1, x_2 \in X$, let $\mathscr{D}(x_1, x)$, $\mathscr{D}(x, x')$, $\mathscr{D}(x_2, x')$ be differentiable families of curves, $x \in a$ coordinate neighborhood U,

$x' \in a$ coordinate neighborhood V, then

$$\chi_0(\mathscr{C}_1 \mathscr{D}(x_1, x)\mathscr{D}(x, x')\mathscr{D}(x_2, x')^{-1}\mathscr{C}_2^{-1}) \in G,$$
$$\mathscr{C}_1 \in \Omega_{x_0, x_1}, \quad \mathscr{C}_2 \in \Omega_{x_0, x_2},$$

is differentiable with respect to x, x'.

Proof. Let $\{U_\alpha\}$ be a covering system of coordinate neighborhoods such that if $U_\alpha \ni x = (x^1, \cdots\cdots, x^n)$, then $U \ni (tx^1, \cdots\cdots, tx^n)$, $0 \leqslant t \leqslant 1$. Let x_α be the point whose coordinates in U_α are $(0, \cdots, 0)$, and for $x \in U_\alpha$, let $\mathscr{C}(x_\alpha, x)$ be the curve which is the locus of points whose coordinates are $(tx^1, \cdots\cdots, tx^n)$, $0 \leqslant t \leqslant 1$, in U_α. For each point x_α, we take a fixed curve $\mathscr{C}_\alpha \in \Omega_{x_0, x_\alpha}$.

In $U_\alpha \cap U_\beta \neq \phi$, define $g_{\beta\alpha}: U_\alpha \cap U_\beta \to G$ by

$$g_{\beta\alpha}(x) = \chi_0(\mathscr{C}_\beta \mathscr{C}(x_\beta, x)\mathscr{C}(x_\alpha, x)^{-1}\mathscr{C}_\alpha^{-1}), \qquad x \in U_\alpha \cap U_\beta. \tag{30}$$

By iii), $g_{\beta\alpha}$ is differentiable. For any point $x \in U_\alpha \cap U_\beta \cap U_\gamma$, we get by i), ii)

$$\begin{aligned} g_{\gamma\beta}(x)g_{\beta\alpha}(x) &= \chi_0(\mathscr{C}_\gamma(\mathscr{C}(x_\gamma, x)\mathscr{C}(x_\beta x)^{-1}\mathscr{C}_\beta^{-1}) \\ &\quad \chi_0(\mathscr{C}_\beta \mathscr{C}(x_\beta, x)\mathscr{C}(x_\alpha, x)^{-1}\mathscr{C}_\beta^{-1}) \\ &= \chi_0(\mathscr{C}_\gamma(\mathscr{C}(x_\gamma, x)\mathscr{C}(x_\beta, x)^{-1}\mathscr{C}_\beta^{-1}\mathscr{C}_\beta \mathscr{C}(x_\beta, x)\mathscr{C}(x_\alpha, x)^{-1}\mathscr{C}_\alpha^{-1}) \\ &= \chi_0(\mathscr{C}_\gamma \mathscr{C}(x_\gamma, x)\mathscr{C}(x_\alpha, x)^{-1}\mathscr{C}_\alpha^{-1}) = g_{\gamma\alpha}(x), \end{aligned}$$

that is

$$g_{\gamma\beta}(x)g_{\beta\alpha}(x) = g_{\gamma\alpha}(x).$$

Hence, there exists a fibre bundle $\mathfrak{B} = \{B, p, X, Y, G\}$ with fibre Y, group of bundle G whose coordinate transformations are $g_{\beta\alpha}(x)$ with respect to the covering $\{U_\alpha\}$.[1)]

In the next place, for any curve $\mathscr{D}(x, x') \subset U_\alpha$, $\mathscr{D}(x, x') \in \Omega_{x, x'}$, define g_α by

$$g_\alpha(\mathscr{D}(x, x')) = \chi_0(\mathscr{C}_\alpha \mathscr{C}(x_\alpha, x)\mathscr{D}(x, x')\mathscr{C}(x_\alpha, x')^{-1}\mathscr{C}_\alpha^{-1}) \tag{31}$$

and define $\rho(\mathscr{D}(x, x')): Y_{x'} \to Y_x$ by

$$\rho(\mathscr{D}(x, x')) = \phi_{\alpha, x} g_\alpha(\mathscr{D}(x, x'))p_{\alpha, x'}. \tag{32}$$

If $\mathscr{D}(x, x') \subset U_\alpha \cap U_\beta$, then by (30), (31), i), ii) we get

1) See [2], §3.

$$\begin{aligned}
\phi_{\beta,x} g_\beta(\mathscr{D}(x,x'))p_{\beta,x'} &= \phi_{\alpha,x} g_{\alpha\beta}(x) g_\beta(\mathscr{D}(x,x')) g_{\beta\alpha}(x')p_{\alpha,x'} \\
&= \phi_{\alpha,x}\chi_0(\mathscr{C}_\alpha\mathscr{C}(x_\alpha,x)\mathscr{C}(x_\beta,x)^{-1}\mathscr{C}_\beta^{-1}) \\
&\quad \chi_0(\mathscr{C}_\beta\mathscr{C}(x_\beta,x)\mathscr{D}(x,x')\mathscr{C}(x_\beta,x')^{-1}\mathscr{C}_\beta^{-1}) \\
&\quad \chi_0(\mathscr{C}_\beta\mathscr{C}(x_\beta,x')\mathscr{C}(x_\alpha,x')^{-1}\mathscr{C}_\alpha^{-1})p_{\alpha,x'} \\
&= \phi_{\alpha,x}\chi_0(\mathscr{C}_\alpha\mathscr{C}(x_\alpha,x)\mathscr{D}(x,x')\mathscr{C}(x_\alpha,x')^{-1}\mathscr{C}_\alpha^{-1})p_{\alpha,x'} \\
&= \phi_{\alpha,x} g_\alpha(\mathscr{D}(x,x'))p_{\alpha,x'}.
\end{aligned}$$

This shows that $\rho(\mathscr{D}(x,x'))$ is independent of $U_\alpha \supset \mathscr{D}(x,x')$.

Now, we will show that $\rho(\mathscr{D}(x,x'))$ commutes with right translations of $\mathfrak{B}$.

Let $\bar{\mathfrak{B}} = \{\bar{B}, \bar{p}, X, G, G\}$ be the associated principal fibre bundle of $\mathfrak{B}$ and by means of (32), define $\bar{\rho}(\mathscr{D}(x,x')) : G_{x'} \to G_x$ by

$$\begin{aligned}
(33) \qquad \bar{\rho}(\mathscr{D}(x,x'))(\phi_{\alpha,x'}g) &= \phi_{\alpha,x} g_\alpha(\mathscr{D}(x,x'))p_{\alpha,x'}\phi_{\alpha,x'}g \\
&= \phi_{\alpha,x} g_\alpha(\mathscr{D}(x,x'))g \in G_x.
\end{aligned}$$

This shows that

$$\bar{\rho}(\mathscr{D}(x,x'))\mathrm{r}(g_0) = \mathrm{r}(g_0)\bar{\rho}(\mathscr{D}(x,x')), \qquad g_0 \in G.$$

If $\mathscr{D}(x,x')$ is a differentiable family of curves, then $g_\alpha(\mathscr{D}(x,x'))$ is differentiable with respect to x, x' by iii). Hence, we can obtain an infinitesimal connection Γ in $\mathfrak{B}$ such that the holonomy map ρ with respect to Γ coincides with the transformation as above for $\mathscr{D}(x,x') \subset U_\alpha$.

It follows that for $\mathscr{C} \in \Omega_{x_0}$ such that

$$\mathscr{C} = \mathscr{D}_0\mathscr{D}_1 \cdots\cdots \mathscr{D}_m, \qquad \mathscr{D}_\lambda \subset U_{\alpha_\lambda}, \quad \lambda = 0, 1, \cdots\cdots, m,$$

$$(34) \qquad \rho(\mathscr{C}) = \rho(\mathscr{D}_0)\rho(\mathscr{D}_1)\cdots\cdots\rho(\mathscr{D}_m).$$

Lastly, we will prove $\chi_0 \approx \chi_{x_0}$. For any points $x, x' \in X$, let $\mathscr{D} \in \Omega_{x,x'}$ and

$$\mathscr{D} = \mathscr{D}_1\mathscr{D}_2\cdots\cdots\mathscr{D}_m,$$
$$\mathscr{D}_\alpha \subset U_\alpha, \quad \mathscr{D}_\alpha \in \Omega_{x'_{\alpha-1}x'_\alpha} \quad x = x'_0, \ x' = x'_m.$$

By (32), we get

$$\begin{aligned}
\rho(\mathscr{D}_\alpha) &= \phi_{\alpha,x'_{\alpha-1}} g_\alpha(\mathscr{D}_\alpha)p_{\alpha,x'_\alpha}, \\
\rho(\mathscr{D}_\alpha)\rho(\mathscr{D}_{\alpha+1}) &= \phi_{\alpha,x'_{\alpha-1}} g_\alpha(\mathscr{D}_\alpha) g_{\alpha,\alpha+1}(x'_\alpha) g_{\alpha+1}(\mathscr{D}_{\alpha+1})p_{\alpha+1,x'_{\alpha+1}}
\end{aligned}$$

and

$$\rho(\mathscr{D}_1)\rho(\mathscr{D}_2)\cdots\cdots\rho(\mathscr{D}_m)$$
$$= \phi_{1,x'_0}g_1(\mathscr{D}_1)g_{12}(x'_1)g_2(\mathscr{D}_2)\cdots\cdots g_m(\mathscr{D}_m)p_{m,x'_m}.$$

By i), ii), (30), (31), we get

$$g_1(\mathscr{D}_1)g_{12}(x'_1)g_2(\mathscr{D}_2)g_{23}(x'_2)\cdots\cdots g_{m-1,m}(x'_{m-1})g_m(\mathscr{D}_m)$$
$$= \chi_0(\mathscr{C}_1\mathscr{C}(x_1, x'_0)\mathscr{D}_1\mathscr{C}(x_1, x'_1)^{-1}\mathscr{C}_1^{-1})\chi_0(\mathscr{C}_1\mathscr{C}(x_1, x'_1)\mathscr{C}(x_2, x'_1)^{-1}\mathscr{C}_2^{-1})$$
$$\chi_0(\mathscr{C}_2\mathscr{C}(x_2, x'_1)\mathscr{D}_2\mathscr{C}(x_2, x'_2)^{-1}\mathscr{C}_2^{-1})\chi_0(\mathscr{C}_2\mathscr{C}(x_2, x'_2)\mathscr{C}(x_3, x'_2)^{-1}\mathscr{C}_3^{-1})$$
$$\vdots$$
$$\chi_0(\mathscr{C}_m\mathscr{C}(x_m, x'_{m-1})\mathscr{D}_m\mathscr{C}(x_m, x'_m)\mathscr{C}_m^{-1})$$
$$= \chi_0(\mathscr{C}_1\mathscr{C}(x_1, x'_0)\mathscr{D}_1\mathscr{D}_2\cdots\cdots\mathscr{D}_m\mathscr{C}(x_m, x'_m)^{-1}\mathscr{C}_m^{-1}).$$

Accordingly, we get the relation

$$(35) \qquad \rho(\mathscr{D}_1)\rho(\mathscr{D}_2)\cdots\cdots\rho(\mathscr{D}_m) = \rho(\mathscr{D})$$
$$= \phi_{1,x}\chi_0(\mathscr{C}_1\mathscr{C}(x_1, x)\mathscr{D}\mathscr{C}(x_m, x')^{-1}\mathscr{C}_m^{-1})p_{m,x'}.$$

Especially, if we put $x = x' = x_0$, $x_0 \in U_1$, then

$$\chi_{x_0}(\mathscr{C}) = \phi_{1,x_0}\chi_0(\mathscr{C})p_{1,x_0}, \qquad \mathscr{C} \in \Omega_{x_0},$$

that is

$$\chi_{x_0} \approx \chi_0. \qquad \text{Q.E.D.}$$

§7. **Lemma 3.** *Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a differentiable fibre bundle with an infinitesimal connection Γ whose structure group is G and let H be the holonomy group of Γ at $x_0 \in X$. Then $\mathfrak{B}$ with Γ is G-equivalent to another fibre bundle $\mathfrak{B}' = \{B', p', X, Y, H\}$ with an infinitesimal connection Γ' whose structure group is G.*

Proof. We will use the same notations as before. Using Lemme 2, we can obtain a differentiable fibre bundle $\mathfrak{B}' = \{B', p', X, Y, H\}$ with an infinitesimal connection Γ'' whose structure group is H, and whose holonomy map $\chi''_{x_0} \approx \chi_{x_0}$ of Γ. By means of Lemma 1, $\mathfrak{B}$ and $\mathfrak{B}'$ is G-equivalent as fibre bundles. Let $h: B \to B'$ be the differentiable bundle mapping satisfying the condition $p'h = p$. Then, we can obtain a differentiable family Γ' of tangent subspaces to B' by $\Gamma' = h_*\Gamma$. Since h is a bundle mapping, Γ' define an infinitesimal connection in $\mathfrak{B}'$. For any points $x, x' \in X$ and any curve $\mathscr{C} \in \Omega_{x,x'}$, the mapping $\rho'(\mathscr{C}): Y'_{x'} \to Y'_x$ is clearly given by $\rho'(\mathscr{C}) = h\rho(\mathscr{C})h^{-1}$, where Y'_x denotes the fibre of $\mathfrak{B}'$ at x and ρ' is the map defined for the fibre bundle with the infinitesimal connection Γ' as in $\mathfrak{B}$ (see §1). Thus, $\mathfrak{B}$ with the infinitesimal connection Γ is G-equivalent to $\mathfrak{B}'$ with the infinitesimal connection Γ' whose structure group is G. Q.E.D.

Now, we shall deal with the theorem of E. Cartan stated in Introduction. Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a differentiable fibre bundle with an infinitesimal connection Γ whose structure group is G. Let $\{U_\alpha\}$ be a system of coordinate neighborhoods which is an open covering of X, and let θ_α be the $L(G)$-valued linear differential form in U_α derived from Γ as in §§2-4. For each U_α, let x_α be the origin of the coordinate neighborhood. Then $H_\alpha \equiv H_{x_\alpha} = \overline{k_{\alpha\alpha}(\Omega_{x_\alpha, x_\alpha})}$ is the holonomy group of Γ at x_α. For any curve $\mathscr{C} \in \Omega_{x_\alpha, x_\beta}$, we have by means of (22) the relation

$$H_\alpha = k_{\alpha\beta}(\mathscr{C}) H_\beta k_{\alpha\beta}(\mathscr{C})^{-1}. \tag{36}$$

This shows that H_α are homologous each other. Let K be the minimal invariant subgroup of G which contains H_α. We may suppose that each U_α is a coordinate neighborhood as U in §5. Let $\mathfrak{B}_\alpha$ be the portion of $\mathfrak{B}$ over U_α and Γ_α be the subfamily of Γ on $B \cap p^{-1}(U_\alpha)$, then the holonomy group of Γ_α at x_α is clearly a subgroup of H_α. Hence, by virtue of the consideration in §5, for each U_α, we can obtain a mapping $f_\alpha : U_\alpha \to G$ such that $\hat{\theta}_\alpha = \mathrm{ad}(f_\alpha)\theta_\alpha + (f_\alpha^{-1})^* \omega$ is a $L(H_\alpha)$-valued linear differential form and $f_\alpha(x_\alpha) = e$. If $U_\alpha \cap U_\beta \neq \phi$, we have

$$\hat{\theta}_\beta = \mathrm{ad}(\hat{g}_{\beta\alpha})\hat{\theta}_\alpha + (\hat{g}_{\alpha\beta})^* \omega,$$

where

$$\hat{g}_{\alpha\beta}(x) = f_\alpha(x) g_{\alpha\beta}(x) f_\beta(x)^{-1}, \qquad x \in U_\alpha \cap U_\beta.$$

Now, it may be suppose that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to H_1$ by means of Lemma 3, and that if $U_\alpha \cap U_\beta \neq \phi$, then $U_\alpha \cap U_\beta$ is connected. Then, the above relations imply that $\hat{g}_{\alpha\beta}$ can be written as

$$\hat{g}_{\alpha\beta}(x) = \lambda_{\alpha\beta} h_{\alpha\beta}(x), \qquad h_{\alpha\beta}(x) \in K,\ \lambda_{\alpha\beta} \in G, \quad x \in U_\alpha \cap U_\beta. \tag{37}$$

For each U_α, define a mapping $h_\alpha : U_\alpha \to G$ by $h_\alpha(x) = \tau_\alpha f_\alpha(x)$, where τ_α is a fixed element of G, and define a $L(G)$-valued linear differential from $\tilde{\theta}_\alpha$ by

$$\tilde{\theta}_\alpha = \mathrm{ad}(h_\alpha)\theta_\alpha + (h_\alpha^{-1})^* \omega.$$

Since $h_\alpha^{-1} = \mathrm{r}(\tau_\alpha^{-1}) f_\alpha^{-1}$, we have

$$\begin{aligned}\tilde{\theta}_\alpha &= \mathrm{ad}(\tau_\alpha)\mathrm{ad}(f_\alpha)\theta_\alpha + (f_\alpha^{-1})^* \mathrm{r}(\tau_\alpha^{-1})^* \omega \\ &= \mathrm{ad}(\tau_\alpha)\mathrm{ad}(f_\alpha)\theta_\alpha + (f_\alpha^{-1})^* (\mathrm{ad}(\tau_\alpha)\omega) \\ &= \mathrm{ad}(\tau_\alpha)\{\mathrm{ad}(f_\alpha)\theta_\alpha + (f_\alpha^{-1})^*\omega\} \\ &= \mathrm{ad}(\tau_\alpha)\hat{\theta}_\alpha .\end{aligned}$$

Hence, $\tilde{\theta}_\alpha$ is a $L(K)$-valued linear differential form. By this change of coordinate functions, the coordinate transformation $g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ is replaced by

$$\begin{aligned}\tilde{g}_{\alpha\beta}(x) &= h_\alpha(x) g_{\alpha\beta}(x) h_\beta(x)^{-1} \\ &= \tau_\alpha \hat{g}_{\alpha\beta}(x) \tau_\beta^{-1} \\ &= \tau_\alpha \lambda_{\alpha\beta} h_{\alpha\beta}(x) \tau_\beta^{-1} \\ &= \tau_\alpha \lambda_{\alpha\beta} \tau_\beta^{-1} (\tau_\beta h_{\alpha\beta}(x) \tau_\beta^{-1}).\end{aligned}$$

Accordingly, if we can choose $\{\tau_\alpha\}$ so that

(38) $$\tau_\alpha \lambda_{\alpha\beta} \tau_\beta^{-1} \in K, \qquad \text{as} \quad U_\alpha \cap U_\beta \neq \phi,$$

then $\tilde{g}_{\alpha\beta}$ maps $U_\alpha \cap U_\beta$ into K.

On the other hand, if $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$, we have

$$\begin{aligned} e &= \hat{g}_{\alpha\beta}(x) \hat{g}_{\beta\gamma}(x) \hat{g}_{\gamma\alpha}(x) \\ &= \lambda_{\alpha\beta} h_{\alpha\beta}(x) \lambda_{\beta\gamma} h_{\beta\gamma}(x) \lambda_{\gamma\alpha} h_{\gamma\alpha}(x) \\ &= \lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} \{(\lambda_{\beta\gamma}\lambda_{\gamma\alpha})^{-1} h_{\alpha\beta} \lambda_{\beta\gamma}\lambda_{\gamma\alpha}\} \{\lambda_{\gamma\alpha}^{-1} h_{\beta\gamma}(x) \lambda_{\gamma\alpha}\} h_{\gamma\alpha}(x),\end{aligned}$$

from which we obtain the relation

(39) $$\lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} \in K, \qquad \text{as} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \phi,$$

since K is an invariant subgroup of G.

Since X is differentiable manifold, there exists a differentiable simplicial triangulation of X. Let A_α, $\alpha = 1, 2, \cdots\cdots$, be the vertices of this complex $\mathfrak{K}$ and let U_α be the open set defined by the star of A_α of $\mathfrak{K}$. Then, the system $\{U_\alpha\}$ has all the properties above-mentioned. Thus, the above problem is written as follows:

For each oriented 1-simplex $A_\alpha A_\beta$ of $\mathfrak{K}$, let be given an element $\lambda_{\alpha\beta} \in G$ such that

$$\lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} \in K, \qquad \text{for any 2-simplex} \quad A_\alpha A_\beta A_\gamma \text{ of } \mathfrak{K}.$$

Then, can we choose $\tau_\alpha \in G$, $\alpha = 1, 2, \cdots\cdots$, so that

$$\tau_\alpha \lambda_{\alpha\beta} \tau_\beta^{-1} \in K, \qquad \text{for each} \quad A_\alpha A_\beta \in \mathfrak{R}?$$

If X is simply connected, we can easily prove that there exists a system of $\{\tau_\alpha\}$ satisfying the above conditions. By means of $\{\tilde{\theta}_\alpha\}$, $\{\tilde{g}_{\alpha\beta}\}$, we can obtain a fibre bundle $\tilde{\mathfrak{B}} = \{\tilde{B}, \tilde{p}, X, Y, K\}$ with an infinitesimal connection $\tilde{\Gamma}$ whose structure group is K, the $L(K)$-valued linear differential form on U_α is $\tilde{\theta}_\alpha$ and the coordinate transformations are $\tilde{g}_{\alpha\beta}$. $\tilde{\mathfrak{B}}$ with $\tilde{\Gamma}$ is clearly G-equivalent to $\mathfrak{B}$ with Γ. For the holonomy groups of $\tilde{\Gamma}$, we have by (27)

$$\tilde{H}_{x_\alpha} = h_\alpha(x_\alpha) H_{x_\alpha} h_\alpha(x_\alpha)^{-1} = \tau_\alpha H_{x_\alpha} \tau_\alpha^{-1}.$$

Since we can put $\tau_1 = e$, we have $\tilde{H}_{x_1} = H_{x_1} = H$. Accordingly, by virtue of Lemma 3, $\tilde{\mathfrak{B}}$ with $\tilde{\Gamma}$ is K-equivalent to a fibre bundle $\mathfrak{B}' = \{B', p', X, Y, H\}$ with an infinitesimal connection Γ' whose structure group is K.

Thus, we obtained a following theorem.

Theorem 1. *Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a differentiable fibre bundle with an infinitesimal connection Γ whose structure group is G. Let H be the holonomy group of Γ at a point $x_0 \in X$, and K be the minimal invariant subgroup of G which contains H. Then $\mathfrak{B}$ with Γ is G-equivalent to another fibre bundle $\mathfrak{B}' = \{B', p', X, Y, H\}$ with an infinitesimal connection Γ' whose structure group L, where*

i) *if X is an n-cell, then $L = H$:*

ii) *if X is simply connected, then $L = K$;*

iii) *otherwise, $L = G$.*

From this theorem, we see that the *theorem of E. Cartan on holonomy groups holds good, in the large, at least in the following cases:*

i) *X is an n-cell.*

ii) *X is simply connected and H is an invariant subgroup of G.*

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