ON THE REGULARLY CONVEX HULL OF A SET IN A CONJUGATE BANACH SPACE

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The convex hull of a finite set $(p_1, p_2, \cdots\cdots, p_n)$ in a linear space is the set of all $\sum_{i=1}^{n} \alpha_i p_i$ so that $\alpha_i \geqq 0$ and $\sum \alpha_i = 1$. In this paper we shall observe an analogous representation for the regularly convex hull of a bounded weakly closed set in the conjugate space of a Banach space.

Theorem 1. *Let* $\mathbf{E}$ *be a real Banach space, and* $\mathfrak{X}$ *be a bounded weakly closed sub-set of the conjugate Banach space. Let* $Co(\mathfrak{X})$ *denote the smallest regularly convex set which contains* $\mathfrak{X}$. *Then* $Co(\mathfrak{X})$ *is the set of all* $\int_{\mathfrak{X}} \lambda d\varphi(\lambda)$ *so that* φ *are non-negative Borel measures on* $\mathfrak{X}$ *with* $\varphi(\mathfrak{X}) = 1$, *where* $p = \int_{\mathfrak{X}} \lambda d\varphi(\lambda)$ *are Pettis integrals defined by* $p(A) = \int_{\mathfrak{X}} \lambda(A) d\varphi(\lambda)$.

Proof. Put $A^{\vee}(\lambda) = \lambda(A)$ for every $A \in \mathbf{E}$ and $\lambda \in \mathfrak{X}$. $\mathfrak{X}$ is bounded and contained in a sphere $(|\varphi| \leqq r)$. Then we have $|A^{\vee}(\lambda)| = |\lambda(A)| \leqq r|A|$. And $A \to A^{\vee}$ is a bounded linear transform of $\mathbf{E}$ in the Banach space $\mathbf{R}$ of all real weakly continuous functions on $\mathfrak{X}$. For every linear functional f on $\mathbf{R}$ define the linear functional $f^{\vee}$ on $\mathbf{E}$ by $f^{\vee}(A) = f(A^{\vee})$. Then $f \to f^{\vee}$ is the conjugate transform of $A \to A^{\vee}$, and it is weakly continuous. Let $\mathfrak{P}$ denote the set of all linear functionals p on $\mathbf{R}$ so that $p(I) = 1$ and $p(f) \geqq 0$ for every $0 \leqq f \in \mathbf{R}$. Let $\mathfrak{N}$ denote the set $(\delta_{\lambda}: \lambda \in \mathfrak{X})$, where δ_{λ} denotes the linear functional on $\mathbf{R}$ so that $\delta_{\lambda}(f) = f(\lambda)$ for each fixed point $\lambda \in \mathfrak{X}$. Then clearly $\mathfrak{P} \geqq \mathfrak{N}$. We show that $Co(\mathfrak{N}) = \mathfrak{P}$. Let $f \in \mathbf{R}$, then every $p \in \mathfrak{P}$ satisfies $|p(f)| \leqq |f|$. But from the weak compactness of the set $\mathfrak{X}$, there exists $\lambda \in \mathfrak{X}$ so that $|\delta_{\lambda}(f)| = |f(\lambda)| = \sup_{\mu \in \mathfrak{X}} |f(\mu)| = |f|$. Hence by the theorem of Krein-Smulian, $Co(\mathfrak{N})$ contains $\mathfrak{P}$. Since $\mathfrak{P}$ is weakly closed and convex, it is regularly convex. Then $Co(\mathfrak{N}) = \mathfrak{P}$. Now $p \to p^{\vee}$ mapps $\mathfrak{P}$ on the smallest weakly compact convex set which contains all $(\delta_{\lambda}^{\vee}: \delta_{\lambda} \in \mathfrak{N})$. But every $\delta_{\lambda}^{\vee}$ at $\lambda \in \mathfrak{X}$ coincides with λ. Then the image $(p^{\vee}: p \in \mathfrak{P})$ of $\mathfrak{P}$ coincides with $Co(\mathfrak{X})$.

Now it is well-known that every positive functional p on $\mathbf{R}$ is an indefinite integral $p(f)=\int f\,d\varphi$, where φ is a non-negative Borel measure on X. Then every p^{ν} for $p\in\mathfrak{P}$ satisfies

$$p^{\nu}(A) = p(A^{\nu}) = \int_{\mathfrak{X}} A^{\nu}(\lambda)\,d\varphi(\lambda) = \int_{\mathfrak{X}} \lambda(A)\,d\varphi(\lambda).$$

Hence $p^{\nu}=\int \lambda\,d\varphi(\lambda)$, where $\varphi(\mathfrak{X})=p(1)=1$. This concludes the theorem.

Let $\mathfrak{X}$ be a bounded regularly convex set in the conjugate space of a Banach space. By the theorem of Krein-Milman, $\mathfrak{X}$ is the smallest regularly convex set which contains all the extreme points of $\mathfrak{X}$. Then we obtain

Theorem 2. *Let $\mathfrak{X}$ be a bounded regularly convex set in the conjugate space of a Banach space. Let $\mathfrak{E}$ denote the set of all extreme points of $\mathfrak{X}$, and $\overline{\mathfrak{E}}$ denote its weak closure. Then every $f\in\mathfrak{X}$ is expressible by a Pettis integral*

$$f = \int_{\overline{\mathfrak{E}}} \lambda\,d\varphi(\lambda),$$

where φ is a non-negative Borel measure on $\overline{\mathfrak{E}}$ with $\varphi(\overline{\mathfrak{E}})=1$.

Let $\mathbf{A}$ be a uniformly closed self-adjoint algebra of operators on a Hilbert space which contains the identity I. A linear functional f on $\mathbf{A}$ is said a state if it satisfies $f(A^*)=\overline{f(A)}$ and $f(A^*A)\geqq 0$. Then the set $\mathfrak{P}$ of all states p with $p(I)=1$ is a bounded regularly convex set conjugate to the real Banach space $\mathbf{A}^{H}$ of all Hermitian operators in $\mathbf{A}$[1]. A state p is said irreducible if there is no pair (q,r) of states with $p=q+r$ other than $q=\alpha p$ and $r=(1-\alpha)p$. Now $p\in\mathfrak{P}$ is irreducible if and only if it is an extreme point of $\mathfrak{P}$. Therefore

Theorem 3. *Let $\mathbf{A}$ be a uniformly closed self-adjoint algebra of operators on a Hilbert space which contains the identity I. Let $\mathfrak{N}$ denote the set of all irreducibe states u with $u(I)=1$, and $\overline{\mathfrak{N}}$ denote its weak closure. Then every state p on $\mathbf{A}$ is expressed by a Pettis integral*

$$p = \int_{\overline{\mathfrak{N}}} \lambda\,d\varphi(\lambda),$$

where φ is a suitable non-negative Borel measure on $\overline{\mathfrak{N}}$.

In fact, every state p with $p(I)=1$ is expressed as the Theorem using Theorem 2. But every state p is denoted as αq, where q is a state with $q(I)=1$. Then Theorem 3 is valid.

The last theorem is closely related to my immediately subsequent paper of this Journal.

Footnote.

1). Let p be a state. Given an Hermitian $0 \neq A \in \mathbf{A}$, we put $B = |A|^{\frac{1}{2}} (\sum_0^\infty c_i (A/|A|)^i)$, where $\sum_0^\infty c_i x^i$ is the power-series expansion of $(1-x)^{\frac{1}{2}}$. B converges uniformly, and satisfies $B^* B = |A| I - A$. Then $p(A) \leqq |A| p(I)$. Hence p is a bounded linear functional on $\mathbf{A}^H$ with $|p| = p(I)$. Therefore $\mathfrak{P}$ is a bounded weakly closed convex subset of the conjugate space of $\mathbf{A}^H$.

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