SOME REMARKS ON RADICAL IDEALS

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Let R be a (non-commutative) ring. As is well known, every ideal A in R decides a uniquely determined ideal $\bar{A}$ called the *radical* of A, which is defined as the intersection of all (minimal) prime divisors of A. Clearly, the operation: $A \rightarrow \bar{A}$ defined in the set $\mathfrak{J}$ consisting of all ideals in R possesses the following properties:

1) $A \subseteq \bar{A}$,

2) $\bar{A} = \bar{\bar{A}}$,

3) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, where $A, B \in \mathfrak{J}$.

In this note, we consider several properties of ideals A with $\bar{A} = A$, which are called *radical ideals* by N. H. McCoy [3][1].

Theorem 1. *Let C be an ideal in R. Then the following conditions are equivalent to each other:*

a) *C is a radical ideal.*

b) *If an ideal A is nilpotent modulo C, then A is contained in C.*

c) *$AB \subseteq C$ implies $A \cap B \subseteq C$, where A, B are ideals.*

d) *C is an intersection of some prime ideals.*

Proof. Cor. 4 to Th. 2 of [2] shows that the intersection of all the prime ideals of a ring is $\{0\}$ if and only if the zero ideal is the only nilpotent ideal of the ring. Hence, the equivalence between a) and b) may be easily seen.

Let C be a radical ideal. Then, for each prime divisor P of C, $AB \subseteq C$ implies $P \supseteq A$ or B. Hence $A \cap B \subseteq P$, accordingly, $A \cap B \subseteq C$. This shows that a) implies c). Conversely, C be not a radical ideal. Then, by b), there exists an ideal $A \nsubseteq C$ such that $A^2 \subseteq C$. Thus, c) does not hold.

The equivalence of d) to a) is also easy.

In general, let $A \rightarrow A^*$ be a operation defined in $\mathfrak{J}$ which satisfies the following axioms:

1′) $A \subseteq A^*$,

2′) $(A^*)^* = A^*$,

3′) $B \subseteq A$ implies $B^* \subseteq A^*$, where A, B are in $\mathfrak{J}$.

For arbitrary ideals A, B in R we define the join $A \cup B$ as the ideal

1) Numbers in brackets refer to the references cited at the end of this note. The term " ideal " will maen a two-sided ideal.

$(A+B)^*$, and the meet $A\cap B$ as the intersection of A and B. We set $\mathfrak{C}=\{A\in\mathfrak{J}\mid A^*=A\}$.

Lemma. *$\mathfrak{C}$ forms a distributive lattice with respect to the above-defined join and meet if one of the following (trivially equivalent) conditions is satisfied:*

a) *For any $C\in\mathfrak{C}$, $(AB)^*\subseteq C$ implies $A^*\cap B^*\subseteq C$, where $A, B\in\mathfrak{J}$.*

b) *$(AB)^*=A^*\cap B^*$, where $A, B\ni\mathfrak{J}$.*

Proof. As our operation $*$ satisfies the axioms 1′), 2′) and 3′) we have $(A^*+B^*)^*=(A+B)^*$. Let A, B and C be in $\mathfrak{C}$, then $(A\cap B)\cup(A\cap C)=((A\cap B)+(A\cap C))^*=((AB)^*+(AC)^*)^*=(AB+AC)^*=(A(B+C))^*=A\cap(B+C)^*=A\cap(B\cup C)$.

Theorem 2[1]. *The set consisting of all radical ideals in a ring R forms a distributive lattice, where the join and meet are defined as in the above lemma.*

Proof. Let C be a radical ideal. Then $\overline{(AB)}\subseteq C$ implies $AB\subseteq C$. For any prime divisor P of C, $AB\subseteq C$ implies A or $B\subseteq P$, accordingly $\bar{A}$ or $\bar{B}\subseteq P$. Hence, we have $\bar{A}\cap\bar{B}\subseteq P$, that is, $\bar{A}\cap\bar{B}\subseteq C$. This fact shows that our operation: $A\to\bar{A}$ satisfies the condition a) of the lemma.

Next, we prove the following theorem which has been proved in the commutative case by S. Mori [4, Satz 1]:

Theorem 3. *The maximum condition is satisfied for radical ideals in R if and only if the following conditions are satisfied:*

1) *The maximum condition is satisfied for prime ideals in R.*

2) *Every radical ideal is represented as the intersection of a finite number of prime ideals in R.*

Proof. Necessity: As 1) is a special case of our assumption it is desired only to prove 2).

Let $C=\cap P_\sigma$ be a radical ideal, where P_σ is a minimal prime divisor of C. Here, we may assume that C is not a prime ideal. Then, there exist two ideals A, B not contained in C such that $AB\subseteq C$. We can easily see that $R\supset C_1=CB^{-1}$[2] $=\cap(P_\sigma B^{-1})=\cap P_{\sigma'}\supset C$, where $\{P_{\sigma'}\}$ is a subset of $\{P_\sigma\}$. Clearly C_1 is a radical ideal.

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1) Theorem 2 has been proved for commutative rings by J-C. Herz [1].

2) $CB^{-1}[B^{-1}C]$ is defined as the totality of elements x such that $xB\subseteq C[Bx\subseteq C]$. The following properties of quotients are easily verified (see [5]):

(1) $(AB^{-1})C^{-1}=A(CB)^{-1}$.

(2) $(\cap A_\sigma)B^{-1}=\cap A_\sigma B^{-1}$.

(3) For any prime ideal P, $PA^{-1}=P$ or R itself in accordance with $A\nsubseteq P$ or $A\subseteq P$ respectively.

If C_1 is not prime, then we repeat the above procedure for C_1 instead of C and obtain $C_2 = CB_1^{-1}$, and so on. From our assumption, our procedures must terminate after a finite number of steps. Hence we have a prime ideal $(R \supset) C_n = CB_{n-1}^{-1}$, where B_{n-1} is not contained in C_n as C is a radical ideal. We now consider all different prime ideals $P'_\mu (\subset R)$ of the form: $P'_\mu = CU'^{-1}_\mu$. As C is a radical ideal U'_μ is not contained in P'_μ, and P'_μ is a minimal prime divisor of C. If $\mu \neq \lambda$, then $U'_\mu \subseteq P'_\lambda$. For, if not, $U'_\mu \not\subseteq P'_\lambda$ and $P'_\mu U'_\mu \subseteq C \subseteq P'_\lambda$ imply that $P'_\mu \subseteq P'_\lambda$. As P'_λ is a minimal prime divisor of C, $P'_\mu = P'_\lambda$, contradicting with our assumption. The fact proved now shows that the representation $D_1 = \cap P'_\mu$ is irredundant. The maximum condition for radical ideals implies that the set $\{P'_\mu\}$ is finite.

By the same manner as in the above, we can see that the set of all different prime ideals $P''_\nu = U''^{-1}_\nu C$ with $U''_\nu \not\subseteq P''_\nu$ is finite, and we set $D_2 = \cap P''_\nu$.

Clearly, $D = D_1 \cap D_2$ contains C. We shall prove now that $D = C$. If $D \supset C$, as $U''_\nu D \subseteq C$ and $U''_\nu \not\subseteq C$, we can construct a prime ideal $(R \supset) P' = CU'^{-1}$ with $U' \not\subseteq P'$, $\subseteq D$ by using the same argument as in the first part of the proof. Hence, for some μ, $P' = P'_\mu$. On the other hand, $U' \subseteq D$ implies that $U' \subseteq P'_\mu = P'$. This is a contradiction.

Sufficiency: We assume now the conditions 1) and 2). Let $C_1 \subset C_2 \subset \cdots\cdots$ be an infinite ascending chain of radical ideals, where C_i has a short representation $P_{i1} \cap \cdots\cdots \cap P_{in_i}$ with its minimal prime divisors P_{ij}[1]. Then each P_{ij} $(j = 1, \cdots\cdots, n_i)$ is a divisor of some $P_{i-1,k}$ $(k = 1, \cdots\cdots, n_{i-1})$. Now, we call an ascending chain $(R \neq) Q_1 \subseteq Q_2 \subseteq \cdots\cdots$ of which each Q_i is some P_{ij} or R itself a branch of the chain $C_1 \subset C_2 \subset \cdots\cdots$, and Q_1 the starting point of the branch. In case $Q_1 \subseteq Q_2 \subseteq \cdots\cdots \subseteq Q_n \subset Q_{n+1} = R$ we say that the length of the branch is n. In the other case, it is infinite. A prime divisor P_{ij} of C_i is called trivial if P_{ij} is some P_{hk} for each $h > i$. If the lengths of all branches with the starting point P_{ij} of the sub-chain $C_i \subset C_{i+1} \subset \cdots\cdots$ are bounded, we say that P_{ij} is finite, and in the other case it is infinite.

Here, without loss of generality, we may assume that each P_{ij} is non-trivial. As $C_1 \subset C_2 \subset \cdots\cdots$ is infinite, there exists at least one infinite P_{rs} for each r.

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1) If an ideal is represented as the intersection of a finite number of prime ideals, then there exists a unique short representation [5, Theorem 3].

Let P_{1l_1} be infinite. Then there exists a branch $Q_1 = \cdots\cdots = Q_{m_1-1} \subset Q_{m_1} \subseteq \cdots\cdots$ with the starting point P_{1l_1} such that $Q_{m_1} \neq R$. As $P_{m_1,l}$ is a minimal prime divisor of C_{m_1}, for each branch $Q'_1 \subseteq Q'_2 \subseteq \cdots\cdots$ with the starting point $Q'_1 = P_{1l_1}$, we have $Q'_1 \subset Q'_{m_1}$. Hence, there exists at least one infinite P_{m_1,l_2} properly containing P_{1,l_1}. We can repeat the above argument for P_{m_1,l_2} instead of P_{1l_1} and obtain an infinite P_{m_2,l_3} properly containing P_{m_1,l_2}, and so on. Thus, we obtain an infinite ascending chain of prime ideals $P_{1l_1} \subset P_{m_1,l_2} \subset P_{m_2,l_3} \subset \cdots\cdots$. But this is a contradiction.

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