SUPPLEMENT TO MY PREVIOUS PAPER "ON PRIMARY IDEAL DECOMPOSITIONS IN NON-COMMUTATIVE RINGS"

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In his previous paper [3][1], the author proved the following theorem; Every ideal in a ring $\mathfrak{R}$ is represented as the intersection of a finite number of s-primary ideals if and only if the following conditions are satisfied:

(A) The radical of any ideal $\mathfrak{a}$ is nilpotent modulo $\mathfrak{a}$.

(B) For any ideals $\mathfrak{a}$, $\mathfrak{b}$, there exists the limit ideal of $\mathfrak{a}$ by $\mathfrak{b}$, and there exists a finite number $n(\mathfrak{a})$ of ideals which, starting from $\mathfrak{a}$, are obtained by repeating the procedures to make limit ideals successively.

(C) Each minimal prime divisor of any ideal $\mathfrak{a}\subset\mathfrak{R}$ is non-prime to $\mathfrak{a}$.

(D) If $\mathfrak{p}$ is an arbitrary prime ideal associated with an ideal $\mathfrak{a}$, there exists an s-primary ideal $\mathfrak{q}\supseteq\mathfrak{a}$ belonging to $\mathfrak{p}$ such that, for any ideal $\mathfrak{b}\subseteq\mathfrak{q}$, $\nsubseteq\mathfrak{a}$, $\mathfrak{a}\mathfrak{b}^{-1}$ is no primary ideal belonging to $\mathfrak{p}$.

Recently, he found that *the condition* (A) *was derived from* (B) *and* (C), and that the condition (B) might be restated as follows:

(B′) For any ideals $\mathfrak{a}$, $\mathfrak{b}$, there exists the limit ideal of $\mathfrak{a}$ by $\mathfrak{b}$, and the number of all the limit ideals of $\mathfrak{a}$ is finite.

In this note, we prove first these facts (1°), and next we consider a necessary condition which is similar to (B) (2°).

1°. We call an ideal $\mathfrak{c}$ a *component* of $\mathfrak{a}$ if there exists an ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b}^{-1}=\mathfrak{a}\mathfrak{b}^{-2}=\cdots\cdots=\mathfrak{b}^{-2}\mathfrak{a}=\mathfrak{b}^{-1}\mathfrak{a}=\mathfrak{c}$. Clearly, the limit ideal of $\mathfrak{a}$ by $\mathfrak{b}$, if it exists, is a component of $\mathfrak{a}$, and conversely, every component of $\mathfrak{a}$ is the limit ideal of $\mathfrak{a}$ by some ideal. Moreover, in case $\mathfrak{a}$ is represented as the intersection of a finite number of s-primary ideals, a component of $\mathfrak{a}$ is an isolated component of $\mathfrak{a}$ or $\mathfrak{R}$ itself, *vice versa*.

Now we set the next lemma:

Lemma 1. *Let* $\mathfrak{a}$ *be an ideal of which the limit ideal by any ideal exists. If* $\mathfrak{a}_1$ *is a component of* $\mathfrak{a}$ *and* $\mathfrak{a}_2$ *is a component of* $\mathfrak{a}_1$, *then* $\mathfrak{a}_2$ *is a component of* $\mathfrak{a}$.

1) Numbers in brackets refer to the references cited at the end of this note.

2) For the definitions of s-primary ideals, limit ideals etc., see [3, pp. 39-40].

Proof. Since $\mathfrak{a}_1$, $\mathfrak{a}_2$ are components of $\mathfrak{a}$, $\mathfrak{a}_1$ respectively there exist ideals $\mathfrak{b}$, $\mathfrak{c}$ such that

$$\mathfrak{a}_1 = \mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{a}\mathfrak{b}^{-2} = \cdots\cdots = \mathfrak{b}^{-2}\mathfrak{a} = \mathfrak{b}^{-1}\mathfrak{a}$$

and

$$\mathfrak{a}_2 = \mathfrak{a}_1\mathfrak{c}^{-1} = \mathfrak{a}_1\mathfrak{c}^{-2} = \cdots\cdots = \mathfrak{c}^{-2}\mathfrak{a}_1 = \mathfrak{c}^{-1}\mathfrak{a}_1.$$

Now, we can take an integer k such that $\mathfrak{a}\mathfrak{c}^{-k} = \mathfrak{c}^{-k}\mathfrak{a}$ is the limit ideal of $\mathfrak{a}$ by $\mathfrak{c}$. Then

$$\mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-1} = (\mathfrak{a}\mathfrak{b}^{-1})\mathfrak{c}^{-k} = \mathfrak{a}_1\mathfrak{c}^{-k} = \mathfrak{a}_2,$$

and

$$\mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-2} = (\mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-1})(\mathfrak{c}^k\mathfrak{b})^{-1} = (\mathfrak{a}_1\mathfrak{c}^{-k})(\mathfrak{c}^k\mathfrak{b})^{-1} = \mathfrak{c}^{-k}\mathfrak{a}_1\mathfrak{b}^{-1}\mathfrak{c}^{-k}$$
$$= \mathfrak{c}^{-k}\mathfrak{a}_1\mathfrak{c}^{-k} = \mathfrak{c}^{-k}\mathfrak{a}_2 = \mathfrak{a}_2.$$

On the other hand,

$$(\mathfrak{c}^k\mathfrak{b})^{-1}\mathfrak{a} = \mathfrak{b}^{-1}(\mathfrak{c}^{-k}\mathfrak{a}) = \mathfrak{b}^{-1}(\mathfrak{a}\mathfrak{c}^{-k}) = (\mathfrak{b}^{-1}\mathfrak{a})\mathfrak{c}^{-k} = \mathfrak{a}_1\mathfrak{c}^{-k}$$
$$= \mathfrak{a}_2 = \mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-1}.$$

Hence, we obtain

$$\mathfrak{a}_2 = \mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-1} = \mathfrak{a}(\mathfrak{c}^k\mathfrak{b})^{-2} = \cdots\cdots = (\mathfrak{c}^k\mathfrak{b})^{-2}\mathfrak{a} = (\mathfrak{c}^k\mathfrak{b})^{-1}\mathfrak{a},$$

i.e. $\mathfrak{a}_2$ is a component of $\mathfrak{a}$.

By this lemma, we can easily see that *the condition* (B) *is equivalent to* (B′).

The fact that (A) depends on (B) and (C) is proved as the next theorem:

Theorem 1. *If, for any ideal* $\mathfrak{a}$ *in* $\mathfrak{R}$, *there exists the right limit ideal of* $\mathfrak{a}$ *by its radical, then the condition* (C) *implies the condition* (A).

Proof. Let $\mathfrak{p}$ be a minimal prime divisor of any ideal $\mathfrak{a}\subset\mathfrak{R}$. Then, by (C), $\mathfrak{a}\subset\mathfrak{a}\mathfrak{p}^{-1}\subseteq\mathfrak{a}\bar{\mathfrak{a}}^{-1}$, where $\bar{\mathfrak{a}}$ is the radical of $\mathfrak{a}$. If $\mathfrak{a}\bar{\mathfrak{a}}^{-1}$ is not $\mathfrak{R}$ itself, then we have

$$\mathfrak{a}\bar{\mathfrak{a}}^{-1} \subset \mathfrak{a}\bar{\mathfrak{a}}^{-1}(\overline{\mathfrak{a}\bar{\mathfrak{a}}})^{-1} \subseteq \mathfrak{a}\bar{\mathfrak{a}}^{-2}.$$

Continuing the above argument, we reach to the right limit ideal $\mathfrak{a}\bar{\mathfrak{a}}^{-k}$ of $\mathfrak{a}$ by $\bar{\mathfrak{a}}$. To be easily verified, $\mathfrak{a}\bar{\mathfrak{a}}^{-k} = \mathfrak{R}$, that is, $\bar{\mathfrak{a}}^{k+1}\subseteq\mathfrak{a}$.

2°. We say that an ideal $\mathfrak{a}$ satisfies the *chain condition for right quotients* (abr. CCRQ) if each ascending chain

$$(*) \qquad \mathfrak{a} \subset \mathfrak{a}\mathfrak{b}_1^{-1} \subset \cdots\cdots \subset \mathfrak{a}\mathfrak{b}_1^{-1} \cdots\cdots \mathfrak{b}_k^{-1} \subset \cdots\cdots,$$

where $\mathfrak{b}$'s are ideals in a ring $\mathfrak{R}$, terminates after a finite number of quotients. If, moreover, there exists a positive integer $N(\mathfrak{a})$ such that, for each choice of the ideals $\mathfrak{b}$'s, the chain $(*)$ does not contain more thae $N(\mathfrak{a})$ different terms, then we say that $\mathfrak{a}$ satisfies the *strong chain condition for right quotients* (abr. SCCRQ). In this case, we denote by $L(\mathfrak{a})$ (the *length* of $\mathfrak{a}$) the lower limit of $N(\mathfrak{a})$'s.

Lemma 2. *If* $\mathfrak{a}\mathfrak{b}^{-1} \subseteq \mathfrak{a}\mathfrak{c}^{-1}$ *and* $\mathfrak{a}\mathfrak{c}^{-1} = \mathfrak{a}\mathfrak{c}^{-2}$, *then* $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^{-1} = \mathfrak{a}\mathfrak{c}^{-1}$.

Proof. $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^{-1} = \mathfrak{a}(\mathfrak{c}\mathfrak{b})^{-1} \supseteq \mathfrak{a}\mathfrak{c}^{-1}$. On the other hand, as $\mathfrak{a}\mathfrak{b}^{-1} \subseteq \mathfrak{a}\mathfrak{c}^{-1}$, $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^{-1} \subseteq \mathfrak{a}\mathfrak{c}^{-2} = \mathfrak{a}\mathfrak{c}^{-1}$. Hence, $\mathfrak{a}\mathfrak{c}^{-1} = \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}^{-1}$.

By Lemma 2, CCRQ for an ideal $\mathfrak{a}$ implies the ascending chain condition for components of $\mathfrak{a}$, and it secures the existence of the right limit ideal of $\mathfrak{a}$ by any ideal. Moreover, SCCRQ for $\mathfrak{a}$ implies the descending chain condition for components of $\mathfrak{a}$.

Theorem 2. *Every ideal in* $\mathfrak{R}$ *is represented as the intersection of a finite number of s-primary ideals if and only if the following condition* (B″) *is satisfied besides* (C) *and* (D):

(B″) *For any ideal* $\mathfrak{a}$, SCCRQ *is satisfied and, for any ideals* $\mathfrak{a}$, $\mathfrak{b}$, *there exists the left limit ideal of* $\mathfrak{a}$ *by* $\mathfrak{b}$, *which coincides with the right one.*

Proof. The necessity is proved as below. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$ be a short representation of $\mathfrak{a}$. If $\mathfrak{a} \subset \mathfrak{a}\mathfrak{b}^{-1}$, then, for some i, $\mathfrak{b} \subseteq \bar{\mathfrak{q}}_i$, where $\bar{\mathfrak{q}}_i$ denotes the radical of $\mathfrak{q}_i$. (Notice that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{q}_1\mathfrak{b}^{-1} \cap \cdots\cdots \cap \mathfrak{q}_n\mathfrak{b}^{-1}$). If $\mathfrak{b} \not\subseteq \mathfrak{q}_i$, then, by Lemma 2 of [3] and the definition of primary ideals, $\mathfrak{q}_i\mathfrak{b}^{-1}$ is a primary ideal belonging to $\bar{\mathfrak{q}}_i$. Let k_s be the nilpotency index of $\bar{\mathfrak{q}}_s$ modulo $\mathfrak{q}_s$ $(s = 1, \cdots\cdots, n)$ and let

$$(*) \qquad \mathfrak{a} \subset \mathfrak{a}\mathfrak{b}_1^{-1} \subset \cdots\cdots \subset \mathfrak{a}\mathfrak{b}_1^{-1} \cdots\cdots \mathfrak{b}_k^{-1} \subset \cdots\cdots$$

be an ascending chain of right quotients. Then, for some j, k_j of $\mathfrak{b}$'s in $\mathfrak{b}_1, \cdots\cdots, \mathfrak{b}_{k_1 + \cdots\cdots + k_n}$ are contained in $\bar{\mathfrak{q}}_j$. This shows that the chain $(*)$ does not contain more than $n\,(k_1 + \cdots\cdots + k_n) + 1$ different terms.

As is easily seen, Lemma 3, Lemma 4 and Theorem 8 of [3] are also true under the conditions (B″), (C) and (D). This remark shows the sufficiency of our conditions.

Remark. For commutative case, SCCRQ for all ideals is the necessary and sufficient condition that every ideal is represented as the intersection of a finite number of s-primary ideals[1]. But, for non-commutative case, as is well known, it is not yet sufficient.

As our condition (D) is much complicated, it is desirable to replace the condition by some simpler one.

We wish to conclude this note by giving the following theorem:

Theorem 3. *Every ideal in a ring* $\mathfrak{R}$ *is represented as the intersection of a finite number of s-primary ideals if and only if the following condition is satisfied besides* (C) *and* (B″):

(E) *Every non-right primary ideal* $\mathfrak{n}$ *is represented as the intersection of some proper component of* $\mathfrak{n}$ *with some proper divisor of* $\mathfrak{n}$.

Proof. The necessity is easy and the sufficiency is proved by using the same method as in the proof by M. Rabin [2].

Remark. The condition (E) is satisfied, for example, if $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a} = \mathfrak{c}\mathfrak{b}$ for some $\mathfrak{c}$ and the condition (B″) is assumed. Needless to say, in case $\mathfrak{R}$ satisfies the ascending chain condition for ideals, the condition (E) is sufficient too.

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(*Received January 20, 1954*)

1) See, Theorem 9 of [1] or [2, p. 545].