NOTES ON BASIC RINGS, II

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Let R be a ring with unit element. In §1 we study the R-endomorphism ring of an R-module. In §2 we consider an algebra with a finite rank over a field K and discuss the connection between an algebra and its basic algebra.

1. Let V be a left R-module. We denote by $E_V(R)$ or simply by $E(R)$ the R-endomorphism ring of V. We consider $E(R)$ as right operator system of V. Evidently $E(R)$ has the unit element $1'$ and V is faithful as a right $E(R)$-module. If $a \in R$, $a' \in E(R)$, $v \in V$ then

$$a(va') = (av)a'. \tag{1}$$

Let $V = V_1 + V_2$ denote the decomposition of V into a direct sum of left R-modules V_1 and V_2. If $v = v_1 + v_2$, $(v_1 \in V_1, v_2 \in V_2)$ then the mapping e' defined by $ve' = v_1$, is an R-endomorphism of V. We see that e' is idempotent and $V_1 = Ve'$, $V_2 = V(1' - e')$:

$$V = Ve' + V(1' - e'). \tag{2}$$

Lemma 1. *An idempotent element e' of $E(R)$ is primitive if and only if Ve' is indecomposable as a left R-module.*

Lemma 2. *Let e' and f' be two idempotent elements of $E(R)$. If $e' \cong f'$ then $Ve' \cong Vf'$, and conversely.*

Proof. If $e' \cong f'$ there exist two elements a', b' such that $e' = a'b'$ and $f' = b'a'$ $(a' \in e'E(R)f', b' \in f'E(R)e')$. Hence $Va' = Vf'$. Then $Ve' \cong Vf'$ under the correspondence $ve' \to va'$. Conversely suppose $Ve' \cong Vf'$ under the correspondence $ve' \to (ve')^\sigma \in Vf'$. There exist two elements a', b' of $E(R)$ such that

$$(ve')a' = (ve')^\sigma, \qquad (v(1' - e'))a' = 0,$$
$$(vf')b' = (vf')^{\sigma^{-1}}, \qquad (v(1' - f'))b' = 0.$$

We then see that $e' = a'b'$ and $f' = b'a'$.

Lemma 3. $E_{Ve'}(R) = e'E(R)e'$ *for any idempotent element e' of* $E(R)$.

In what follows we assume always that a left R-module V satisfies the following condition:

(**) V is decomposed into a direct sum of a finite number of indecomposable R-modules and this decomposition is unique up to an R-isomorphism.

Let

$$(3) \qquad V = \sum_{\kappa=1}^{m} \sum_{\mu=1}^{\varphi(\kappa)} V_{\kappa,\mu}$$

denote the decomposition of V into a direct sum of indecomposable R-modules $V_{\kappa,\mu}$, where

$$(4) \qquad V_{\kappa,1} \cong V_{\kappa,2} \cong \cdots\cdots \cong V_{\kappa,\varphi(\kappa)}$$

and $V_{\kappa,\mu} \not\cong V_{\lambda,\nu}$ for $\kappa \neq \lambda$. Corresponding to the decomposition (3) we have the decomposition of the unit element $1'$ of $E(R)$ into mutually orthogonal primitive idempotent elements:

$$(5) \qquad 1' = \sum_{\kappa} \sum_{\mu} e'_{\kappa,\mu}.$$

Here $V_{\kappa,\mu} = Ve'_{\kappa,\mu}$ and

$$(6) \qquad V = \sum_{\kappa} \sum_{\mu} Ve'_{\kappa,\mu}.$$

It follows from Lemma 2 that if V satisfies the condition (**) then $E(R)$ satisfies the condition (*) [4, p. 106], and conversely.

We set $e'_{\kappa,1} = e'_{\kappa}$, $e' = \sum_{\kappa} e'_{\kappa}$. $e'E(R)e' = (E(R))^0$ is a basic ring of $E(R)$ and if $\{c'_{\kappa,\mu\nu}\}$ is a system of matrix units corresponding to the decomposition (5) then

$$(7) \qquad E(R) = \sum_{\kappa,\lambda} \sum_{\mu,\nu} c'_{\kappa,\mu 1}(E(R))^0 c'_{\lambda,1\nu}.$$

The left R-module Ve' is called the reduced module of V. We have by Lemma 3 the following

Lemma 4. $E_{Ve'}(R) = (E(R))^0$.

We assume that V is a faithful left R-module. Denote by $E(E(R))$ the $E(R)$-endomorphism ring of V (considered as a right $E(R)$-module). If $a' \in E(R)$, $a'' \in E(E(R))$, $v \in V$ then

$$(8) \qquad (a''v)a' = a''(va').$$

Since V is a faithful left R-module it follows from (1) that

$$(9) \qquad R \subseteq E(E(R)).$$

Further we see easily that

$$E(R) = E(E(E(R))). \tag{10}$$

We assume hereafter that R satisfies the condition $(*)$ and $e_{i,\alpha}$, e_i, $c_{i,\alpha\beta}$, e, $R^0 = eRe$ have the same meaning as in [4]. Moreover we assume that $1v = v$ for any element v in V. We then have $AV = V$ and

$$V = \sum_{i,\alpha} e_{i,\alpha}V = \sum_{i,\alpha} c_{i,\alpha 1}V, \tag{11}$$

where $e_iV \cong c_{i,\alpha 1}V$ and $c_{i,\alpha 1}V \not\cong c_{j,\beta 1}V$ for $i \neq j$.

Lemma 5. *If $E(E(R)) = R$ then the right $E(R)$-modules e_iV ($i = 1, 2, \cdots\cdots, n$) are indecomposable.*

Lemma 6. $E_{eV}(R^0) = E(R)$.

Proof. The mapping a^* defined by $(ev)a^* = (ev)a'$ $(a' \in E(R))$ is an R^0-endomorphism of eV since

$$eae((ev)a^*) = eae((ev)a') = (eaev)a' = (eaev)a^*.$$

We have

$$(e_iv)a' = e_i(va') \in e_iV,$$
$$(c_{i,\alpha 1}v)a' = c_{i,\alpha 1}(va') \in c_{i,\alpha 1}V.$$

Hence, if $a' \neq b'$ there exists at least an element $v \in V$ such that $(ev)a' \neq (ev)b'$. This implies $a^* \neq b^*$. Conversely let $a^* \in E_{eV}(R^0)$. We set $(e_iv)a' = (e_iv)a^*$, $(c_{i,\alpha 1}v)a' = c_{i,\alpha 1}((e_iv)a^*)$. We then have

$$\begin{aligned} c_{j,\beta 1}ac_{i,1\alpha}((c_{i,\alpha 1}v)a') &= c_{j,\beta 1}ac_{i,1\alpha}(c_{i,\alpha 1}(e_iv)a^*) \\ &= c_{j,\beta 1}(e_jae_i)((e_iv)a^*) = c_{j,\beta 1}((e_jae_iv)a^*) \\ &= (c_{j,\beta 1}ae_iv)a' = (c_{j,\beta 1}ac_{i,1\alpha}c_{i,\alpha 1}v)a'. \end{aligned}$$

Similarly

$$\begin{aligned} c_{j,\beta 1}ac_{i,1\gamma}((c_{i,\alpha 1}v)a') &= (c_{j,\beta 1}ac_{i,1\gamma}(c_{i,\alpha 1}v))a' \\ &= 0 \qquad (\gamma \neq \alpha), \\ c_{j,\beta 1}ac_{k,1\gamma}((c_{i,\alpha 1}v)a') &= (c_{j,\beta 1}ac_{k,1\gamma}(c_{i,\alpha 1}v))a' \\ &= 0 \qquad (k \neq i). \end{aligned}$$

Hence $a' \in E(R)$ and $E_{eV}(R^0) = E(R)$.

Lemmas 4 and 6 are also valid for a right R-module V:

$$E_{e'V}(R) = (E(R))^0, \tag{12}$$

(13) $$E_{V_0}(R^0) = E(R).$$

Theorem 1. $E_{Ve'}(E_{Ve'}(R)) = E(E(R))$.

Proof. Applying (13) to the right $E(R)$-module V, we have $E_{Ve'}((E(R))^0) = E(E(R))$. Lemma 4 yields $E_{Ve'}((E(R))^0) = E_{Ve'}(E_{Ve'}(R))$.

Corollary. *If $E_{Ve'}(E_{Ve'}(R)) = R$ then $E(E(R)) = R$, and conversely.*

Lemma 7. *If $E_{eV}(E_{eV}(R^0)) = R^0$ then $E(E(R)) = R$, and conversely.*

Proof. By Lemmas 6 and 3

$$E_{eV}(E(R)) = E_{eV}(E_{eV}(R^0)),$$
$$eE(E(R))e = E_{eV}(E(R)),$$

whence $eE(E(R))e = E_{eV}(E_{eV}(R^0))$. Suppose $E_{eV}(E_{eV}(R^0)) = R^0$. We then have $eE(E(R))e = R^0$ and

$$E(E(R)) = \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}(eE(E(R))e)c_{j,1\beta}$$
$$= \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}R^0 c_{j,1\beta} = R.$$

Conversely if $E(E(R)) = R$ then $eRe = R^0 = E_{eV}(E_{eV}(R^0))$.

Theorem 2. $E_{V_0}(R^0) = (E(R))^0$, *where* $V_0 = eVe'$.

Proof. Applying Lemma 4 to the left R^0-module eV, we have $E_{V_0}(R^0) = (E_{eV}(R^0))^0$, whence $E_{V_0}(R^0) = (E(R))^0$ by Lemma 6.

Theorem 3. *If $E_{V_0}(E_{V_0}(R^0)) = R^0$ then $E(E(R)) = R$, and conversely.*

Proof. Suppose $E_{V_0}(E_{V_0}(R^0)) = R^0$. By Corollary to Theorem 1, $E_{eV}(E_{eV}(R^0)) = R^0$ and hence $E(E(R)) = R$ by Lemma 7. The converse may be proved easily.

As is well known, if $V \cong R$ (considered as a left R-module) then

(14) $$E_V(E_V(R)) = R.$$

By Corollary to Theorem 1, we obtain

(15) $$E_{Re}(E_{Re}(R)) = R.$$

Generally we have the following

Theorem 4. *Let V be a direct sum of any left R-module V_1 and $V_2 = Re$. Then $E_V(E_V(R)) = R$.*

Proof. $eV = eV_1 + eRe = eV_1 + R^0$. We then have $E_{eV}(E_{eV}(R^0)) = R^0$ by Theorem II - E [3]. Hence $E_V(E_V(R)) = R$ by Lemma 7.

2. We consider always an algebra with a finite rank over a field K and with unit element. Denote by A^0 a basic algebra of an algebra A. Then

$$A = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} A^0 c_{j,1\beta}.$$

Let B be a second algebra over the same field K. A and B are called similar if $A^0 \cong B^0$. We write then $A \sim B$. This is a reflexive, symmetric, and transitive relation, by means of which algebras over K are classified into disjoint classes.

Lemma 8. $A \times B \sim A^0 \times B^0$.

Proof. Let $1' = \sum_{j,\beta} e'_{j,\beta}$ denote the decomposition of the unit element $1'$ of B into mutually orthogonal primitive idempotent elements. We set as usual $e'_{j,1} = e'_j$, $\sum e'_j = e'$. Then $B^0 = e'Be'$. The idempotent elements $e_{i,\alpha} \times e'_{j,\beta}$ of $A \times B$ are not necessarily primitive. Since $e_{i,\alpha} \times e'_{j,\beta} \cong e_i \times e'_j$, $e \times e'$ may be decomposed into two mutually orthogonal idempotent elements e'' and f'': $e \times e' = e'' + f''$, where e'' is the sum of a maximal system of mutually orthogonal, mutually non-isomorphic primitive idempotent elements of $A \times B$. We then have

$$\begin{aligned}(A \times B)^0 &= e''(A \times B)e'' = e''((e \times e')(A \times B)(e \times e'))e'' \\ &= e''(eAe \times e'Be')e'' = e''(A^0 \times B^0)e'' = (A^0 \times B^0)^0.\end{aligned}$$

Theorem 5. *If $A \sim B$, $C \sim D$ then $A \times C \sim B \times D$.*

Let A_m denote the complete matric algebra of degree m with coefficients from A. We see easily that $A_m \sim A$.

Theorem 6. *If $A \sim B$ then $A_L \sim B_L$ for any extension field L of K.*

Proof. Let $A^0 = eAe$. Evidently $e_i \cong e_{i,\alpha}$ in A_L since $e_i \cong e_{i,\alpha}$ in A. Then $e = e^* + f^*$ $(e^*f^* = f^*e^* = 0)$, where e^* is the sum of a maximal system of mutually orthogonal, mutually non-isomorphic primitive idempotent elements of A_L. We have

$$(A_L)^0 = e^*A_Le^* = e^*(eA_Le)e^* = e^*((A^0)_L)e^* = ((A^0)_L)^0.$$

We consider an A-B-module V such that $1v = v1' = v$ $(v \in V)$, where $1'$ denotes the unit element of B. Let $\{c_{i,\alpha\beta}\}$ and $\{c'_{\kappa,\mu\nu}\}$ be the system of matrix units of A and B respectively. Then

$$V = \sum_{i,\alpha} \sum_{\kappa,\nu} c_{i,\alpha 1} V c'_{\kappa,1\nu}. \tag{16}$$

Denote by N and N' the radicals of A and B. We set $\bar{A} = A/N$, $\bar{B} = B/N'$. If

$$Ve'_\kappa = V_1 e'_\kappa \supset V_2 e'_\kappa \supset \cdots\cdots \supset V_t e'_\kappa \supset 0$$

is a composition series for Ve'_κ considered as a left A-module, then

$$eVe'_\kappa = eV_1 e'_\kappa \supset eV_2 e'_\kappa \supset \cdots\cdots \supset eV_t e'_\kappa \supset 0$$

is a composition series for the left A^0-module eVe'_κ and if $V_u e'_\kappa / V_{u+1} e'_\kappa \cong \bar{A}\bar{e}_\iota$ then $eV_u e'_\kappa / eV_{u+1} e'_\kappa \cong \bar{e}_\iota \bar{A} \bar{e}_\iota$.

Now we assume that V is finite-dimensional over K. Let $h_{\iota\kappa}$ be the number of the factor groups $\cong \bar{A}\bar{e}_\iota$ in a composition factor group series of the left A-module Ve'_κ and let $h'_{\kappa\iota}$ be the number of the factor groups $\cong \bar{e}'_\kappa \bar{B}$ in a composition factor group series of the right B-module $e_\iota V$. We can prove in a similar manner as Theorem 3 [1] the following

Theorem 7. $h_{\iota\kappa}$ (Rank $\bar{e}_\iota \bar{A} \bar{e}_\iota$) $= h'_{\kappa\iota}$ (Rank $\bar{e}'_\kappa \bar{B} \bar{e}'_\kappa$).

If K is algebraically closed then $h_{\iota\kappa} = h'_{\kappa\iota}$. We denote by $\mathfrak{A}$ the representation of A defined by the left A-module V. Let F_ι be the irreducible representation of A defined by $\bar{A}\bar{e}_\iota$ and let U_κ be the representation of A defined by Ve'_κ. Similarly we define $\mathfrak{A}'$, F'_κ, and U'_ι of B. Then

$$(17) \qquad U_\kappa \leftrightarrow \sum_\iota h_{\iota\kappa} F_\iota, \qquad U'_\iota \leftrightarrow \sum_\kappa h_{\iota\kappa} F'_\kappa,$$

where the sign $\leftrightarrow$ indicates that we have the same irreducible constituents on both sides and so

$$\mathfrak{A} \leftrightarrow \sum_\kappa \sum_\iota f'(\kappa) h_{\iota\kappa} F_\iota, \qquad \mathfrak{A}' \leftrightarrow \sum_\iota \sum_\kappa f(\iota) h_{\iota\kappa} F'_\kappa.$$

This shows that the multiplicity of F_ι in $\mathfrak{A}$ is the degree of U'_ι and the multiplicity of F'_κ in $\mathfrak{A}'$ is the degree of U_κ.

If V is a left A-module then V may be considered as an A-$E(A)$-module. Hence above arguments are valid for the A-$E(A)$-module V.

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