SOME STUDIES ON FROBENIUS ALGEBRAS, II

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This note is a continuation of the previous paper [11]. In §§1 and 2 we shall give by a slightly modified method of original one simpler proofs of some results obtained in [4], [8], and [9]. We shall then discuss the connection between a Frobenius algebra and its basic algebra. In §3 a characterization of Frobenius algebras is given. As an application, we can prove Theorem 1 [12] in general case. We shall study, in the same section, the connection between Nakayama automorphisms of a Frobenius algebra and those of its basic algebra. In §4 we obtain a new proof of Theorem 5 [3]. This gives an alternative proof of Theorem 4.1 [2]. §5 deals with some properties of quasi-Frobenius algebras.

1. Let A be a ring with minimum condition for left and right ideals. Let N be the radical of A and let us assume that $A \neq N$. $A/N = \bar{A} = \bar{A}_1 + \bar{A}_2 + \cdots\cdots + \bar{A}_k$ is a direct decomposition of $\bar{A}$ into simple two-sided ideals $\bar{A}_\kappa$ and let $f(\kappa)$, $e_{\kappa,i}$, $e_\kappa = e_{\kappa,1}$, $c_{\kappa,ij}$, and $E_\kappa = \sum_{i=1}^{f(\kappa)} e_{\kappa,i}$ have the same meaning as in [5]; namely $e_{\kappa,i}$ are mutually orthogonal primitive idempotent elements whose sum is a principal idempotent element E of A:

$$(1) \qquad E = \sum_{\kappa=1}^{k} \sum_{i=1}^{f(\kappa)} e_{\kappa,i}.$$

Hence

$$(2) \qquad A = \sum_{\kappa} \sum_{i} Ae_{\kappa,i} + l(E) \qquad (= \sum_{\kappa} \sum_{i} e_{\kappa,i}A + r(E))$$

is a direct decomposition of A into directly indecomposable left (right) ideals $Ae_{\kappa,i}$ $(e_{\kappa,i}A)$ and $l(E)$ $(r(E))$, where $l(E)$ $(r(E))$ denotes the left (right) annihilator of E[1]. $Ae_{\kappa,i}$ and $Ae_{\lambda,j}$ are isomorphic if and only if $\kappa = \lambda$. For each κ, $c_{\kappa,ij}$ $(i, j = 1, 2, \cdots\cdots, f(\kappa))$ are a system of matric units with $c_{\kappa,ii} = e_{\kappa,i}$ and $c_{\kappa,ij}$ satisfy

$$(3) \qquad c_{\kappa,ij}c_{\lambda,lm} = \delta_{\kappa,\lambda}\delta_{j,l}c_{\kappa,im}.$$

The residue class $\bar{E}_\kappa = E_\kappa (\mathrm{mod}\ N)$ is the unit element of simple ring $\bar{A}_\kappa$.

1) We denote by $l(*)$ $(r(*))$ the left (right) annihilator in A.

Theorem 1 ([4]). *If in a ring A with minimum condition for left and right ideals the duality* $l(r(\mathfrak{l}))=\mathfrak{l}$ *is valid for every nilpotent simple left ideal and zero, then A possesses the unit element and there exists a permutation* π *of* $(1, 2, \cdots\cdots, k)$ *such that for each* κ:

(i) *the largest completely reducible right subideal of* $e_\kappa A$ *is a direct sum of simple right ideals isomorphic to* $e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$,

(ii) $Ae_{\pi(\kappa)}$ *has a unique simple left subideal which is isomorphic to* Ae_κ/Ne_κ, *and*

(iii) *if* $f(\pi(\kappa))>1$ *then the largest completely reducible right subideal of* $e_\kappa A$ *is simple.*

Proof. We proceed stepwise:

1) $l(r(0))=l(A)=0$ implies $A\neq N$.

2) $l(r(\mathfrak{l}))=\mathfrak{l}$ for every non-nilpotent simple left ideal (see [5]).

3) $R=RE$, where R denotes $l(N)$ (see [9]).

4) Since $e_\kappa R$ is the largest completely reducible right subideal, $e_\kappa R\neq 0$ for each κ. $e_\kappa R=e_\kappa RE$ implies that there exists at least one E_λ such that $e_\kappa RE_\lambda\neq 0$.

5) We denote $r(N)$ by L. For each λ, Le_λ is the largest completely reducible left subideal of Ae_λ. Let $\mathfrak{l}$ be any simple left subideal of Le_λ. Since $\mathfrak{l}=\mathfrak{l}e_\lambda$, $r(\mathfrak{l})\subseteq N\cup(E-e_\lambda)A$. Then $\mathfrak{l}=l(r(\mathfrak{l}))\supseteq Re_\lambda$. Hence if $Re_\lambda\neq 0$ then $Le_\lambda=Re_\lambda$ and Re_λ is simple. According to 4) $e_\kappa Re_\lambda\neq 0$ and hence we have $Re_\lambda\cong Ae_\kappa/Ne_\kappa$. This implies $e_\mu Re_\lambda=0$ $(\mu\neq\kappa)$. Consequently if we set $\lambda=\pi(\kappa)$ then $\pi(\kappa)$ is uniquely determined by κ and π is a permutation of $(1, 2, \cdots\cdots, k)$. Thus we have

$$E_\kappa RE_{\pi(\kappa)}\neq 0,\qquad E_\kappa RE_\mu=0\quad(\mu\neq\pi(\kappa)).$$

Moreover $Le_\kappa=Re_\kappa$ for every κ. These facts show the validities of (i) and (ii).

6) It follows from $Re_{\pi(\kappa),i}\cong Ae_\kappa/Ne_\kappa$ that $ERe_{\pi(\kappa),i}=Re_{\pi(\kappa),i}$, whence $ERE=ER=RE=R$. We then have $r(E)\cap R=r(E)\cap ER=0$ and hence $r(E)=0$. Then $A=EA+r(E)=EA$. Finally we have $A=AE$ since $l(E)=l(A)=0$. Thus E is the unit element of A. We denote E by 1. According to 5) we have $R=L$. We denote this by M.

7) Suppose that $f(\pi(\kappa))>1$. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be any two simple right ideals of $e_\kappa M$. Since $\mathfrak{r}_i\cong e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$ $(i=1,2)$, $e_\kappa\mathfrak{r}_1e_{\pi(\kappa),1}\neq 0$ and $e_\kappa\mathfrak{r}_2e_{\pi(\kappa),2}\neq 0$. Let $d_1\in e_\kappa\mathfrak{r}_1e_{\pi(\kappa),1}$ and $d_2\in e_\kappa\mathfrak{r}_2e_{\pi(\kappa),2}$ be non-zero elements. Then $d_1A=\mathfrak{r}_1$ and $d_2A=\mathfrak{r}_2$. We set $d_3=d_1+d_2$. Since $d_3e_{\pi(\kappa),1}=d_1$ and $d_3e_{\pi(\kappa),2}=d_2$, $Ad_3\neq Ad_i$ $(i=1,2)$. We see that

$$Ad_i \cong Ae_\kappa / Ne_\kappa \qquad (i = 1, 2, 3).$$

Since Ad_3 is a simple left ideal, $r(Ad_3) = r(d_3)$ $(\supset N \cup (1 - e_{\pi(\kappa),1} - e_{\pi(\kappa),2})A)$ is a maximal right ideal of A. Then $d_3A \cong A/r(d_3)$ implies that d_3A is simple. Now $d_1A = d_3e_{\pi(\kappa),1}A \subseteq d_3A$ and similarly $d_2A \subseteq d_3A$, whence $d_1A = d_2A = d_3A$. This completes the proof of (iii).

We consider a ring with unit element and with minimum condition for left and right ideals. The subring $A^0 = eAe$ with unit element $e = \sum_\kappa e_\kappa$ is called the basic ring of A. As was shown in [13], the basic ring of A is determined by A up to an inner automorphism of A. We denote by $l^0(*)$ $(r^0(*))$ the left (right) annihilator in A^0. Let $\mathfrak{A}$ be a two-sided ideal of A. $\mathfrak{A}^0 = \mathfrak{A} \cap A^0 = e\mathfrak{A}e$ is the two-sided ideal of A^0 and

$$\mathfrak{A} = \sum_{\kappa, i} \sum_{\lambda, j} c_{\kappa, i1} \mathfrak{A}^0 c_{\lambda, 1j}.$$

Thus $\mathfrak{A} \to \mathfrak{A}^0$ gives a (1-1) correspondence between the two-sided ideals of A and those of A^0. We see easily that

$$(4) \qquad (r(\mathfrak{A}))^0 = r^0(\mathfrak{A}^0), \qquad (l(\mathfrak{A}))^0 = l^0(\mathfrak{A}^0).$$

Corollary. *Under the assumptions of Theorem* 1, *the two-sided ideal M^0 of A^0 is a principal left ideal*: $M^0 = A^0d^0$ $(d^0 \in A^0)$.

Proof. $M^0 = M \cap A^0 = \sum_\kappa e_\kappa M e_{\pi(\kappa)}$. We choose a non-zero element d_κ^0 of $M^0 e_{\pi(\kappa)} = e_\kappa M e_{\pi(\kappa)}$ and set $d^0 = \sum_\kappa d_\kappa^0$. Then $d^0 e_{\pi(\kappa)} = e_\kappa d^0 = d_\kappa^0$ and

$$(5) \qquad M^0 = \sum_\kappa A^0 d_\kappa^0 = A^0 d^0.$$

2. In what follows we consider an algebra A with a finite rank over a field F.

Theorem 2 ([8]). *An algebra A is a Frobenius algebra if (and only if) A possesses a right unit element and $L = r(N)$ is a principal left ideal*: $L = Ad$.

Proof. $L = Ad$ is left-homomorphic to A by $E \to d$ (E being the right unit element) and indeed, to A/N, since $Nd = 0$. $L = \sum Le_{\kappa,i}$ and each $Le_{\kappa,i}$ is the largest completely reducible left subideal of $Ae_{\kappa,i}$, whence L is a direct sum of at least $\sum_\kappa f(\kappa)$ simple left ideals. Hence we have necessarily $L \cong A/N$, and each $Le_{\kappa,i}$ must be simple. Since $Le_{\kappa,i} \cong Le_{\kappa,j}$ and $Ae_{\kappa,i}/Ne_{\kappa,i} \not\cong Ae_{\lambda,j}/Ne_{\lambda,j}$ for $\kappa \neq \lambda$, there must exist a permutation π of $(1, 2, \cdots\cdots, k)$ such that

$$Le_{\kappa,i} \cong Ae_{\pi(\kappa),i}/Ne_{\pi(\kappa),i}$$

and $f(\kappa)=f(\pi(\kappa))$. Let U_κ (V_κ) be the directly indecomposable representations of A belonging to Ae_κ ($e_\kappa A$). Since Ae_κ has a unique simple left subideal Le_κ, we have by Lemma 1 [5]

$$V_{\pi(\kappa)} \cong \begin{pmatrix} * & 0 \\ * & U_\kappa \end{pmatrix} \tag{6}$$

and hence $(Ae_\kappa : F) \leq (e_{\pi(\kappa)}A : F)$. Since $A = EA + r(E)$

$$(A:F) = \sum_\kappa f(\kappa)\,(Ae_\kappa : F) = \sum_\kappa f(\pi(\kappa))\,(e_{\pi(\kappa)}A : F) + (r(E) : F).$$

This implies $(Ae_\kappa : F) = (e_{\pi(\kappa)}A : F)$ and $r(E)=0$. Thus E is the unit element of A and $U_\kappa \cong V_{\pi(\kappa)}$.

Corollary 1 ([8]). *An algebra is a quasi-Frobenius algebra if (and only if) A possesses a right unit element E and for each* κ *left ideal* $r(N)e_\kappa$ *is simple and isomorphic to* $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, *where* π *is a permutation of* $(1, 2, \cdots\cdots, k)$.

Proof. We denote $r(N)$ by L. $L = LE = \sum Le_{\kappa,i}$. As Le_κ is simple and isomorphic to $Ae_{\pi(\kappa)}/Ne_{\pi(\kappa)}$, we have $ELE = EL = LE = L$. Hence $r(E) \cap L = r(E) \cap EL = 0$. Then $r(E)=0$ since $r(E)$ is a two-sided ideal of A. Thus E is the unit element of A. Let $A^0 = eAe$ be the basic algebra of A as before. By Corollary to Theorem 1 we have $L^0 = A^0 d^0$ ($d^0 \in A^0$). Hence A^0 is a Frobenius algebra and then A is a quasi-Frobenius algebra.

From Corollary to Theorem 1 and Theorem 2 we obtain readily the following

Corollary 2 ([4]). *An algebra A is a quasi-Frobenius algebra if (and only if) the duality* $l(r(\mathfrak{l})) = \mathfrak{l}$ *is valid for every nilpotent simple left ideal and zero.*

Theorem 3. *An algebra with unit element is an almost symmetric (or weakly symmetric) algebra if and only if its basic algebra* A^0 *is almost symmetric (or weakly symmetric).*

Proof. Denote by C the center of A. A is called almost symmetric if $L = r(N) = Ad$ ($d \in C$)[1]. If A is almost symmetric then

$$L^0 = eLe = A^0 de$$

where de lies in the center C^0 of A^0. Conversely suppose A^0 is

1) See [1]. Cf. also [11].

almost symmetric: $L^0 = A^0 d^0$ $(d^0 \in C^0)$. We set $d = \sum_{\kappa, l} c_{\kappa, l1} d^0 c_{\kappa, 1l}$. For any element a^0 in A^0 we have

$$c_{\lambda, j1} a^0 c_{\rho, 1l} d = c_{\lambda, j1} a^0 e_\rho d^0 c_{\rho, 1l} = c_{\lambda, j1} d^0 a^0 c_{\rho, 1l} = d c_{\lambda, 1j} a^0 c_{\rho, 1l}.$$

Since for any element a in A

$$a = \sum_{\lambda, \rho} \sum_{j, l} c_{\lambda, j1}(c_{\lambda, 1j} a c_{\rho, l1}) c_{\rho, 1l} \qquad (c_{\lambda, 1j} a c_{\rho, l1} \in A^0)$$

d lies in C. Evidently $Ad \subseteq L$. On the other hand $eAde = A^0 d^0 = L^0$. Hence $L = Ad$ and so A is almost symmetric.

Since A is weakly symmetric if and only if $L = Ad$, where $E_\kappa d = dE_\kappa$ for every E_κ[1], we see easily that our theorem is valid for a weakly symmetric algebra.

3. Let us denote by $A \times A$ the Kronecker product of an algebra A.

Theorem 4. *An algebra A with unit element is a Frobenius algebra if and only if there exists a pair of bases (a_i), $(\bar{a}_i)$ of A such that*

$$\sum_{i=1}^{n} a a_i \times \bar{a}_i = \sum_{i=1}^{n} a_i \times \bar{a}_i a \qquad (a \in A). \tag{7}$$

Proof. We denote by $S(a)$ $(R(a))$ the left (right) regular representation of A defined by the basis (a_j):

$$a(a_j) = (a_j) S(a), \qquad (a_j) a = R(a)(a_j).$$

If we set $S(a) = (s_{ij}(a))$ then

$$\sum_j a a_j \times \bar{a}_j = \sum_j (\sum_i s_{ij} a_i) \times \bar{a}_j) = \sum_i (a_i \times (\sum_j s_{ij} \bar{a}_j)).$$

This implies $\bar{a}_i a = \sum_j s_{ij} \bar{a}_j$, whence $a \to S(a)$ is the right regular representation of A defined by the basis $(\bar{a}_i)$. Let $(\bar{a}_i) = P(a_i)$. We then have

$$S(a) = PR(a)P^{-1} \qquad (a \in A). \tag{8}$$

Thus A is a Frobenius algebra. Conversely suppose A is a Frobenius algebra: $S(a) = QR(a)Q^{-1}$. If we set $(\bar{a}_i) = Q(a_i)$, then (a_i), $(\bar{a}_i)$ satisfy the condition (7).

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1) See [11], Corollary to Theorem 14.

We say that (a_i) and $(\bar{a}_i)$ in Theorem 4 are *dual bases.* If (a_i), $(\bar{a}_i)$ are dual then (a_i), $(b\bar{a}_i)$ are also dual for any regular element b in A. Moreover if (a_i), (a_i') are dual then there exists a regular element b such that $a_i' = b\bar{a}_i$ $(i = 1, 2, \cdots\cdots, n)$. It is also easily seen that for any basis (u_i) of a Frobenius algebra A, there exists a basis $(\bar{u}_i)$ such that (u_i), $(\bar{u}_i)$ are dual. It follows from (8) that $P'R(a)(P')^{-1} = S(a^\varphi)$ $(a^\varphi \in A)$, where P' denotes the transpose of the matrix P. The mapping $a \rightarrow a^\varphi$ forms an automorphism of A which is called Nakayama automorphism. We see that $(a_i^\varphi) = (P')^{-1}(\bar{a}_i) = (P')^{-1}P(a_i)$. If (a_i), $(\bar{a}_i)$ are dual then $(\bar{a}_i)$, (a_i^φ) are also dual:

$$(9) \qquad \sum_i a\bar{a}_i \times a_i^\varphi = \sum_i \bar{a}_i \times a_i^\varphi a.$$

Further we have

$$(10) \qquad \sum_i a^\varphi \bar{a}_i \times a_i = \sum_i \bar{a}_i \times a_i a.$$

Theorem 5. *An algebra A with unit element is a symmetric algebra if and only if there exists a pair of bases (a_i), $(\bar{a}_i)$ such that*

$$(11) \qquad \begin{aligned} \sum_i aa_i \times \bar{a}_i &= \sum_i a_i \times \bar{a}_i a, \\ \sum_i a\bar{a}_i \times a_i &= \sum_i \bar{a}_i \times a_i a. \end{aligned}$$

(a_i), $(\bar{a}_i)$ in Theorem 5 are called *quasi-complementary bases.* If (a_i), $(\bar{a}_i)$ are quasi-complementary then $(\bar{a}_i)$, (a_i) are also quasi-complementary. For any basis (u_i) of a symmetric algebra A, there exists a basis $(\bar{u}_i)$ such that (u_i), $(\bar{u}_i)$ are quasi-complementary.

Let A^0 be the basic algebra of an algebra A and let us take a basis $(u_{\kappa\lambda,\alpha})$ of A^0 in accord with the decomposition

$$(12) \qquad A^0 = \sum_{\kappa,\lambda} e_\kappa A^0 e_\lambda .$$

Here $u_{\kappa\lambda,\alpha} \in e_\kappa A e_\lambda$, that is, $e_\kappa u_{\kappa\lambda,\alpha} e_\lambda = u_{\kappa\lambda,\alpha}$. Then the elements

$$(13) \qquad c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}$$

form a basis of A which is called a Cartan basis. Let us take a Cartan basis (13) of a symmetric algebra A and let $(\overline{c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}})$ be a quasi-complementary basis. It follows from (11) that

$$(14) \qquad \overline{c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}} = c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i},$$

where the elements of $(v_{\lambda\kappa,\alpha})$ form a basis of A^0 and $v_{\lambda\kappa,\alpha} \in e_\lambda A e_\kappa$.

Theorem 6. *An algebra is symmetric if and only if its basic algebra is symmetric*[1].

Proof. Suppose that A is symmetric. It follows from (11), (14) that for any element a in A

$$\begin{aligned}\sum_{\kappa,\lambda,\alpha} a^0 u_{\kappa\lambda,\alpha} \times v_{\lambda\kappa,\alpha} &= \sum_{\kappa,\lambda,\alpha} u_{\kappa\lambda,\alpha} \times v_{\lambda\kappa,\alpha} a^0, \\ \sum_{\kappa,\lambda,\alpha} a^0 v_{\lambda\kappa,\alpha} \times u_{\kappa\lambda,\alpha} &= \sum_{\kappa,\lambda,\alpha} v_{\lambda\kappa,\alpha} \times u_{\kappa\lambda,\alpha} a^0,\end{aligned} \tag{15}$$

whence A^0 is symmetric. Conversely we assume that A^0 is symmetric. Let us take a basis $(u_{\kappa\lambda,\alpha})$ of A^0 as before and let $(u_{\kappa\lambda,\alpha})$, $(\bar{u}_{\kappa\lambda,\alpha})$ be quasi-complementary bases of A^0. We have $\bar{u}_{\kappa\lambda,\alpha} = v_{\lambda\kappa,\alpha}$, where $v_{\lambda\kappa,\alpha} \in e_\lambda A e_\kappa$. We then see that

$$(c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}), \qquad (c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i})$$

satisfy the condition (11), that is, A is symmetric. Observe that for any element a in A

$$\begin{aligned}&\sum_{\kappa,\lambda,\alpha,i,j} (c_{\mu,l1} a^0 c_{\nu,1m}) c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j} \times c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i} \\ &\qquad = \sum_{\kappa,\lambda,\alpha,i,j} c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j} \times c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i} (c_{\mu,l1} a^0 c_{\nu,1m}), \\ &\sum_{\kappa,\lambda,\alpha,i,j} (c_{\mu,l1} a^0 c_{\nu,1m}) c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i} \times c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j} \\ &\qquad = \sum_{\kappa,\lambda,\alpha,i,j} c_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i} \times c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j} (c_{\mu,l1} a^0 c_{\nu,1m}).\end{aligned}$$

Let φ be a Nakayama automorphism of a Frobenius algebra A. If A^0 is the basic algebra of A, then $(A^0)^\varphi = e^\varphi A e^\varphi$ is also the basic algebra of A. Since there exists a regular element b such that $b^{-1} e^\varphi b = e$, the Nakayama automorphism $a \to b^{-1} a^\varphi b$ induces an automorphism of A^0.

Theorem 7. *Let φ be a Nakayama automorphism of a Frobenius algebra A such that φ induces an automorphism φ_0 of the basic algebra A^0. Then φ_0 is a Nakayama automorphism of A^0. Conversely let φ_0 be a Nakayama automorphism of A^0 then φ_0 is extended to a Nakayama automorphism of A.*

Proof. We take a Cartan basis (13) as a basis of A. Let $(\overline{c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}})$ be a dual basis. It follows from (7), (10) that

$$\overline{c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}} = c^\varphi_{\lambda,j1} v_{\lambda\kappa,\alpha} c_{\kappa,1i},$$

1) In case of algebras over an algebraically closed field this fact was proved in [10], [12].

where $(v_{\lambda\kappa,\alpha})$ is a basis of A^0 such that $v_{\lambda\kappa,\alpha} \in e_\lambda^\varphi A e_\kappa$ ($e_\lambda^\varphi A e_\kappa \subset A^0$, since $e_\lambda^\varphi = e e_\lambda^\varphi e \in A^0$). We see from (7), (10) that

$$\sum_{\kappa,\lambda,\alpha} a^0 u_{\kappa\lambda,\alpha} \times v_{\lambda\kappa,\alpha} = \sum_{\kappa,\lambda,\alpha} u_{\kappa\lambda,\alpha} \times v_{\lambda\kappa,\alpha} a^0,$$

$$\sum_{\kappa,\lambda,\alpha} (a^0)^{\varphi_0} v_{\lambda\kappa,\alpha} \times u_{\kappa\lambda,\alpha} = \sum_{\kappa,\lambda,\alpha} v_{\lambda\kappa,\alpha} \times u_{\kappa\lambda,\alpha} a^0,$$

whence φ_0 is a Nakayama automorphism of A^0 determined by dual bases $(u_{\kappa\lambda,\alpha})$, $(v_{\lambda\kappa,\alpha})$. We shall prove the converse. Let $(u_{\kappa\lambda,\alpha})$, $(\bar{u}_{\kappa\lambda,\alpha})$ be dual bases of A^0 and φ_0 be a Nakayama automorphism determined by these bases. Here $\bar{u}_{\kappa\lambda,\alpha} \in e_\lambda^{\varphi_0} A e_\kappa$. If we set $(\bar{u}_{\kappa\lambda,\alpha}) = Q(u_{\kappa\lambda,\alpha})$ then $(u_{\kappa\lambda,\alpha}^{\varphi_0}) = (Q')^{-1} Q(u_{\kappa\lambda,\alpha})$. Since $e_\lambda^{\varphi_0} e_{\kappa,i} = e_\lambda^{\varphi_0} e e_{\kappa,i} = 0$, $e_{\kappa,i} e_\lambda^{\varphi_0} = e_{\kappa,i} e e_\lambda^{\varphi_0} = 0$ for $i > 0$,

$$1 = \sum_{\kappa=1}^{k} e_\kappa^{\varphi_0} + \sum_{\kappa=1}^{k} \sum_{i>1} e_{\kappa,i} \tag{16}$$

is a decomposition of 1 into mutually orthogonal primitive idempotent elements. Let $Ae_\kappa^{\varphi_0} \cong Ae_{\pi(\kappa)}$, where π is a permutation of $(1, 2, \cdots\cdots, k)$. Corresponding to the decomposition (16), we can construct a system of matric units $c_{\kappa,ij}^*$ such that

$$c_{\kappa,11}^* = e_\kappa^{\varphi_0}, \qquad c_{\kappa,ii}^* = e_{\pi(\kappa),i} \qquad (i > 1)$$

and

$$c_{\kappa,ij}^* c_{\lambda,lm}^* = \delta_{\kappa\lambda} \delta_{jl} c_{\kappa,im}^*.$$

Thus we see that

$$(c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}), \qquad (c_{\lambda,j1}^* \bar{u}_{\kappa\lambda;\alpha} c_{\kappa,1i}) \tag{17}$$

satisfy the condition (7), that is, (17) are dual bases of A. If we set

$$(c_{\lambda,j1}^* \bar{u}_{\kappa\lambda,\alpha} c_{\kappa,1i}) = P(c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j})$$

then

$$((c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j})^\varphi) = (P')^{-1} P(c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j}),$$

where φ is a Nakayama automorphism of A determined by dual bases (17). If we arrange the elements of basis $(c_{\kappa,i1} u_{\kappa\lambda,\alpha} c_{\lambda,1j})$ so that first the elements $u_{\kappa\lambda,\alpha}$ of A^0 appear then we have

$$P = \begin{pmatrix} Q & 0 \\ 0 & * \end{pmatrix}$$

and hence

$$(P')^{-1}P = \begin{pmatrix} (Q')^{-1}Q & 0 \\ 0 & * \end{pmatrix}$$

This implies $u^{\varphi}_{\kappa\lambda,\alpha} = u^{\varphi_0}_{\kappa\lambda,\alpha}$ for every $u_{\kappa\lambda,\alpha}$.

4. We shall give a new proof for a characterization of a separable algebra obtained in [3].

Theorem 8. *A Frobenius algebra is separable if and only if there exist dual bases* (a_i), $(\bar{a}_i)$ *such that* $\sum_i a_i\bar{a}_i = 1$.

Proof. The set

$$\mathfrak{c}(A) = \{\sum_i a_i x \bar{a}_i \mid x \in A\} \tag{18}$$

forms an ideal of the center C of A such that $\mathfrak{c}(A) \subset r(N)$, where N denotes as usual the radical of A [14, Theorem 1]. By our assumption $1 \in r(N)$, whence $N = 0$ and so A is semisimple. Since the condition $\sum a_i\bar{a}_i = 1$ is independent with the underlying field F, A is separable[1]. Conversely let A be separable. Then A is a symmetric algebra. To prove our theorem, we may assume without restriction that A is simple. We consider first a division algebra $\mathfrak{K}$ and let (u_i), $(\bar{u}_i)$ be dual bases of $\mathfrak{K}$. Since $\mathfrak{K}$ is separable we have

$$\mathfrak{c}(\mathfrak{K}) = \mathfrak{z} \tag{19}$$

where $\mathfrak{z}$ is the center of $\mathfrak{K}$[2]. Hence $\sum u_i(b\bar{u}_i) = 1$ for dual bases (u_i), $(b\bar{u}_i)$ with a suitable element $b \neq 0$ in $\mathfrak{K}$. Next let A be a total matric algebra F_m with matric units (e_{ij}). We can take the bases (e_{ij}), (e_{ji}) as quasi-complementary bases of F_m. For an element $a = \sum_{p,q} \alpha_{pq}e_{pq}$ in F_m, we have

$$\sum_{i,j} e_{ij}ae_{ji} = (\sum_q \alpha_{qq})1.$$

If we choose a regular element a such that $\sum \alpha_{qq} = 1$, then $\sum_{i,j} e_{ij}(ae_{ji}) = 1$ for dual bases (e_{ji}), (ae_{ji}). Then we can find easily dual bases (a_i), $(\bar{a}_i)$ of a simple separable algebra $A = \mathfrak{K}_m$ which satisfy $\sum a_i\bar{a}_i = 1$.

Corollary ([3]). *An algebra with unit element is separable if and*

1) Note that A_L is a Frobenius algebra for any extension field L of F.

2) See [14], Theorem 3.

only if there exists a pair of bases (a_i), $(\bar{a}_i)$ *such that*

(20) $$\sum_i aa_i \times \bar{a}_i = \sum_i a_i \times \bar{a}_i a, \qquad \sum_i a_i\bar{a}_i = 1.$$

By Hochschild [3], the conditions (20) are equivalent to vanishing of one-dimensional cohomology group of A for every two-sided A-module. Thus we have an alternative proof of the following

Theorem 9 ([2]). *A necessary and sufficient condition for an algebra to be separable is that one-dimensional cohomology group of A for every two-sided A-module vanishes.*

5. Let A be an algebra with unit element over a field F and let L be any extension field of F. We consider the algebra A_L over L. Let $\mathfrak{l}$ be a left ideal of A. If we take a basis $a_1, a_2, \cdots\cdots, a_n$ of A such that $\mathfrak{l}$ is spanned by $a_{s+1}, a_{s+2}, \cdots\cdots, a_n$. Then $\mathfrak{l}_L$ is a left ideal of A_L. In a similar manner as Theorem 14 [6], we have

(21) $$r^*(\mathfrak{l}_L) = (r(\mathfrak{l}))_L,$$

where $r^*(*)$ denotes the right annihilator in A_L. Similarly we have for a right ideal $\mathfrak{r}$ of A

(22) $$l^*(\mathfrak{r}_L) = (l(\mathfrak{r}))_L.$$

Theorem 10. *An algebra is a quasi-Frobenius algebra if and only if* A_L *is a quasi-Frobenius algebra.*

Proof. If A is a quasi-Frobenius algebra then the basic algebra A^0 is a Frobenius algebra and hence $(A^0)_L$ is also a Frobenius algebra. Then the basic algebra $((A^0)_L)^0$ of $(A^0)_L$ is a Frobenius algebra. Since $((A^0)_L)^0$ is also the basic algebra of A_L, A_L must be a quasi-Frobenius algebra. Conversely suppose A_L is a quasi-Frobenius algebra. It follows from (21), (22) that for any left ideal $\mathfrak{l}$ of A

$$\mathfrak{l}_L = l^*(r^*(\mathfrak{l}_L)) = l^*((r(\mathfrak{l}))_L) = (l(r(\mathfrak{l}))_L,$$

whence $\mathfrak{l} = l(r(\mathfrak{l}))$, and so A is a quasi-Frobenius algebra.

Let A be an algebra and let $\mathfrak{A}$ be a two-sided ideal in A, we have $(A/\mathfrak{A})^0 = A^0/\mathfrak{A}^0$. This, combined with Theorem 2 [8], yields the following

Theorem 11. *Let A be a quasi-Frobenius algebra and let* $\mathfrak{A}$ *be a two-sided ideal in A. The residue class algebra* $A/\mathfrak{A}$ *is a quasi-Frobenius algebra if and only if the two-sided ideal* $l^0(\mathfrak{A}^0)$ *in* A^0 *is a principal ideal* $A^0b^0 = b^0A^0$ $(b^0 \in A^0)$.

By Lemma 2 [7], we have

Theorem 12. *If every residue class algebra $A/\mathfrak{A}$ is a quasi-Frobenius algebra then A is uni-serial, and conversely.*

We see by the same way that Theorem 12 is also valid for a ring with minimum condition for left and right ideals as was shown in [4].

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