A CONSTRUCTION OF CLOSED SURFACES OF NEGATIVE CURVATURE IN E^4

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Introduction. The author recently proved a theorem[1] as follows.

Theorem. *Let*

$$A_{\alpha ij}x^i x^j = 0^{2)}, \qquad A_{\alpha ij} = A_{\alpha ji},$$
$$i, j = 1, 2, \cdots\cdots, n; \quad \alpha = 1, 2, \cdots\cdots, N$$

be a system of quadratic equations in x^i. If

$$\sum_\alpha (A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh})x^i y^j x^h y^k \leqq 0$$

for any x^i, y^i, it has a non-trivial real solution in x^i, when $N < n$.

This gives the corollary[3]:

Corollary. *Let*

$$\Psi_\alpha(x, x) \equiv A_{\alpha ij}x^i x^j, \qquad A_{\alpha ij} = A_{\alpha ji},$$
$$i, j = 1, 2, \cdots\cdots, n; \quad \alpha = 1, 2, \cdots\cdots, N$$

be N quadratic forms of x^i. If

$$\sum_\alpha (A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh})x^i y^j x^h y^k < 0$$

for any two linearly independent vectors $x = (x^i)$, $y = (y^i)$, then $N \geqq n - 1$.

Proof. Let us suppose that $N \leqq n - 2$. By virtue of the theorem above, there exists a vector $x = (x^i)$ such that $x \neq 0$ and

$$\Psi_\alpha(x, x) = 0, \qquad \alpha = 1, 2, \cdots\cdots, N.$$

For a system of linear equations in y^i such that

$$x^i y^i = 0,$$
$$\Psi_\alpha(x, y) = 0, \qquad \alpha = 1, 2, \cdots\cdots, N$$

there exists a non-trivial solution $y = (y^i)$. For the x^i, y^i, we have

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1) T. Ōtsuki, On the existence of solutions of a system of quadratic equations and its geometrical application, Proc. Japan Acad., Vol. 29 (1953), pp. 99 - 100.

2) The summation convention of tensor analysis is used throughout.

3) The author was informed by Prof. S. S. Chern of this corollary.

$$\sum_{\alpha}(A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh})x^i y^j x^h y^k = 0,$$

this contradicts to the assumption.

As a geometrical application of the corollary, we have a theorem as follows.

Theorem. *An n-dimensional Riemann manifold with the property that at least at a point, the sectional curvatures for all plane elements are negative cannot be isometrically imbedded in an Euclidean space of dimension* $2n-2$.

Proof. Let V_n be a Riemann manifold with the property stated in the theorem. Assume that V_n can be isometrically imbedded in an Euclidean space of dimension $n+N$. Let the line element of V_n be given by $ds^2 = g_{ij}(u)du^i du^j$ in a coordinate neighborhood (u^i) of the point. Let

$$R_{i\ hk}^{\ j} = \frac{\partial \Gamma_{i\ k}^{\ j}}{\partial u^h} - \frac{\partial \Gamma_{i\ h}^{\ j}}{\partial u^k} - \Gamma_{i\ h}^{\ m}\Gamma_{m\ k}^{\ j} + \Gamma_{i\ k}^{\ m}\Gamma_{m\ h}^{\ j}$$

be the components of the Riemann-Christoffel tensor of V_n, where $\Gamma_{i\ k}^{\ j}$ denote the Christoffel symbols of the second kind made by g_{ij}. Let $\Psi_\alpha = A_{\alpha ij}du^i du^i$, $A_{\alpha ij} = A_{\alpha ji}$, $\alpha = 1, 2, \cdots\cdots, N$, be the second fundamental forms of V_n. As is well known, we have the equations

$$R_{ijhk} = -\sum_{\alpha}(A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh}).$$

By the assumption, it follows that

$$\sum_{\alpha}(A_{\alpha ih}A_{\alpha jk} - A_{\alpha ik}A_{\alpha jh})x^i y^j x^h y^k < 0$$

for any two linearly independent tangent vectors x^i, y^i to V_n at the point. Hence, by means of the corollary above, it must be $N \geqslant n-1$.

Furthermore, we have the theorem:

Theorem. *A compact n-dimensional Riemann manifold with the property that at any point, the sectional curvatures for all plane elements are non-positive cannot be isometrically imbedded in an Euclidean space of dimension* $2n-1$[1].

Pooof. Let V_n be a Riemann manifold with the property stated in the theorem. Let us suppose that V_n can be isometrically imbedded in an Euclidean space of dimension $2n-1$. Let O be a fixed point of the space and let P be a point of V_n whose distance from

1) This is another proof of a special case of Theorem 1 in the paper cited in Footnote 1), p. 95.

O is maximum. We may consider that the second fundamental form $\Psi_1 = A_{1ij}du^i du^j$ is correspond to the normal direction $\overrightarrow{OP}$ to V_n at P. Since dist (O, P) is maximum on V_n, we have $\overrightarrow{OP} \cdot dP = 0$, $dP \cdot dP + \overrightarrow{OP} \cdot d^2P \leqslant 0$ at P, hence $e_{n+1}\, d^2P = -A_{1ij}du^i du^j < 0$, where a dot indicates the inner product and e_{n+1} is the unit vector of the direction $\overrightarrow{OP}$. Accordingly, Ψ_1 is positive definite at P. We must have the following relation

$$(A_{1ih}A_{1jk} - A_{1ik}A_{1jh})\, x^i y^j x^h y^k > 0$$

for any two linearly independent tangent vectors x^i, y^i to V_n at P. Hence we obtain from the assumption

$$\sum_{\beta=2}^{n-1} (A_{\beta ih}A_{\beta ik} - A_{\beta ik}A_{\beta jh})\, x^i y^j x^h y^k < 0.$$

By virtue of the corollary, we must have $N = n - 2 \geqslant n - 1$, which is a contradiction.

It is well known that there exists no closed surface with non-positive curvature at every point in a 3-dimensional Euclidean space. According to the argument above, for any compact n-dimensional Riemann manifold V_n with the property that at any point, the sectional curvatures for all plane elements are non-positive, the minimum of the dimensions of the Euclidean spaces in which V_n can be isometrically imbedded is not smaller than $2n$. *Can such V_n be always isometrically imbedded in an Euclidean space of dimension $2n$?* As is well known, any n-dimensional Riemann manifold can be locally and isometrically imbedded in an Euclidean space of dimension $n(n+1)/2$. For $n = 2, 3$, we have $n(n+1)/2 \leqslant 2n$. Especially, can a compact 2-dimensional V_2 everywhere with non positive curvature be always isometrically imbedded in an Euclidean space of dimension 4? The author don't know any literatures on this question. In the present paper, we shall show that *there exist closed orientable surfaces everywhere with negative curvature in an Euclidean space of dimension 4.* This will show that the question above is reasonable.

Let V_2 be a compact, orientable Riemann manifold of dimension 2. Let $ds^2 = \omega_1\omega_1 + \omega_2\omega_2$ be the line element of V_2. Let K be the Gaussian total curvature of V_2. Then, as is well known, we have the equation

$$\frac{1}{2\pi} \int_{V_2} K\, \omega_1 \wedge \omega_2 = \chi(V_2)$$

where $\chi(V_2)$ denotes the Euler-Poincaré characteristic of V_2. If we put the genus of $V_2 = p$, we have

$$\frac{1}{2\pi}\int_{V_2} K\,\omega_1 \wedge \omega_2 = 2(1-p).$$

Hence, if $K \leqslant 0$, it must be $1 \leqslant p$ and if $K \leqslant 0$, $K \not\equiv 0$, it must be $2 \leqslant p$. This shows that the only compact, orientable and flat Riemann manifold of dimension 2 is a topological torus.

1. Tubiform surfaces in an Euclidean space of dimension 4.

Let $\mathscr{C}$ be a curve of class C^r[1] in a 3-dimensional hyperplane of an Euclidean space E of dimension 4. For any point $\bar{P} \in \mathscr{C}$, let $\{\bar{P}, e_i\}$, $i = 1, 2, 3, 4$, be the Frenet frame of $\mathscr{C}$, that is

$$(1)\qquad \begin{cases} dP = du\,e_1, \\ de_1 = \sigma(u)du\;e_2, \\ de_2 = -\sigma(u)du\;e_1 + \tau(u)du\;e_3, \\ de_3 = -\tau(u)du\;e_2, \\ de_4 = 0 \end{cases}$$

where u denotes the parameter which represents the arc length measured from a fixed point of $\mathscr{C}$ to $\bar{P}(u)$ on $\mathscr{C}$, $\sigma(u)$ and $\tau(u)$ denote the first and second curvatures of the curve respectively.

Making use of a positive function $R(u)$, let us construct a tubiform surface M as

$$(2)\qquad P(u,\theta) = \bar{P}(u) + R(u)(\cos\theta\; e_3 + \sin\theta\; e_4)^{2)}.$$

If we take a suitable $R(u)$[1], M is a regular submanifold in E. Since we have

$$P_u = \frac{\partial P}{\partial u} = e_1 + \dot{R}(\cos\theta\; e_3 + \sin\theta\; e_4) - R\tau\cos\theta\; e_2,$$

$$P_\theta = \frac{\partial P}{\partial \theta} = R(-\sin\theta\; e_3 + \cos\theta\; e_4),$$

1) As for the differentiability assumptions we suppose that the class r of $\mathscr{C} \geqslant 4$ and the one of $R(u) \geqslant 2$.

2) P, $\bar{P}$ denote also their position vectors.

$$(3)\qquad \begin{cases} E = P_u\cdot P_u = 1 + \dot{R}^2 + R^2\tau^2\cos^2\theta, \\ F = P_u\cdot P_\theta = 0, \\ G = P_\theta\cdot P_\theta = R^2 \end{cases}$$

where the dotes indicate derivatives with respect to u, the line element of M is written as

$$(4)\qquad ds^2 = (1 + \dot{R}^2 + R^2\tau^2\cos^2\theta)du^2 + R^2 d\theta^2.$$

Let us put

$$W^2 = EG - F^2 = R^2(1 + \dot{R}^2 + R^2\tau^2\cos^2\theta),$$

then, according to the well-known formula on the Gaussian total curvature:

$$K = -\frac{1}{4W^4}\begin{vmatrix} E & E_u & E_\theta \\ F & F_u & F_\theta \\ G & G_u & G_\theta \end{vmatrix} - \frac{1}{2W}\left\{\frac{\partial}{\partial\theta}\frac{E_\theta - F_u}{W} - \frac{\partial}{\partial u}\frac{F_\theta - G_u}{W}\right\},$$

we have

$$\begin{aligned} K &= -\frac{1}{2W}\left\{\frac{\partial}{\partial\theta}\frac{E_\theta}{W} + \frac{\partial}{\partial u}\frac{G_u}{W}\right\} \\ &= -\frac{1}{2W^4}\left\{W^2E_{\theta\theta} - \frac{1}{2}(W^2)_\theta E_\theta + W^2G_{uu} - \frac{1}{2}(W^2)_u G_u\right\}. \end{aligned}$$

Since we have $E_\theta = -2R^2\tau^2\cos\theta\sin\theta$, $E_{\theta\theta} = -2R^2\tau^2(\cos^2\theta - \sin^2\theta)$, we get

$$\begin{aligned} 2W^4K &= -\left\{GEE_{\theta\theta} - \frac{1}{2}GE_\theta E_\theta + GG_{uu}E - \frac{1}{2}(GE_u + G_uE)G_u\right\} \\ &= 2R^4\tau^2(1 + \dot{R}^2 + R^2\tau^2\cos^2\theta)(\cos^2\theta - \sin^2\theta) \\ &\quad + 2R^6\tau^4\cos^2\theta\sin^2\theta \\ &\quad - 2R^2(R\ddot{R} + \dot{R}^2)(1 + \dot{R}^2 + R^2\tau^2\cos^2\theta) \\ &\quad + 2R\dot{R}\{R\dot{R}(1 + \dot{R}^2) + R^2\dot{R}\ddot{R} + (2R^3\dot{R}\tau^2 + R^4\tau\dot{\tau})\cos^2\theta\} \\ &= -2R^3\{\ddot{R} + R^2\tau^2(1 + \dot{R}^2)\} \\ &\quad + 2R^4\{\tau^2(2 + 3\dot{R}^2) + R\dot{R}\tau\dot{\tau} - \tau^2R\ddot{R}\}\cos^2\theta + 2R^6\tau^4\cos^4\theta, \end{aligned}$$

that is

$$(5)\qquad \begin{aligned} &R^4(1 + \dot{R}^2 + R^2\tau^2\cos^2\theta)^2K \\ &= -2R^3\{\ddot{R} + R\tau^2(1 + \dot{R}^2)\} \\ &\quad + 2R^4\{\tau^2(2 + 3\dot{R}^2) + R\dot{R}\tau\dot{\tau} - \tau^2R\ddot{R}\}\cos^2\theta + 2R^6\tau^4\cos_4\theta. \end{aligned}$$

When $\tau = 0$ and $\ddot{R} = 0$, we have $K = 0$ by the equation above. Especially, if $\mathscr{C}$ is a closed curve, it must be $R =$ constant. Thus we have the following theorem.

Theorem. *Let $\mathscr{C}$ be a differentiable closed Jordan curve of class C^r on a plane E^2 in E^4. Let M be a tubiform surface which is the locus of circles with constant radius, lying in the normal planes to E^2 through all the points of $\mathscr{C}$ and whose centers are these points. Then M has zero Gaussian total curvature.*

We can easily see that any locally Euclidean, closed, orientable 2-dimensional Riemann manifold can be isometrically imbedded in E^4.

Now, since the right hand side of (5) is a quadratic form with respect to $\cos^2\theta$, in order that $K \leqslant 0$ for $0 \leqslant \theta \leqslant 2\pi$, it is necessary and sufficient that

$$\ddot{R} + R\tau^2(1 + \dot{R}^2) \geqslant 0, \tag{6}$$

and

$$\begin{aligned} & -2R^3\{\ddot{R} + R\tau^2(1 + \dot{R}^2)\} \\ & + 2R^4\{\tau^2(2 + 3\dot{R}^2) + R\dot{R}\tau\dot{\tau} - \tau^2 R\ddot{R}\} + 2R^6\tau^4 \\ = & -2R^3(1 + R^2\tau^2)\ddot{R} + 2R^4\tau^2(1 + 2\dot{R}^2) + 2R^5\dot{R}\tau\dot{\tau} + 2R^6\tau^4 \leqslant 0, \end{aligned}$$

that is

$$\ddot{R} \geqslant \frac{1}{1 + \tau^2 R^2}\{\tau^2 R(1 + 2\dot{R}^2) + \tau\dot{\tau}R^2\dot{R} + \tau^4 R^3\}, \tag{7}$$

putting $\cos\theta = 0, 1$ respectively in the right hand side of (5). Especially, in order that $K < 0$ for $0 \leqslant \theta \leqslant 2\pi$, it is necessary and sufficient to take only the signs of inequality in (6) and (7).

2. A constrution of surfaces of negative curvature.

The genus of a closed orientable surface everywhere with negative Gaussian total curvature $\geqslant 2$. Making use of argument stated in the last section, we will construct such a surface of genus 7 in E^4 as follows.

We take a fixed rectangular coordinate system in E^4 and let E^3 be the 3-dimensional subspace spanned by the first three axises of the coordinate system.

Firstly we shall constitute a figure composed of curves as indicated in Fig. 1 which has the properties as follows:

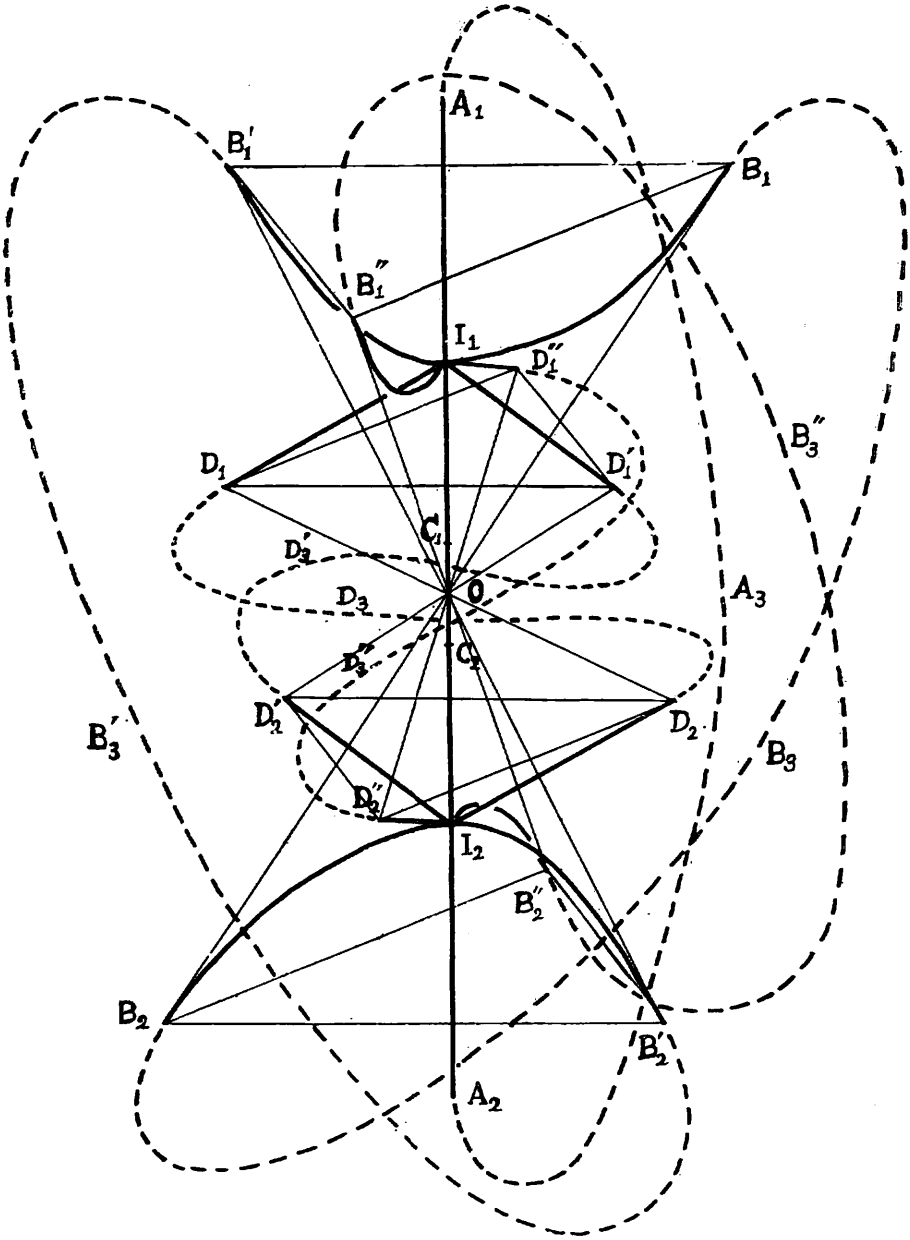

Fig. 1.

1) The part indicated by full lines lies in E^3 and is symmetric with respect to the point O.

2) The line segments $\overline{I_1A_1}$, $\overline{I_1D_1}$, the arc $\widehat{I_1B_1}$ lie on a plane, $\overline{OB_1}$ is tangent to the arc $\widehat{I_1B_1}$ at B_1 and $\angle OI_1D_1 < \pi/2$. $\overline{I_1D_1'}$, $\overline{I_1B_1'}$; $\overline{I_1D_1''}$, $\overline{I_1B_1''}$ are obtained from $\overline{I_1D_1}$, $\overline{I_1B_1}$ by rotating these through angles of $2\pi/3$, $4\pi/3$ around the axis OA_1 respectively in E^3.

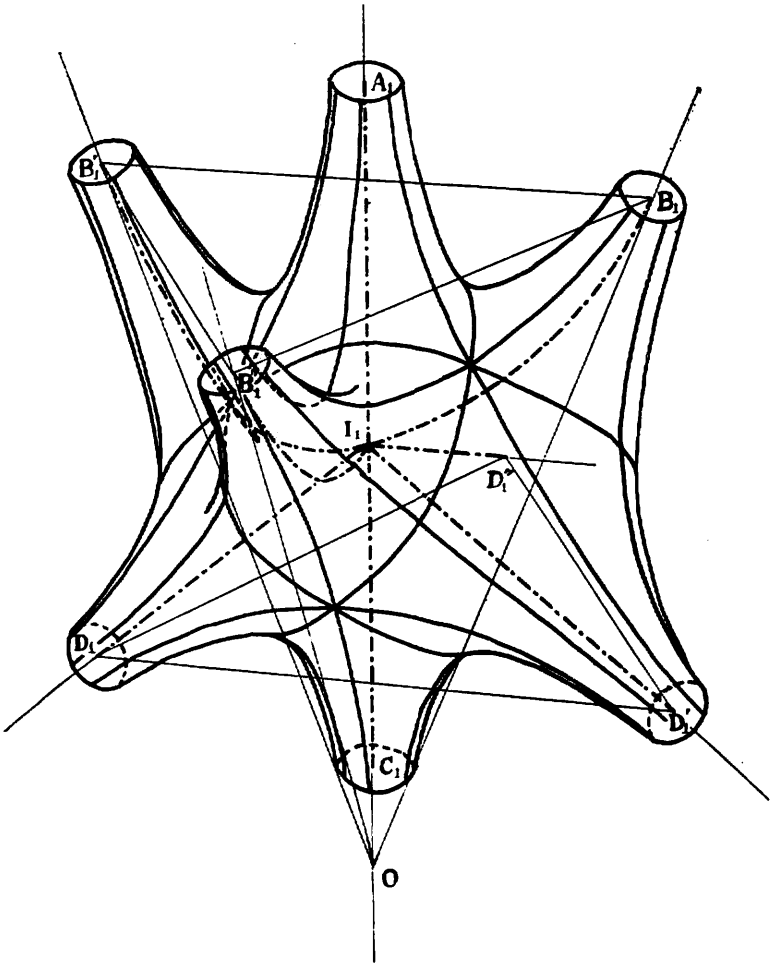

Fig. 2.

3) The curve $\widehat{A_1A_3A_2}$ lies on a plane which contains the line A_1A_2 and is parallel to the fourth axis of the coordinate system. The curve $\widehat{A_1A_3A_2}$ osculates of a suitable order with the straight line A_1A_2 at A_1 and A_2.

4) The curves $\widehat{B_1B_3B_2}$, $\widehat{B_1'B_3'B_2'}$, $\widehat{B_1''B_3''B_2''}$ are related to the straight lines B_1B_2, $B_1'B_2'$, $B_1''B_2''$ in the same manner as $\widehat{A_1A_3A_2}$ is related to the straight line A_1A_2 as stated above.

5) The curve $\widehat{D_1D_3D_2}$ lies in the 3-dimensional subspace which contains the plane $(D_1D_2I_1I_2)$ and is parallel to the fourth axis of the coordinate system. The curve $\widehat{D_1D_3D_2}$ is spiral and osculates of a suitable order with the lines I_1D_1, I_2D_2 at D_1, D_2 respectively. $\angle OD_1I_1$ is an obtuse angle.

6) The curves $\widehat{D_1'D_3'D_2'}$, $\widehat{D_1''D_3''D_2''}$ have the same property as $\widehat{D_1D_3D_2}$.

7) These curves does not intersect each other save for the points I_1, I_2.

In the next place, we shall constitute surfaces $\mathfrak{F}_1$, $\mathfrak{F}_2$ around the points I_1, I_2 in E^3 as indicated in Fig. 2 which have the properties as follow:

1) They have everywhere negative curvature.

2) The sections of the surfaces by the normal planes in E^3 to the curves at A_i, B_i, B_i', B_i'', C_i, D_i, D_i', D_i'', $i = 1, 2$, are circles with centers at these points. Along these circles the surfaces are approximately surfaces of revolution.

Lastly, we shall constitute tubiform surfaces everywhere with negative curvature around the curves $\widehat{A_1A_3A_2}$, $\widehat{B_1B_3B_2}$, $\widehat{B_1'B_3'B_1'}$, $\widehat{B_1''B_3''B_2''}$, $\widehat{D_1D_3D_2}$, $\widehat{D_1'D_3'D_2'}$, $\widehat{D_1''D_3''D_2''}$ and $\overline{I_1I_2}$ by the manner stated in Section 1.

We shall show in the following section that we can, in fact, construct a surface as stated above.

3. Spiral curves and inequalities (6), (7).

We can join D_1, D_2 by a suitable helix such that the straight lines I_1D_1, I_2D_2 are tangent to it at D_1, D_2 respectively. While the torsion of a helix is a constant $\neq 0$ and the one of a straight line is zero. Accordingly, making use of the curve, the metric of tubiform surface constituted as in Sections 1, 2 *around it becomes discontinuous at each point of the circles corresponding to* D_1, D_2 *by* (4). Hence it is necessary to compensate it so that the continuity of metric and also Gaussian total curvature is preserved at the joints.

Let us consider a spiral curve $\mathscr{C}$ in E^3 as follows:

$$(8)\qquad x = a\,\rho(\theta)\cos\theta,\quad y = a\,\rho(\theta)\sin\theta,\quad z = a\tan\beta\int_0^\theta \sqrt{\rho^2+\rho'^2}\,d\theta,$$

where the dash denotes the derivative with respect to θ, $\beta = \angle OD_1I_1 - \pi/2$, $a > 0$, $a\tan\beta\int_0^{2\pi}\sqrt{\rho^2+\rho'^2}\,d\theta = h = \overline{D_1D_2}$ and $\rho(\theta)$ is a function of class C^{r+1} as follows: For small positive constants ε_0, θ_0,

$$\rho(\theta) \neq 0,$$

$$(9)\qquad \begin{cases} \rho(\theta) = \dfrac{1+\varepsilon_0}{\cos\theta} & \text{for } -\dfrac{1}{2}\pi < \theta \leqq 0,\ 2\pi \leqq \theta < \dfrac{5}{2}\pi, \\ \rho(\theta) = 1 & \text{for } \theta_0 \leqq \theta \leqq 2\pi - \theta_0. \end{cases}$$

By the method of construction of the figure, we can take any positive angle near 0 for β.

Let us put

$$\max_{0\leqq\theta\leqq 2\pi}\sqrt{\rho(\theta)^2+\rho'(\theta)^2} = M.$$

Then, for the length $2l$ of the subarc $(0 \leqq \theta \leqq 2\pi)$ of $\mathscr{C}$, we have easily

$$(10)\qquad \begin{cases} l \leqq \pi a M \sec\beta, \\ h = l\sin\beta \leqq \pi a M\tan\beta. \end{cases}$$

The torsion of the curve $\mathscr{C}$ is given by the equation

$$\tau = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} \Big/ \{(y'z''-y''z')^2 + (z'x''-z''x')^2 + (x'y''-x''y')^2\}$$

as is well known. Putting $\tau = \varphi(\theta,\beta)/a$, $\varphi(\theta,\beta)$ is of class C^1 with respect to θ, β and especially analytic with respect to β. Furthermore it has the properties:

$$\begin{aligned} \varphi(\theta,\beta) &= 0 && \text{for } \theta \leqq 0,\ \theta \geqq 2\pi, \\ \varphi(\theta,\beta) &= \cos\beta\,\sin\beta && \text{for } \theta_0 \leqq \theta \leqq 2\pi - \theta_0. \end{aligned}$$

Since $\varphi(\theta, 0) = 0$, there exists a positive constant N_0 for $\beta_0 > 0$ such that

$$(11)\qquad |\varphi(\theta,\beta)| < N_0\beta \qquad \text{for } 0 < \beta \leqq \beta_0.$$

On the other hand, for the arclength u of the curve $\mathscr{C}$ we have

$$du = a \sec \beta \sqrt{\rho^2 + \rho'^2}\, d\theta,$$

hence

$$\frac{d\tau}{du} = \dot{\tau} = \frac{\cos \beta}{a^2} \frac{1}{\sqrt{\rho^2 + \rho'^2}} \frac{\partial \varphi(\theta, \beta)}{\partial \theta}.$$

Accordingly, there exists a positive constant N_1 such that

$$(12) \qquad |\tau| \leqslant \frac{N_0 \beta}{a}, \qquad |\dot{\tau}| \leqslant \frac{N_1 \beta}{a^2} \qquad \text{for} \quad 0 \leqslant \beta \leqslant \beta_0.$$

Now, by means of (6), (7), it is sufficient to verify that we can choose a function $R(u) > 0$ such that

$$(6') \qquad \ddot{R} > 0,$$

$$(7) \qquad (1 + R^2\tau^2)\ddot{R} > \tau^2 R(1 + 2\dot{R}^2) + \tau\dot{\tau}R^2\dot{R} + \tau^4 R^3$$

for $0 \leqslant u \leqslant l$, $\dot{R}(0) = 0$ and $\dot{R}(l)$ is a suitably small positive number. Furthermore, since we have $\dot{R}(u) > 0$ for $0 < u \leqslant l$, making use of (11), (12), it is more sufficient to replace (7) by the inequality

$$(13) \qquad \ddot{R} > \frac{N_0^2 \beta^2}{a^2} R(1 + 2\dot{R}^2) + \frac{N_0^4 \beta^4}{a^4} R^3 + \frac{N_0 N_1 \beta^2}{a^3} R^2 \dot{R}.$$

Now, putting $R = b_0 + b_1 u^2$, $b_0, b_1 > 0$, we shall verify that we can choose the constants b_0, b_1 such that R satisfies the condition (13) and the requirements stated above are fulfilled.

Substituting $R = b_0 + b_1 u^2$, $\dot{R} = 2b_1 u$, $\ddot{R} = 2b_1$ in (13), we get

$$2b_1 > \frac{N_0^2 \beta^2}{a^2}(b_0 + b_1 u^2)(1 + 8b_1^2 u^2) + \frac{N_0^4 \beta^4}{a^4}(b_0 + b_1 u^2)^3 + \frac{2N_0 N_1 \beta^2}{a^3}(b_0 + b_1 u^2)^2 b_1 u.$$

By means of (10), it is sufficient for our purpose if we can determine b_0, b_1 so that

$$2b_1 > \frac{N_0^2 \beta^2}{a^2}(b_0 + b_1 u^2)(1 + 8b_1^2 u^2) + \frac{N_0^4 \beta^4}{a^4}(b_0 + b_1 u^2)^3 + \frac{2\pi N_0 N_1 M \beta^2 \sec \beta}{a^3}(b_0 + b_1 u^2)^2 b_1.$$

We write the inequality above as

$$B_0 - B_1 u^2 - B_2 u^4 - B_3 u^6 > 0$$

where

$$(14)\quad \begin{cases} B_0 = 2b_1 - \dfrac{N_0^2\beta^2}{a^2}b_0 - \dfrac{N_0^4\beta^4}{a^4}b_0^3 - \dfrac{2\pi N_0 N_1 M\beta^2 \sec\beta}{a^2}b_0^2 b_1, \\ B_1 = \dfrac{N_0^2\beta^2}{a^2}b_1 + \left\{\dfrac{3N_0^4\beta^4}{a^4}b_0 + 4\left(\dfrac{2N_0^2\beta^2}{a^2} + \dfrac{\pi N_0 N_1 M\beta^2 \sec\beta}{a^2}\right)b_1\right\}, \\ B_2 = \left\{\dfrac{3N_0^4\beta^4}{a^4}b_0 + 2\left(\dfrac{4N_0^2\beta^2}{a^2} + \dfrac{\pi N_0 N_1 M\beta^2 \sec\beta}{a^2}\right)b_1\right\}b_1^2 \\ B_3 = \dfrac{N_0^4\beta^4}{a^4}b_1^3. \end{cases}$$

To simplify the evaluation, we shall make the function

$$f(v) = B_0 - B_1 v - B_2 v^2 - B_3 v^3$$

has no extreme value. The condition is evidently

$$B_2^2 - 3B_1 B_3 < 0.$$

If we can do so, $f(v)$ is monotone decreasing, hence $f(v) > 0$ for $0 \leqslant v \leqslant l^2$ when $f(l^2) > 0$.

Now, from (14) we have

$$B_2^2 - 3B_1 B_3 = \frac{N_0^2\beta^4}{a^4}\left\{-\frac{3N_0^4\beta^2}{a^2} + \frac{24N_0^4\beta^4}{a^2}b_0 b_1 + 4(4N_0 + \pi N_1 M\sec\beta)^2 b_1^2\right\}b_1^4.$$

Since $b_1, N_0, \beta \neq 0$, if

$$(15)\quad -\frac{3N_0^4\beta^2}{a^2} + \frac{24N_0^4\beta^2}{a^2}b_0 b_1 + 4(4N_0 + \pi N_1 M\sec\beta)^2 b_1^2 < 0,$$

$f(v)$ becomes a monotone decreasing function.

On the other hand, from the circumstance which is described in Sections 1, 2 we may put $\beta_0 = \pi/3$, that is $\sec\beta \leqslant 2$. Inequality (15) is satisfied by b_0, b_1 such that

$$(16)\quad \begin{cases} 4(4N_0 + 2\pi N_1 M)^2 b_1^2 < \dfrac{3N_0^4\beta^2}{a^2}, \\ \dfrac{24N_0^4\beta^2}{a^2}b_0 b_1 < \dfrac{3N_0^4\beta^2}{a^2} - 4(4N_0 + 2\pi N_1 M)^2 b_1^2, \end{cases}$$

of which the first gives

$$(17)\quad b_1 < \frac{\sqrt{3}\, N_0^2\beta}{4(2N_0 + \pi N_1 M)a}.$$

If $f(v)$ is monotone decreasing, $f(l^2) > 0$ holds good when $f(4\pi^2 a^2 M) > 0$ since $l \leqslant \pi a M \sec \beta \leqslant 2\pi a M$. The last inequality is fulfilled furthermore if we have

$$
(18)\qquad \begin{aligned}
&2b_1 - \frac{N_0^2\beta^2}{a^2} b_0 - \frac{N_0^4\beta^4}{a^4} b_0^3 - \frac{4N_0 N_1 M\beta^2}{a^2} b_0^2 b_1 \\
&- \left\{\frac{N_0^2\beta^2}{a^2} b_1 + \left(\frac{3N_0^4\beta^4}{a^4} b_0 + 8\left(\frac{N_0^2\beta^2}{a^2} + \frac{\pi N_0 N_1 M\beta^2}{a^2}\right) b_1\right) b_0 b_1\right\} (2\pi a M)^2 \\
&- \left\{\frac{N_0^4\beta^4}{a^4} b_0 + 4\left(\frac{2N_0^2\beta^2}{a^2} + \frac{\pi N_0 N_1 M\beta^2}{a^2}\right) b_1\right\} b_1^2 (2\pi a M)^4 \\
&- \frac{N_0^4\beta^4}{a^4} b_1^3 (2\pi a M)^6 > 0.
\end{aligned}
$$

Thus, we see that the conditions (6′), (7) are satisfied by $R = b_0 + b_1 u^2$ such that b_0, b_1 satisfy the inequalities (16), (17), (18).

For a constant b_1 satisfying (17), a sufficiently small positive constant b_0 satisfies (16). If we put $b_0 = 0$ in (18), we have

$$
\begin{aligned}
&2b_1 - 4\pi^2 N_0^2 M^2 \beta^2 b_1 - 64\pi^4 a^2 M^4 N_0 \beta^2 (2N_0 + \pi N_1 M) b_1^3 \\
&- 64\pi^6 a^2 N_0^4 M^6 \beta^4 b_1^3 > 2b_1 \Big[1 - 2\pi^2 N_0^2 M^2 \beta^2 \\
&- \frac{3N_0}{2(2N_0 + \pi N_1 M)} (2\pi^2 N_0^2 M^2 \beta^2)^2 - \frac{3N_0^2}{4(2N_0 + \pi N_1 M)^2} (2\pi^2 N_0^2 M^2 \beta^2)^3\Big],
\end{aligned}
$$

making use of (17).

This relation shows that the left hand side of (18) in which we put $b_0 = 0$ is positive for sufficiently small $\beta > 0$. In fact, the quantity inclosed in the brackets in the right hand side of the inequality above is rewritten as

$$
\begin{aligned}
&1 + \frac{2(2N_0 + \pi N_1 M)}{9N_0} \\
&\quad - \frac{2(2N_0 + \pi N_1 M)}{9N_0} \left\{1 + \frac{3N_0}{2(2N_0 + \pi N_1 M)} (2\pi^2 N_0^2 M^2 \beta^2)\right\}^3.
\end{aligned}
$$

Hence, this is positive for β such that

$$
(19)\qquad \beta < \left\{\frac{2N_0 + \pi N_1 M}{3\pi^2 N_0^3 M^2}\left(\left(1 + \frac{9N_0}{2(2N_0 + \pi N_1 M)}\right)^{\frac{1}{3}} - 1\right)\right\}^{\frac{1}{2}}.
$$

In conclusion, if we take β satisfying (19), in the next place b_1 satisfying (17) and lastly sufficiently small b_0 so that (16), (18) are fulfilled, then $R(u) = b_0 + b_1 u^2$ satisfies (6), (7).

On the other hand, from the manner in which the surface is constructed, it may be desirable that $R(l)$ is small. Let r, m be given positive numbers. We may here require the following conditions for $R(u)$:

$$R(l) < r, \qquad \dot{R}(l) < m.$$

They are sufficiently satisfied if we have

$$b_0 + 4\pi^2 a^2 M^2 b_1^2 < r,$$
$$4\pi a M b^1 < m.$$

Hence the conditions above are satisfied by b_0, b_1 such that

$$b_1 < \min\left(\frac{m}{4\pi a M}, \frac{\sqrt{r}}{2\pi a M}\right),$$
$$b_0 < r - 4\pi^2 a^2 M^2 b_1^2$$

which clearly do not contradict to the circumstance under which b_0, b_1 have been determined so that (16), (17), (18), (19) hold good. Now, since a does not appear in (19), we may condier in our construction of the surface that

$$OD_1 = h = l \sin\beta = a \tan\beta \int_0^\pi \sqrt{\rho^2 + \rho'^2}\, d\theta$$

can take any positive value.

Finally we call attention to the fact that when we take a curve $\widehat{D_2 D_3 D_1}$ in Fig. 1 such that it is represented as (8), the direction of the second vector of the Frenet frame at D_1 is the one of the fourth axis of the coordinate system, which can be easily verified. This shows that the surfaces $\mathfrak{F}_1$, $\mathfrak{F}_2$ and the tubiform surface constructed as above around the curve $\widehat{D_2 D_3 D_1}$ can be joined smoothly with one another.

From the argument above we obtain the theorem:

Theorem. *There exist closed surfaces everywhere with negative curvature and of genus 7 in Euclidean space of dimension 4.*

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(*Received January 5, 1954*)