SOME THEOREMS ON A SYSTEM OF MATRICES AND A GEOMETRICAL APPLICATION

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In this paper, we shall prove some theorems on a system of (n, m)-matrices and apply them to the proof of a theorem on isometrical imbedding of Riemann spaces in Euclidean spaces.

1. Let $\mathfrak{M}_{n,m}$ be the set of (n, m)-matrices. We define a multiplication as follows:

$$\mathfrak{M}_{n,m} \ni M_1, M_2, \qquad M_1 \circ M_2 = M_1 M_2' (\in \mathfrak{M}_{n,n}),$$

where M_2' denotes the transposed matrix of M_2. Let us denote the full linear group and the orthogonal group in n variables by $L(n)$ and $O(n)$ respectively. We will say that M_1 and M_2 is commutative if $M_1 \circ M_2 = M_2 \circ M_1$.

Lemma 1. *If M_1 and M_2 in $\mathfrak{M}_{n,m}$ is commutative, then AM_1B and AM_2B is also commutative for any $A \in L(n)$, $B \in O(n)$.*

The proof is evident from the definition of the multiplication of M_1 and M_2.

Let $\varphi_{A,B}: \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}$ $(A \in L(n),\ B \in L(m))$ be a linear transformation of $\mathfrak{M}_{n,m}$ onto itself such that $\varphi_{A,B}(M) = AMB$, $M \in \mathfrak{M}_{n,m}$. Then we have a corollary.

Corollary. *$\varphi_{A,B}$ $(A \in L(n),\ B \in O(m))$ is a linear homeomorphism on $\mathfrak{M}_{n,m}$ preserving the commutativity with respect to the multiplication $M_1 \circ M_2$.*

Now, we will say that a linear subspace $\mathfrak{M}$ of $\mathfrak{M}_{n,m}$ is an α-system if any two elements of $\mathfrak{M}$ are commutative. Our main theorem is as follows:

Theorem 1. *If $\mathfrak{N}$ is an α-system in $\mathfrak{M}_{n,m}$, then dim $\mathfrak{N} \leqslant m$.*

Proof. We shall prove inductively the theorem. In order to prove the theorem, it is sufficient that if we can prove the following propositions depending on positive integers n, m.

$(q_{n,m})$: *Let $\mathfrak{N}$ be any α-system in $\mathfrak{M}_{n,m}$. If $M_1, M_2, \cdots\cdots, M_h \in \mathfrak{N}$, $h \leqslant m$, are linearly independent, then*

$$rank\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_h \end{pmatrix} \geqslant h, \qquad (n, m = 1, 2, 3, \cdots\cdots).$$

$(p_{n,m})$: *The dimension of any α-system in $\mathfrak{M}_{n,m} \leqslant m$.*

Since $\mathfrak{M}_{1,m}$ is an m-dimensional vector space and our product of $M_1, M_2 \in \mathfrak{M}_{1,m}$ is the inner product of M_1, M_2, the propositions $(q_{1,m})$, $(p_{1,m})$, $m = 1, 2, 3, \cdots\cdots$ are evident. We suppose that the propositions $(q_{j,m})$, $(p_{j,m})$ $j = 1, 2, \cdots\cdots, n-1$, $m = 1, 2, \cdots\cdots$ are true.

Since $\mathfrak{M}_{n,1}$ is an n-dimensional vector space, it is evident that the propositions $(q_{n,1})$, $(p_{n,1})$ hold good.

Now, we suppose that the propositions $(q_{n,1})$, $(p_{n,1})$, $\cdots\cdots$, $(p_{n,m-1})$, $(q_{n,m})$ have already been proved.

Proof of Proposition $(p_{n,m})$. Let $\mathfrak{N}$ be an α-system in $\mathfrak{M}_{n,m}$. Let $M_1, M_2, \cdots\cdots, M_m, M_{m+1}$ be a set of elements of $\mathfrak{N}$ such that any m elements of them are linearly independent. We shall prove that M_1, $M_2, \cdots\cdots, M_m, M_{m+1}$ are linearly dependent.

Taking any transformation

$$N_i = \sum_{j=1}^{m+1} c_{ij} M_j, \qquad i = 1, 2, \cdots\cdots, m+1,$$

where $|c_{ij}| \neq 0$, we may investigate the linear dependency of N_1, $\cdots\cdots, N_m, N_{m+1}$ instead of the one of $M_1, M_2, \cdots\cdots, M_{m+1}$. Accordingly, we may consider that M_{m+1} is of the form $\begin{pmatrix} 0 \cdots\cdots 0 \\ * \end{pmatrix}$. The relation $M_i \circ M_{m+1} = M_{m+1} \circ M_i$ $(i = 1, 2, \cdots\cdots, m)$ implies that if we consider each row of any element of $\mathfrak{M}_{n,m}$ indicates a vector in an m-dimensional Euclidean vector space, then the first rows of M_1, $\cdots\cdots, M_m$ are orthogonal to each rows of M_{m+1}. Since $M_{m+1} \neq 0$, the first rows of $M_1, \cdots\cdots, M_m$ lie in an $(m-1)$-dimensional linear subspace in the m-dimensianal Euclidean vector space. Hence, we may also put the first row of M_m is the zero vector. The first rows of $M_1, \cdots\cdots, M_{m-1}$ are orthogonal to each rows of M_m, M_{m+1} by the same reason above. By virtue of $(q_{n,m})$, rang $\begin{pmatrix} M_m \\ M_{m+1} \end{pmatrix} \geqslant 2$. Accordingly, the first rows of $M_1, \cdots\cdots, M_{m-1}$ lie in an $(m-2)$-dimensional linear subspace in the $(m-1)$-dimensional Euclidean vector space above. Hence, we may also put the first row of M_{m-1} is the zero vector. By virtue of $(q_{n,m})$, we can repeat this process, hence we may consider the first rows of $M_1, \cdots\cdots, M_{m+1}$ are the zero vector. If we perform inductively the same consideration with respect to n, we see that we may put $M_{m+1}=0$. This contradicts to $M_{m+1}\neq 0$. The proof of $(p_{n,m})$ is complete.

Proof of Proposition $(q_{n,m+1})$. Let $\mathfrak{N}$ be any α-system in $\mathfrak{M}_{n,m+1}$. Let us suppose that $M_1, \cdots\cdots, M_h \in \mathfrak{N}$, $h \leqslant m+1$, are linearly independent and

$$rank\begin{pmatrix} M_1 \\ \vdots \\ M_h \end{pmatrix} = r < h.$$

Then, if we take a suitable $B \in O(m+1)$ and we transform $\mathfrak{N}$ to $\varphi_{1,B}(\mathfrak{N})=\mathfrak{N}B$, then we may put $M_i B = (N_i\ 0)$ where $N_i \in \mathfrak{M}_{n,r}$. Hence we may put $M_i = (N_i\ 0)$ from the beginning. Then, $N_1, \cdots\cdots, N_h$ belong to an α-system in $\mathfrak{M}_{n,r}$. Since $r \leqslant m$, the proposition $(p_{n,r})$ holds good by induction and the proposition $(p_{n,m})$ which has been proved above. Accordingly, it follows that $N_1, N_2, \cdots\cdots, N_h$ are linearly dependent. Hence $M_1, M_2, \cdots\cdots, M_h$ must be also linearly dependent. This contradicts to the assumption above. Thus Proposition $(q_{n,m+1})$ holds good. The proof of Theorem 1 is complete.

2. Let $\mathfrak{B}$ be a vector space over the real field $\mathfrak{k}$. An exterior form of order 2 on $\mathfrak{B}$ is defined by a form $\sum_{i,j=1}^{r}\alpha_{ij}\, u_i \wedge u_j$, where $\alpha_{ij} = -\alpha_{ji} \in \mathfrak{B}$ and u_i are real variables. We define $\sum \alpha_{ij}\, u_i \wedge u_j = \sum \beta_{ij}\, v_i \wedge v_j$ if $u_i = \sum a_{ik} v_k$, $a_{ik} \in \mathfrak{k}$, $|\, a_{ik}\,| \neq 0$ and $\sum_{i,j} \alpha_{ij}\, a_{ik} a_{jh} = \beta_{kh}$, $h, k = 1, 2, \cdots\cdots, r$. Then, we have a lemma as a simple generalization of the ordinary one as follows.

Lemma 2. *Let k be the minimum of numbers of variables such that an exterior form $\sum_{i,j=1}^{r} \alpha_{ij}\, u_i \wedge u_j$ of order 2 can be expressed by them, then $r-k$ is the dimension of the set of solutions of the system of linear equations over $\mathfrak{B}$ with respect to real variables $x_1, \cdots\cdots, x_r$; $\sum_{j=1}^{r} \alpha_{ij}\, x_j = 0$, $i = 1, 2, \cdots\cdots, r$.*

We can easily prove the lemma in the same manner as the ordinary case in which $\mathfrak{B}$ is $\mathfrak{k}$.

Now let $\mathfrak{M} \subset \mathfrak{M}_{n,m}$ be a linear subspace. Let $M_1, \cdots\cdots, M_r$ be a base of $\mathfrak{M}$ and put $N_{ij} = M_i \circ M_j - M_j \circ M_i$ $(\in \mathfrak{M}_{n,n})$. We construct an exterior form of the vector space $\mathfrak{M}_{n,n}$ over the real field $\mathfrak{k}$ such that

$$\Phi(\mathfrak{M}, u) = \sum N_{ij} \ u_i \wedge u_j .$$

This form is dependent on $\mathfrak{M}$ but independent on the choice of bases. For, we have

$$(\sum x_i M_i) \circ (\sum y_j M_j) - (\sum y_i M_i) \circ (\sum x_j M_j) = \frac{1}{2} \sum N_{ij}(x_i y_j - x_j y_i).$$

We call the exterior form $\Phi(\mathfrak{M}, u)$ the *exterior form associated with* $\mathfrak{M}$.

Now let $(x_{i(\lambda)})$, $\lambda = k+1, \cdots\cdots, r$ be a base of the system of solutions of the following linear equations in r variables

$$\sum N_{ij} x_j = 0, \qquad i = 1, 2, \cdots\cdots, r.$$

Then, $K_\lambda = \sum x_{i(\lambda)} M_i$, $\lambda = k+1, \cdots\cdots, r$, are linearly independent. For otherwise, there exist constants c_λ, $\lambda = k+1, \cdots\cdots, r$, such that they are not all zero and $\sum c_\lambda K_\lambda = 0$. This implies $\sum c_\lambda x_{i(\lambda)} M_i = 0$. Since $M_1, \cdots\cdots, M_r$ are linearly independent, it follows $\sum_\lambda c_\lambda x_{i(\lambda)} = 0$. This contradicts to the fact that $(x_{i(\lambda)})$, $\lambda = k+1, \cdots\cdots, r$ are linearly independent.

On the other hand, we have

$$\begin{aligned} K_\lambda \circ K_\mu - K_\mu \circ K_\lambda &= (\sum x_{i(\lambda)} M_i) \circ (\sum x_{j(\mu)} M_j) - (\sum x_{i(\mu)} M_i) \circ (\sum x_{j(\lambda)} M_i) \\ &= N_{ij} x_{i(\lambda)} x_{j(\mu)} = 0. \end{aligned}$$

Hence, $K_{k+1}, \cdots\cdots, K_r$ become a base of an α-system in $\mathfrak{M}$. We denote the α-system by α-$(\mathfrak{M})$ and call it the *α-system associated with* $\mathfrak{M}$. We prove the appropriateness of this notation.

Let $\bar{M}_i = \sum M_j a_{ji}$, $| a_{ji} | \neq 0$, be another base of $\mathfrak{M}$. Let us put $\bar{N}_{ij} = \bar{M}_i \circ \bar{M}_j - \bar{M}_j \circ \bar{M}_i = N_{kh} a_{ki} a_{hj}$. Then, the system of linear equations $\sum \bar{N}_{ij} \bar{x}_j = 0$, $i = 1, 2, \cdots\cdots, r$, becomes $\sum_{k,j} N_{ik} a_{kj} \bar{x}_j = 0$, $i = 1, 2, \cdots\cdots, r$. Accordingly, we may put $x_{i(\lambda)} = \sum_j a_{ij} \bar{x}_{j(\lambda)}$. It follows that $K_\lambda = \sum x_{i(\lambda)} M_i = \sum a_{ij} \bar{x}_{j(\lambda)} M_j = \sum \bar{x}_{j(\lambda)} \bar{M}_j = \bar{K}_\lambda$. This shows that the above stated α-system α-$(\mathfrak{M})$ is independent of the choice of base of $\mathfrak{M}$.

Thus, by means of Theorem 1, Lemma 2 and the argument above, we obtain a theorem as follows.

Theorem 2. *Let $\mathfrak{M}$ be a r-dimensional linear subspace in $\mathfrak{M}_{n,m}$ and k be the minimum of numbers of variables such that $\Phi(\mathfrak{M})$ can be expressed by them. Then dim* $\alpha\text{-}(\mathfrak{M}) = r - k \leqslant m$.

3. In this section, we shall apply Theorem 3 to a geometrical problem.

Let V_n be a Riemann manifold of dimension n whose line element is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)\,dx_i dx_j$$

in local coordinates $x_1, x_2, \cdots\cdots, x_n$. Let us put

$$\begin{aligned}\sum g_{ij}(x)\,dx_i dx_j &= \sum_{i=1}^{n} \omega_i(x, dx)\,\omega_i(x, dx),\\ d\omega_i &= \sum \omega_j \wedge \omega_{ji},\\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},\end{aligned}$$

where Ω_{ij} are the curvature forms of V_n as is well known.

Let $k(p)$ be the minimum number of linear differential forms in terms of which the curvature forms at $p \in V_n$ can be expressed, and let $k = max_{p \in V_n} k(p) = k(V_n)$. According to S. S. Chern and N. H. Kuiper, $n - k(p)$ is called the *index of nullity at p.*

Theorem 3. *A compact Riemann manifold V_n of dimenison n can not be isometrically imbedded in an Euclidean space of dimension* $2n - k(V_n) - 1$.

Proof. We suppose that such an imbedding of V_n in E^{2n-k-1} exists. Let $(p, e_1, \cdots\cdots, e_n, e_{n+1}, \cdots\cdots, e_{2n-k-1})$ be a field of orthonormal frames of E^{2n-k-1} defined on a coordinate neighborhood of V_n such that $e_1, e_2, \cdots\cdots, e_n$ are tangent vectors at p to V_n. Then we have

$$\begin{aligned}dp &= \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha,\\ de_\alpha &= \sum_i \omega_{\alpha i} e_i + \sum_\beta \omega_{\alpha\beta} e_\beta,\\ &\qquad i, j = 1, 2, \cdots\cdots, n;\ \alpha, \beta = n+1, \cdots\cdots, 2n-k-1\end{aligned}$$

and $ds^2 = dp\,dp = \sum \omega_i \omega_i$. These relations give

$$0 = \sum_i d\omega_i e_i - \sum_i \omega_i \wedge de_i = \sum_i (d\omega_i - \sum_k \omega_k \wedge \omega_{ki}) e_i - \sum_k \omega_k \wedge \omega_{k\alpha} e_\alpha,$$

that is,

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki}, \qquad \sum_k \omega_k \wedge \omega_{k\alpha} = 0.$$

The second equation implies

$$\omega_{in+\lambda} = \sum A_{\lambda ij}\omega_j, \quad A_{\lambda ij} = A_{\lambda ji}, \qquad \lambda = 1, 2, \cdots\cdots, n-k-1.$$

The quadratic differential forms $\Phi_\lambda = \sum A_{\lambda ij}\omega_i\omega_j$, $\lambda = 1, 2, \cdots\cdots, n-k-1$, are the so-called second fundamental forms of V_n. We have analogously

$$\begin{aligned} 0 &= \sum d\omega_{ij}e_j + \sum d\omega_{i\alpha}e_\alpha - \sum \omega_{ij}\wedge de_j - \sum \omega_{i\alpha}\wedge de_\alpha \\ &= \sum_j (d\omega_{ij} - \sum_k \omega_{ik}\wedge\omega_{kj} - \sum_\alpha \omega_{i\alpha}\wedge\omega_{\alpha j})e_j \\ &\qquad + \sum (d\omega_{i\alpha} - \sum_k \omega_{ik}\wedge\omega_{k\alpha} - \sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha})e_\alpha . \end{aligned}$$

It follows that

$$\Omega_{ij} = d\omega_{ij} - \sum \omega_{ik}\wedge\omega_{kj} = \sum \omega_{i\alpha}\wedge\omega_{\alpha j} = -\sum A_{\lambda it}A_{\lambda jh}\omega_t\wedge\omega_h .$$

Accordingly, $k(p)$ is the number of linearly independent forms of

$$\sum_h (A_{\lambda it}A_{\lambda jh} - A_{\lambda jt}A_{\lambda ih})\,\omega_h .$$

Let M_h be the $(n, n-k-1)$-matrix whose (i, λ)-element is $A_{\lambda ih}$. Then, $\sum (A_{\lambda it}A_{\lambda jh} - A_{\lambda jt}A_{\lambda ih})$ is (i, j)-element of $M_t \circ M_h - M_h \circ M_t = N_{th}$. Hence $n - k(p)$ is the dimension of solutions $(y_1, \cdots\cdots, y_n)$ of linear equations

$$\sum_h N_{th}y_h = 0. \qquad\qquad t = 1, 2, \cdots\cdots, n.$$

On the other hand, $M_1, \cdots\cdots, M_n$ are linearly independent at a point p, since V_n is compact and $V_n \subset E^{2n-k-1}$. By Theorem 3, we must have, at the point p, $n - k(p) \leqslant n-k-1$. This contradicts to the definition of $k(p)$. The proof is complete.

REFERENCES

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