ON SOME RIEMANN SPACES

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In the present paper, we shall investigate the existence of n-dimensional Riemann spaces V_n whose Ricci tensors K_i^j[1] $(i, j = 1, 2, \cdots\cdots, n)$ satisfy the following conditions:

$$\text{(a)} \qquad K_i^k K_k^j = \frac{1}{n-1} K_k^k K_i^j,$$

and

$$\text{(b)} \qquad K_{i;k}^j = 0,$$

where a semicolon ";" denotes the covariant differentiation of V_n. If $n > 2$, Einstein spaces[2] are characterized by

$$K_i^k K_k^j = \frac{1}{n} K_k^k K_i^j,$$

for the relation implies $K_i^j = \frac{1}{n} K_k^k \delta_i^j$, from which we get $K_{i;k}^j = 0$ as is well known. Accordingly, we may formally regard spaces whose Ricci tensors satisfy the condition (a) or the conditions (a), (b) as analogues for Einstein spaces. In a previous paper[3], the author proved that *a Riemann space V_n whose Ricci tensor satisfies the conditions* (a), (b) *can be imbedded, as a hypersurface, in a Riemann space V_{n+1} which has the following property*:

The group of holonomy of the space with a normal projective connexion corresponding to V_{n+1} fixes a hyperquadric and V_n is its image in V_{n+1}, that is, the locus of points lying on the parallel displaced hyperquadrics, regarded as points in the tangent projective spaces.

In Part I, we shall prove the existence of Riemann spaces V_n whose Ricci tensors satisfy the condition (a) and are of non trivial types.

In Part II, we shall investigate spaces whose Ricci tensors satisfy the conditions (a), (b) and prove the existence of such spaces with some additional properties.

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1) L. P. Eisenhart, Riemannian geometry, p. 22.

2) ibid. p. 92.

3) T. Ōtsuki, On the spaces with normal projective connexions and some imbedding problem of Riemannian spaces II, Math. J. of Okayama University, Vol. 2, No. 1, 1952, Theorem 2.

Part I

§1. Preliminaries

Let V_n be an n-dimensional Riemann space with positive definite line element

$$ds^2 = g_{ij}(x)\,dx^i dx^j \qquad (i, j = 1, 2, \cdots\cdots, n)$$

in each of its coordinate neighborhoods. Let Γ^i_{jk} be the Christoffel symbols made by g_{ij}.

$$K_j{}^i{}_{hk} = \frac{\partial \Gamma^i_{jh}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^h} + \Gamma^m_{jh}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{mh},$$

$$K_{jh} = K_j{}^k{}_{hk}, \qquad K = g^{ij}K_{ij}$$

are the components of the curvature tensor, the Ricci tensor and the scalar curvature of V_n.

In a suitable coordinate neighborhood $x^1, x^2, \cdots\cdots, x^{n-1}, y = x^n$, the line element may be represented by the form

$$(1) \qquad ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda dx^\mu + (\psi(x, y)\,dy)^2 \qquad (\lambda, \mu = 1, 2, \cdots\cdots, n-1).$$

Let $V_{n-1}(y)$ be an $(n-1)$-dimensional subspace of V_n on which y is constant and whose line element is $ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda dx^\mu$. Let $R_{\lambda\mu\nu\sigma}$, $R_{\lambda\mu}$, R be the components of the curvature tensor, the Ricci tensor in the coordinates $x^1, x^2, \cdots\cdots, x^{n-1}$, the scalar curvature of $V_{n-1}(y)$. Then we have the following relations

$$(2) \quad \begin{cases} \dfrac{\partial}{\partial y} g_{ab} = -2\psi h_{ab}, \\ \Gamma^a_{bc} = \left\{ {a \atop bc} \right\}, \quad \Gamma^n_{ab} = \dfrac{1}{\psi} h_{ab}, \quad \Gamma^a_{bn} = -\psi h^a_b, \\ \Gamma^n_{an} = \dfrac{1}{\psi}\psi_{,a}, \quad \Gamma^a_{nn} = -\psi g^{ab}\psi_{,b}, \quad \Gamma^n_{nn} = \dfrac{1}{\psi}\dfrac{\partial\psi}{\partial y}, \end{cases}$$

$$(a, b, c = 1, 2, \cdots\cdots, n-1)$$

where $\left\{ {a \atop bc} \right\}$ are the Christoffel symbols made by g_{ab}. In the following, we shall denote the covariant differentiation in $V_{n-1}(y)$ by a comma ",". Making use of Gauss-Codazzi equations[1)] of V_n

$$K_{abcd} = R_{abcd} - h_{ac}h_{bd} + h_{ad}h_{bc},$$

$$\psi K_a{}^n{}_{bc} = h_{ab,c} - h_{ac,b}$$

1) J. A. Schouten and D. J. Struik, Einfuhrung in die neueren Methoden der Differentialgeometrie, 1953, Vol. 2, p. 121.

and

$$K_{a\ bn}^{\ n} = \frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} + h_a^c h_{bc} - \frac{1}{\psi}\psi_{,ab},$$

we have

$$(3)\qquad \begin{cases} K_a^b = \dfrac{1}{\psi}\dfrac{\partial}{\partial y}h_a^b - h h_a^b + R_a^b - \dfrac{1}{\psi}g^{b\lambda}\psi_{,a\lambda}, \\ K_n^a = \psi V^a, \quad K_a^n = \dfrac{1}{\psi}V_a, \\ K_n^n = \dfrac{1}{\psi}\dfrac{\partial h}{\partial y} - h_\lambda^\mu h_\mu^\lambda - \dfrac{1}{\psi}g^{\lambda\mu}\psi_{,\mu\lambda} \\ K = \dfrac{2}{\psi}\dfrac{\partial h}{\partial y} - hh - h_\mu^\lambda h_\mu^\lambda + R - \dfrac{2}{\psi}g^{\lambda\mu}\psi_{,\lambda\mu}, \end{cases}$$

where we put

$$(4)\qquad \begin{cases} V_a = h_{,a} - h_{a,\lambda}^\lambda, \qquad h = h_\lambda^\lambda, \\ Q_a^b = h h_a^b - h_a^\lambda h_\lambda^b - R_a^b. \end{cases}$$

Now, let us suppose that the space V_n satisfies the condition (a). Then, we can easily see that rank of matrix (K_i^j) is 0 or $n-1$ according to $K=0$ or $\neq 0$. If $K=0$, it follows $K_i^j=0$. Hence, we have a theorem.

Theorem 1. *n-dimensional Riemann spaces whose Ricci tensors satisfy the condition* (a) *and whose scalar curvatures are zero are Einstein spaces* ($n>2$) *or locally euclidean spaces* ($n \geqslant 2$).

From now on, we assume that $K \neq 0$, that is, rank $(K_i^j) = n-1$. Under the circumstances above-mentioned, we can rewrite (a), by means of quantities of $V_{n-1}(y)$, as

$$(5)\qquad K_a^\lambda K_\lambda^b + V_a V^b = \frac{1}{n-1}KK_a^b,$$

$$(6)\qquad K_a^\lambda V_\lambda = \left(\frac{1}{n-1}K - K_n^n\right)V_a,$$

$$(7)\qquad V_\lambda V^\lambda = \left(\frac{1}{n-1}K - K_n^n\right)K_n^n.$$

We distinguish two cases as follows.

Case (I): $V_\lambda V^\lambda = 0$.

If $K_n^n = 0$, we get from (5)

(8) $$K_{ab} = \frac{1}{n-1} K g_{ab},$$

since $K = K_\lambda^\lambda \neq 0$. If $K_n^n \neq 0$, from (7) we get $(n-2)K_n^n = K_\lambda^\lambda$. Hence, if $n > 2$, we get

$$\frac{1}{n-1} K = K_n^n = \frac{1}{n-2} K_\lambda^\lambda \neq 0,$$

and then (5) becomes

$$K_a^\lambda K_\lambda^b = \frac{1}{n-2} K_\lambda^\lambda K_a^b.$$

This is the condition (a) replaced n with $n-1$. If $n = 2$, the case is regarded as the case $K_n^n = 0$ since $K_1^1 = 0$.

Case (II): $V_\lambda V^\lambda \neq 0$.

$\frac{1}{n-1} K - K_n^n$ and V_a are an eigen value and an eigen vector of this value of square matrix (K_a^b) of order $n-1$. Choosing a suitable orthonormal frame at any point in $V_{n-1}(y)$, we may put

$$(K_a^b) = \begin{pmatrix} \rho_1 & 0 & \cdots & \cdots & 0 \\ 0 & \rho_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \cdot & 0 \\ 0 & \cdots & \cdots & 0 & \rho_{n-1} \end{pmatrix}, \quad (V_a) = \begin{pmatrix} v \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

$$\rho_1 = \frac{1}{n-1} K - K_n^n.$$

Then, we get from (5) the relations

$$\rho_1\left(\rho_1 - \frac{1}{n-1} K\right) + vv = -\rho_1 K_n^n + vv = 0,$$

$$\rho_2\left(\rho_2 - \frac{1}{n-1} K\right) = 0$$

$$\vdots$$

$$\rho_{n-1}\left(\rho_{n-1} - \frac{1}{n-1} K\right) = 0.$$

(7) becomes

$$vv = \left(\frac{1}{n-1} K - K_n^n\right) K_n^n = \rho_1 K_n^n.$$

Since rank $(K_i^j) = n-1$, it follows that

$$\rho_2 = \rho_3 = \cdots\cdots = \rho_{n-1} = \frac{1}{n-1}K,$$

that is, with respect to the frame, (K_i^j) is of the form

$$(K_i^j) = \begin{pmatrix} \frac{1}{n-1}K - K_n^n & 0 & \cdots\cdots\cdots\cdots & & \frac{1}{\psi}v \\ 0 & \frac{1}{n-1}K & \cdots\cdots\cdots\cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \frac{1}{n-1}K & 0 \\ \psi v & 0 & \cdots\cdots\cdots & 0 & K_n^n \end{pmatrix}.$$

By virtue of these relations, with respect to a natural frame of $V_{n-1}(y)$, we get

$$K_a^b = \frac{1}{n-1}K\delta_a^b - \frac{K_n^n}{V_\lambda V^\lambda}V_a V^b. \tag{9}$$

§2. Systems of differental equations

In the paragraph, we shall derive systems of differential equations for our construction of spaces stated in the introduction from the formulas obtained in §1.

In Case (I), we may put $K_n^a = K_a^n = K_n^n = 0$ and $K_{ab} = \frac{1}{n-1}Kg_{ab}$. Accordingly, making use of (3), we obtain the relations

$$\frac{1}{n-1}K\delta_a^b = \frac{1}{\psi}\frac{\partial}{\partial y}h_a^b - hh_a^b + R_a^b - \frac{1}{\psi}g^{b\lambda}\psi_{,\lambda},$$

$$V_a \equiv h_{,a} - h_{a,\lambda}^\lambda = 0,$$

$$\frac{\partial}{\partial y}h - \psi h_\lambda^\mu h_\mu^\lambda - g^{\lambda\mu}\psi_{,\lambda\mu} = 0.$$

On the other hand, we get by (3) and the last equation above

$$\begin{aligned} K &= \frac{2}{\psi}\frac{\partial}{\partial y}h - hh - h_\lambda^\mu h_\mu^\lambda + R - \frac{2}{\psi}g^{\lambda\mu}\psi_{,\lambda\mu} \\ &= \frac{2}{\psi}\left(\frac{\partial}{\partial y}h - \psi h_\lambda^\mu h_\mu^\lambda - g^{\lambda\mu}\psi_{,\lambda\mu}\right) + h_\lambda^\mu h_\mu^\lambda - hh + R, \end{aligned}$$

that is,

$$K = -Q = -hh + h^\mu_\lambda h^\lambda_\mu + R. \tag{10}$$

Hence, we obtain

$$\frac{\partial}{\partial y} h^b_a = \psi(hh^b_a - R^b_a) - \frac{\psi}{n-1}\delta^b_a(hh - h^\mu_\lambda h^\lambda_\mu - R) + g^{b\lambda}\psi_{,a\lambda}, \tag{11}$$

which is represented only by h^b_a, ψ and R^b_a.

In Case (II), since we get from (3) $K = 2K^n_n - Q$, (7) becomes

$$\frac{n-3}{n-1}(K^n_n)^2 + \frac{1}{n-1}QK^n_n + V_\lambda V^\lambda = 0.$$

If $n \neq 3$, it follows

$$K^n_n = \frac{1}{2(n-3)}\left(-Q \pm \sqrt{QQ - 4(n-1)(n-3)V_\lambda V^\lambda}\right) \equiv F, \tag{12}$$

hence K is a function of h^b_a, R^b_a and V_a. Accordingly, we get

$$\frac{\partial}{\partial y} h = \psi(h^\mu_\lambda h^\lambda_\mu + F) + g^{\lambda\mu}\psi_{,\lambda\mu}.$$

On the other hand, we get also from (3) and (12)

$$\frac{\partial}{\partial y} h^b_a = \psi(hh^b_a - R^b_a) + g^{b\lambda}\psi_{,a\lambda} + \psi\left(\frac{2F-Q}{n-1}\delta^b_a - \frac{F}{V_\lambda V^\lambda}V_a V^b\right), \tag{13}$$

which derives the relation above.

If $n = 3$, (7) becomes

$$QK^n_n + 2V_\lambda V^\lambda = 0.$$

By the assumption $V_\lambda V^\lambda \neq 0$, it must be $Q \neq 0$. Hence we get from (3)

$$\frac{\partial}{\partial y} h = \psi h^\lambda_\mu h^\mu_\lambda + g^{\lambda\mu}\psi_{,\lambda\mu} - 2\psi\frac{V_\lambda V^\lambda}{Q}$$

and

$$K = 2K^n_n - Q = -\left(\frac{4V_\lambda V^\lambda}{Q} + Q\right).$$

Accordingly we get the equation

$$\begin{aligned} \frac{\partial}{\partial y} h^b_a &= \psi(hh^b_a - R^b_a) + g^{b\lambda}\psi_{,a\lambda} \\ &\quad + 2\psi\left\{\frac{1}{Q}(V_a V^b - V_\lambda V^\lambda \delta^b_a) - \frac{Q}{4}\delta^b_a\right\} \end{aligned} \tag{13'}$$

which derives the differential equation above.

§3. Constructions of spaces

In the following, we replace n with $n+1$. Let V_n be a given n-dimensional Riemann space with line element

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu \qquad \lambda, \mu, \cdots\cdots = 1, 2, \cdots\cdots, n^{1)}$$

is each of its coordinate neighborhoods. We shall construct an $(n+1)$-dimensional Riemann space V_{n+1} satisfying the condition (a) whose line element is

$$ds^2 = g_{\lambda\mu}(x, y)dx^\lambda dx^\mu + (\psi(x, y)dy)^2$$

in coordinates $x^1, \cdots\cdots, x^n, y$ and $g_{\lambda\mu}(x) = [g_{\lambda\mu}(x, y)]_{y=0}$.

In Case (I), by means of (2), (11), if we have a solution $g_{ab}(x, y)$, $h_{ab}(x, y)$ of the system of differential equations

$$(14)\qquad \begin{cases} \dfrac{\partial}{\partial y}g_{ab} = -2\psi h_{ab}, \\ \dfrac{\partial}{\partial y}h_a^b = \psi(hh_a^b - R_a^b) - \dfrac{\psi}{n}\delta_a^b(hh - h_\lambda^\mu h_\mu^\lambda - R) + g^{b\lambda}\psi_{,a\lambda} \end{cases}$$

under the condition

$$(15)\qquad V_a \equiv h_{,a} - h_{a,\lambda}^\lambda = 0$$

and the initial condition $g_{ab}(x) = [g_{ab}(x, y)]_{y=0}$, then V_{n+1} with line element $ds^2 = g_{\lambda\mu}(x, y)dx^\lambda dx^\mu + (\psi(x, y)dy)^2$ is a space which satisfy the condition (a).

Now, let $\psi(x, y)$ be a scalar of V_n depending on y which we shall restrict in future. Let $g_{ab}(x, y)$, $h_a^b(x, y)$ be any solution of the system (14), construct the vector V_a by them. Then, by means of (2), (14), (15) we get

$$\begin{aligned} \frac{\partial}{\partial y}V_a &= \left(\frac{\partial}{\partial y}h\right)_{,a} - \left(\frac{\partial}{\partial y}h_a^\lambda\right)_{,\lambda} + (\psi h)_{,\lambda}h_a^\lambda - (\psi h_\lambda^\mu)_{,\mu}h_\mu^\lambda \\ &= \psi h_\lambda^\mu h_{\mu,a}^\lambda + \psi_{,\lambda}R^{\mu\lambda}{}_{\lambda a} + \psi_{,\lambda}R_a^\lambda - \psi\left(hh_{a,\lambda}^\lambda - \frac{1}{2}R_{,a}\right) \\ &\quad + \frac{1}{n}\psi_{,a}Q + \frac{1}{n}\psi(2hh_{,a} - 2h_\lambda^\mu h_{\mu,a}^\lambda - R_{,a}), \end{aligned}$$

that is

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1) In this paragraph, indices take the following values:
$a, b, c, \cdots\cdots;\ \lambda, \mu, \nu, \cdots\cdots = 1, 2, \cdots\cdots, n.$

(16) $$\frac{\partial}{\partial y} V_a = \psi h V_a - \frac{(n-2)}{2n} \psi Q_{,a} + \frac{1}{n} \psi_{,a} Q .$$

If $n > 2$, it is written as

$$\frac{\partial}{\partial y} V_a = \psi h V_a - \frac{n-2}{2n} \psi^{\frac{n}{n-2}} \left(Q \psi^{\frac{-2}{n-2}} \right)_{,a} .$$

If $g_{ab}(x, y)$, $h_a^b(x, y)$ satisfy (15), the following relation must hold good

(17) $$\eta \equiv Q - \rho(y) \psi^{\frac{2}{n-2}} = 0,$$

where $\rho(y)$ is a suitable function of y. Hence, if we put $f = \rho(y) \psi^{\frac{2}{n-2}}$, then we get

$$\begin{aligned} \frac{\partial}{\partial y} \eta = {} & 2h \{ \psi h_\lambda^\mu h_\mu^\lambda + g^{\lambda\mu} \psi_{,\lambda\mu} \} \\ & -2 h_\lambda^\mu \left\{ \psi (h h_\mu^\lambda - R_\mu^\lambda) - \frac{\psi}{n} Q \delta_\mu^\lambda + g^{\lambda\nu} \psi_{,\mu\nu} \right\} \\ & -2 \{ \psi h_\lambda^\mu h_\mu^\lambda + g^{\lambda\mu} (\psi_{,\lambda\mu} h + 2 \psi_{,\lambda} h_{,\mu} + \psi h_{,\lambda\mu}) \\ & \quad - \psi_{,\lambda\mu} h^{\lambda\mu} - 2 \psi_{,\lambda} h^{\lambda\mu}{}_{,\mu} - \psi h^{\lambda\mu}{}_{,\lambda\mu} \} - \frac{\partial f}{\partial y} \end{aligned}$$

that is

(18) $$\frac{\partial}{\partial y} \eta = \frac{2}{n} \psi h \eta - 4 \psi_{,\lambda} V^\lambda - 2 \psi g^{\lambda\mu} V_{\lambda,\mu} + \left(\frac{2}{n} \psi h f - \frac{\partial}{\partial y} f \right).$$

Accordingly, if $g_{ab}(x, y)$, $h_a^b(x, y)$ satisfy (15), it must hold good

(19) $$\frac{\partial}{\partial y} f = \frac{2}{n} \psi h f .$$

On the other hand, if $g_{ab}(x, y)$, $h_a^b(x, y)$ and $f(x, y)$ are any solution of the system of differential equations (14), (19) and if we put

$$\zeta_a \equiv \left(f \psi^{\frac{-2}{n-2}} \right)_{,a} ,$$

then we get

$$\frac{\partial}{\partial y} \zeta_a = \left(\frac{2}{n} \psi^{\frac{n-4}{n-2}} h f - \frac{2}{n-2} f \psi^{\frac{-2}{n-2} - 1} \frac{\partial \psi}{\partial y} \right)_{,a} ,$$

that is

(20) $$\begin{aligned} \frac{\partial}{\partial y} \zeta_a = {} & \zeta_a \left(\frac{2}{n} h \psi - \frac{2}{n-2} \frac{\partial}{\partial y} \log \psi \right) \\ & + f \psi^{\frac{-2}{n-2}} \left(\frac{2}{n} h \psi - \frac{2}{n-2} \frac{\partial}{\partial y} \log \psi \right)_{,a} . \end{aligned}$$

Accordingly, if $g_{ab}(x, y)$, $h_a^b(x, y)$ satisfy (15), it must hold good also

$$\frac{\partial}{\partial y}\psi = \frac{n-2}{n}h\psi^2 + \psi P$$

where $P(y)$ is a suitable auxiliary function depending only on y. Thus we obtain a closed system of differential equations.

In conclusion, to solve (14) under the condition (15) and the initial condition above is equivalent to solve the differential equations

$$(\alpha_{\mathrm{I}}) \quad \begin{cases} \dfrac{\partial}{\partial y}g_{ab} = -2\psi h_{ab}, \\ \dfrac{\partial}{\partial y}h_a^b = \psi(hh_a^b - R_a^b) - \dfrac{\psi}{n}\delta_a^b(hh - h_\lambda^\mu h_\mu^\lambda - R) + g^{b\lambda}\psi_{,a\lambda}, \\ \dfrac{\partial}{\partial y}f = \dfrac{2}{n}\psi hf, \\ \dfrac{\partial}{\partial y}\psi = \dfrac{n-2}{n}h\psi^2 + \psi P \end{cases}$$

where $P(y)$ is a given function depending only on y, under the conditions

$$(\beta_{\mathrm{I}}) \quad \begin{cases} V_a \equiv h_{,a} - h_{a,\lambda}^\lambda = 0, \\ \eta \equiv hh - h_\lambda^\mu h_\mu^\lambda - R - f = 0, \\ \zeta_a \equiv \left(f\psi^{\frac{-2}{n-2}}\right)_{,a} = 0 \end{cases}$$

and the initial condition $[g_{ab}(x, y)]_{y=0} = g_{ab}(x)$.

If $g_{ab}(x, y)$, $h_a^b(x, y)$, $f(x, y)$, $\psi(x, y)$ is any solution of (α_{I}) and V_a, η, ζ_a are the vectors and the scalar made by these tensors and scalars, then we have from (16), (18), (20) the relations

$$\frac{\partial}{\partial y}V_a = \psi hV_a - \frac{n-2}{2n}\psi^{\frac{n}{n-2}}\left\{\left(\eta\psi^{\frac{-2}{n-2}}\right)_{,a} + \zeta_a\right\},$$

$$\frac{\partial}{\partial y}\eta = \frac{2}{n}\psi h\eta - 4\psi_{,\lambda}V^\lambda - 2\psi g^{\lambda\mu}V_{\lambda,\mu},$$

$$\frac{\partial}{\partial y}\zeta_a = -\frac{2}{n-2}P\zeta_a.$$

Hence (β_{I}) holds good, if it does so for $y = 0$. Accordingly, in order to solve our problem, it is sufficient that we can take $h_{ab}(x)$, $f(x) \neq 0$, $\psi(x)$ in the space V_n such that

$$h_{,a} - h_{a,\lambda}^\lambda = 0, \quad hh - h_\lambda^\mu h_\mu^\lambda - R - f = 0, \quad \left(f\psi^{\frac{-2}{n-2}}\right)_{,a} = 0,$$

where $f(x) \neq 0$ is derived from (10). We can take such $h_{ab}(x)$, $f(x) \neq 0$, $\psi(x)$.

Now, we investigate the excluded case $n = 2$. (16) becomes

$$\frac{\partial}{\partial y} V_a = \psi h V_a + \frac{1}{2} \psi_{,a} Q.$$

Hence, if $g_{ab}(x, y)$, $h_a^b(x, y)$ satisfy (15), it follows the relation $\psi_{,a} Q = 0$. By virtue of (10), it must be $\psi_{,a} = 0$ and $Q = -K \neq 0$. We may put $\psi = 1$ without loss of generality. Then, we can easily see that if we can take $h_{ab}(x)$ in the space V_n such that $h_{,a} - h^\lambda_{a,\lambda} = 0$, we can obtain a solution $g_{ab}(x, y)$, $h_a^b(x, y)$ of (14) under the condition (15) and the initial condition $[g_{ab}(x, y)]_{y=0} = g_{ab}(x)$.

In Case (II), as in Case (I), by means of (2), (12), (13), we have the system of differential equations

$$(21) \quad \begin{cases} \dfrac{\partial}{\partial y} g_{ab} = -2\psi h_{ab} \\ \dfrac{\partial}{\partial y} h_a^b = \psi (h h_a^b - R_a^b) + g^{b\lambda} \psi_{,a\lambda} + \psi\left(\dfrac{2F - Q}{n} \delta_a^b - \dfrac{F}{V_\lambda V^\lambda} V_a V^b\right), \end{cases}$$

where

$$F \equiv \frac{1}{2(n-2)}\left(-Q \pm \sqrt{QQ - 4n(n-2)V_\lambda V^\lambda}\right)$$

when $n > 2$, and

$$(21') \quad \begin{cases} \dfrac{\partial}{\partial y} g_{ab} = -2\psi h_{ab}, \\ \dfrac{\partial}{\partial y} h_a^b = \psi (h h_a^b - R_a^b) + g^{b\lambda} \psi_{,a\lambda} \\ \qquad + 2\psi \left\{ \dfrac{1}{Q} (V_a V^b - V_\lambda V^\lambda \delta_b^a) - \dfrac{Q}{4} \delta_a^b \right\} \end{cases}$$

when $n = 2$, with the initial condition $[g_{ab}(x, y)]_{y=0} = g_{ab}(x)$. We can solve the system of differential equations above. Putting together these results in the paragraph, we obtain the following theorems.

Theorem 2. *Let V_n be an n-dimensional Riemann space with non-zero scalar curvature, whose Ricci tensor satisfies the condition* (a). *Then, at each point in V_n, the Ricci tensor has a null direction.*

Accordingly, in such a space, the field of null directions of Ricci tensor determines a family of curves whose tangent directions are null directions of the Ricci tensor and which simply covers the space V_n.

Theorem 3. *Any n-dimensional Riemann space V_n can be imbedded in a suitable $(n+1)$-dimensional Riemann space V_{n+1} whose Ricci tensor satisfies the condition* (a) *and whose scalar curvature $\neq 0$, as a hypersurface. Furthermore, we can do this imbedding as the hypersurface above is one of a family of hypersurfaces with properties as follows*:

(i) *At each point in V_{n+1}, the null direction of the Ricci tensor of V_{n+1} is orthogonal to the tangent n-direction of the hypersurface through the point.*

(ii) *In the n-dimensional tangent linear subspace, the Ricci form of V_{n+1} is proportional to the fundamental form of V_{n+1}.*

As we have proved above, (i) implies (ii).

Corollary. *There exist n-dimensional Riemann spaces whose Ricci tensors satisfy the condition* (a).

Part II

§4. Spaces whose Ricci tensors satisfy (a), (b)

In this paragraph, we shall use the notations in §§1, 2. Let V_n be an n-dimensional Riemann space whose Ricci tensor satisfies the conditions (a) and (b), that is,

$$\text{(a)} \qquad K_i^h K_h^j = \frac{1}{n-1} K K_i^j,$$

$$\text{(b)} \qquad K_{i;h}^j = 0 \qquad (i, j, h = 1, 2, \cdots\cdots, n).$$

By virtue of (b), it follows that $K = K_i^i$ is a constant. As in §1, in a suitable coordinate neighborhood $x^1, \cdots\cdots, x^{n-1}, y = x^n$ the line element of V_n may be represented as

$$ds^2 = g_{\lambda\mu}(x, y) dx^\lambda dx^\mu + (\psi(x, y) dy)^2.$$

Now, making use of quantities of the spaces $V_{n-1}(y)$, by means of (2), (b) becomes

$$K_{a;c}^b = K_{a,c}^b - h_{ac} V^b - h_c^b V_a = 0,$$

$$K_{a,b}^n = \left(\frac{1}{\psi} V_a\right)_{,b} - \frac{1}{\psi} h_{ac} K_n^n + \frac{1}{\psi} h_{\lambda b} K_a^\lambda + \frac{1}{\psi^2} \psi_{,b} V_a = 0,$$

$$K_{n;a}^n = K_{n,a}^n + 2 h_a^\lambda V_\lambda = 0,$$

$$K_{a;n}^b = \frac{\partial}{\partial y} K_a^b - \psi h_\lambda^b K_a^\lambda - g^{b\lambda} \psi_{,\lambda} V_a - \psi_{,a} V^b + \psi h_a^\lambda K_\lambda^b = 0,$$

$$K^n_{a;n} = \frac{\partial}{\partial y}\left(\frac{1}{\psi}V_a\right) + \frac{1}{\psi}\psi_{,\lambda}K^\lambda_a + \frac{1}{\psi^2}V_a\frac{\partial \psi}{\partial y} + h^\lambda_a V_\lambda$$
$$- \frac{1}{\psi}\psi_{,a}K^n_n = 0,$$
$$K^n_{n;n} = \frac{\partial}{\partial y}K^n_n + 2g^{\lambda\mu}\psi_{,\lambda}V_\mu = 0.$$

If we put

$$K^b_a = T^b_a, \qquad K^n_n = S, \tag{22}$$

these relations are written as follows:

$$\xi^b_{ac} \equiv T^b_{a,c} - h_{ac}V^b - h^b_c V_a = 0, \tag{23}$$

$$\eta_{ab} \equiv V_{a,b} - h_{ab}S + T^\lambda_a h_{\lambda b} = 0, \tag{24}$$

$$\frac{\partial}{\partial y}T^b_a = \psi(h^b_\lambda T^\lambda_a - h^\lambda_a T^b_\lambda) + \psi_{,a}V^b + g^{b\lambda}\psi_{,\lambda}V_a, \tag{25}$$

$$\frac{\partial}{\partial y}V_a = -\psi_{,\lambda}T^\lambda_a - \psi h^\lambda_a V_\lambda + \psi_{,a}S \tag{26}$$

and

$$\begin{cases} S_{,a} + 2h^\lambda_a V_\lambda = 0, \\ \dfrac{\partial}{\partial y}S = -2\psi_{,\lambda}V^\lambda \end{cases}$$

which show that $T + S = K$ is constant.

Now, let us consider Case (I) in §1, in which $V_a = 0$, $S = 0$ Then, the relations (23), (24), (25), (26) become

$$T^b_{a,c} = 0,$$
$$T^\lambda_a h_{\lambda b} = 0,$$
$$\frac{\partial}{\partial y}T^b_a = \psi(h^b_\lambda T^\lambda_a - h^\lambda_a T^b_\lambda),$$
$$\psi_{,\lambda}T^\lambda_a = 0.$$

In §1, we have seen that if rank $(K^b_a) \neq n-1$, then the space V_n is an Einstein space with scalar curvature zero ($n > 2$) or a locally euclidean space ($n \geqq 2$). Saving for the case, we may put rank $(K^b_a) = n-1$. Then, from the relations above we obtain $h_{ab} = 0$, $\psi_{,a} = 0$, hence $T^b_a = T^b_a(x)$, $T^b_{a,c} = 0$ and we may put $\psi = 1$. Accordingly, by means of (α_1) and (β_1) replaced n with $n-1$, we get the relations $g_{ab}=g_{ab}(x)$, $R_a^b=\dfrac{1}{n-1}R\delta_a^b$, $f=-R=\text{constant}$, $P=0$. Furthermore, in the case, we get $K_a^b=R_a^b$, $K=R$ by (3). Hence, V_{n-1} is an Einstein space $(n>3)$ or a 2-dimensional Riemann space of constant curvature $(\neq 0)$. Thus we obtain the following theorem.

Theorem 4. *Let V_n be an n-dimensional Riemann space with non zero scalar curvature whose Ricci tensor satisfies the condition* (a), (b). *If the curves whose tangent directions are null directions of the Ricci tensor of V_n are orthogonal trajectories of a family of hypersurfaces, then V_n is a product space of an Einstein space with non zero scalar curvature $(n>3)$ or a surface with non zero constant curvature and a straight line. The converse is also true.*

§5. Basic relations in Case (II)

Let V_n be an $(n+1)$-dimensional Riemann space whose Ricci tensor satisfies the conditions (a) and (b). In a suitable coordinate neighborhood $x^1, \cdots\cdots, x^n, y$, the line element of V_{n+1} may be written as

$$ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda dx^\mu + (\psi(x, y)\,dy)^2 \qquad (\lambda, \mu, \cdots\cdots = 1, 2, \cdots\cdots, n)^{1)}.$$

As in §1, let $V_n(y)$ be an n-dimensional subspace of V_{n+1} on which y is constant, and whose line element is $ds^2=g_{\lambda\mu}(x, y)\,dx^\lambda dx^\mu$. We shall denote the components of the curvature tensor, the Ricci tensor and the scalar curvature of $V_n(y)$ by $R_a{}^b{}_{cd}$, R_{ab}, $R=R_\lambda^\lambda$ $(a, b, c, \cdots\cdots = 1, 2, \cdots\cdots, n)$ respectively as in §1. From now on we assume that $g_{\lambda\mu}V^\lambda V^\mu \neq 0$, where $V_a=\psi K_a^{n+1}$. Then, making use of quantities of the space $V_n(y)$, by means of (2), (3), (25), (26) we have

$$(\alpha_{II}) \qquad \left\{\begin{aligned} \frac{\partial}{\partial y}g_{ab} &= -2\psi h_{ab}, \\ \frac{\partial}{\partial y}h_a^b &= \psi(h h_a^b - R_a^b + T_a^b) + g^{b\lambda}\psi_{,a\lambda}, \\ \frac{\partial}{\partial y}T_a^b &= \psi(h_\lambda^b T_a^\lambda - h_a^\lambda T_\lambda^b) + \psi_{,a}V^b + g^{b\lambda}\psi_{,\lambda}V_a, \\ \frac{\partial}{\partial y}V_a &= -\psi_{,\lambda}T_a^\lambda - \psi h_a^\lambda V_\lambda + (K-T)\psi_{,a}, \end{aligned}\right.$$

1) In this paragraph, indices take the following values:
$a, b, c, \cdots\cdots;\ \lambda, \mu, \nu, \cdots\cdots = 1, 2, \cdots\cdots, n.$

where $T_a^b = K_a^b$, $T = T_\lambda^\lambda$. By (7), (9), we have

$$T_a^b - \frac{1}{n} K\delta_a^b + \frac{S}{V_\lambda V^\lambda} V_a V^b = 0,$$
$$SS - \frac{1}{n} KS + V_\lambda V^\lambda = 0.$$

Since $V_\lambda V^\lambda \neq 0$, it follows that $S \equiv K - T \neq 0, \frac{1}{n} K$, and the first relation may be replaced with

$$\text{(27)} \qquad \left(T_a^b - \frac{1}{n} K\delta_a^b\right)\left(T - \frac{n-1}{n} K\right) + V_a V^b = 0,$$

which implies the second relation above by contraction.

From (3), (α_{II}) we get

$$\frac{\partial}{\partial y} h = \psi(h_\lambda^\mu h_\mu^\lambda + S) + g^{\lambda\mu}\psi_{,\lambda\mu} = \psi(hh - R + T) + g^{\lambda\mu}\psi_{,\lambda\mu},$$

hence

$$hh - h_\lambda^\mu h_\mu^\lambda - R + T - S = 0.$$

Thus, we obtain a system of relations

$$(\beta_{\text{II}}) \qquad \begin{cases} \tau_a^b \equiv \left(T_a^b - \frac{1}{n} K\delta_a^b\right)\left(T - \frac{n-1}{n} K\right) + V_a V^b = 0, \\ \rho_a \equiv V_a - h_{,a} + h_{a,\lambda}^\lambda = 0, \\ \xi_a{}^b{}_c \equiv T_{a,c}^b - h_{ac} V^b - h_c^b V_a = 0, \\ \eta_{ab} \equiv V_{a,b} + T_a^\lambda h_{\lambda b} - h_{ab}(K - T) = 0, \\ \zeta \equiv hh - h_\lambda^\mu h_\mu^\lambda - R - K + 2T = 0. \end{cases}$$

By the argument above, we can easily see that (a), (b) are equivalent to (α_{II}), (β_{II}), $K =$ constant when $g_{\lambda\mu} V^\lambda V^\mu \neq 0$.

Let $g_{ab}(x, y)$, $h_a^b(x, y)$, $T_a^b(x, y)$, $V_a(x, y)$ be any solution of the system of differential equation (α_{II}), and τ_a^b, ρ_a, $\xi_a{}^b{}_c$, η_{ab}, ζ be the quantities made by them according to the left hand side of (β_{II}).

By means of (2), (α_{II}), we get the relations as follows:

$$\begin{aligned} \frac{\partial}{\partial y}\rho_a = & -\psi_{,\lambda} T_a^\lambda - \psi h_a^\lambda V_\lambda + \psi_{,a}(K - T) - \psi_{,a}(hh - R + T) \\ & - \psi(2hh_{,a} - R_{,a} + T_{,a}) - \psi_{,}{}^\lambda{}_{\lambda a} + \psi_{,\lambda}(hh_a^\lambda - R_a^\lambda + T_a^\lambda) \\ & + \psi(h_{,\lambda} h_a^\lambda + hh_{a,\lambda}^\lambda - R_{a,\lambda}^\lambda + T_{a,\lambda}^\lambda) + \psi_{,}{}^\lambda{}_{a\lambda} \\ & - \psi_{,\lambda} hh_a^\lambda - \psi h_{,\lambda} h_a^\lambda + \psi_{,a} h_\lambda^\mu h_\mu^\lambda + \psi h_\lambda^\mu h_{\mu,a}^\lambda \\ = & - \psi h_a^\lambda V_\lambda - \psi_{,a}(Q - K + 2T) - \psi h(h_{,a} - h_{a,\lambda}^\lambda) \end{aligned}$$

$$-\psi\left(\frac{1}{2}Q_{,a}+T_{,a}-T^{\lambda}_{a,\lambda}\right)$$

that is

$$\text{(28)}\quad \frac{\partial}{\partial y}\rho_a = -\psi_{,a}\zeta+\psi h\rho_a-\frac{\psi}{2}\zeta_{,a}+\psi\xi^{\lambda}_{a\lambda}.$$

$$\begin{aligned}\frac{\partial}{\partial y}\zeta &= 2h\{\psi(hh-R+T)+\psi_{,\lambda}{}^{\lambda}\}\\ &\quad -2h^{\mu}_{\lambda}\{\psi(hh^{\lambda}_{\mu}-R^{\lambda}_{\mu}+T^{\lambda}_{\mu})+\psi_{,}{}^{\lambda}{}_{\mu}\}\\ &\quad -2\{\psi h^{\mu}_{\lambda}R^{\lambda}_{\mu}+\psi_{,\lambda}{}^{\lambda}h+2\psi_{,\lambda}h_{,}{}^{\lambda}+\psi h_{,}{}^{\lambda}{}_{\lambda}\}\\ &\quad +2\{\psi_{,\lambda\mu}h^{\lambda\mu}+2\psi_{,\lambda}h^{\lambda\mu}{}_{,\mu}+\psi h^{\lambda\mu}{}_{,\lambda\mu}\}+4\psi_{,\lambda}V^{\lambda}\\ &= 2\psi h(Q+T)-2\psi h^{\mu}_{\lambda}T^{\lambda}_{\mu}-4\psi_{,\lambda}(h_{,}{}^{\lambda}-h^{\lambda\mu}{}_{,\mu})+4\psi_{,\lambda}V^{\lambda}\\ &\quad -2\psi(h_{,}{}^{\lambda}{}_{\lambda}-h^{\mu\lambda}{}_{,\lambda\mu}),\end{aligned}$$

that is

$$\text{(29)}\quad \frac{\partial}{\partial y}\zeta = 2\psi h\zeta+4g^{\lambda\mu}\psi_{,\lambda}\rho_{\mu}+2\psi g^{\lambda\mu}(\rho_{\lambda,\mu}-\eta_{\lambda\mu}).$$

$$\begin{aligned}\frac{\partial}{\partial y}\tau^{b}_{a} &= 2\psi_{,\lambda}V^{\lambda}\left(T^{b}_{a}-\frac{1}{n}K\delta^{b}_{a}\right)\\ &\quad +\left(T-\frac{n-1}{n}K\right)\{\psi(h^{b}_{\lambda}T^{\lambda}_{a}-h^{\lambda}_{a}T^{b}_{\lambda})+\psi_{,a}V^{b}+\psi_{,}{}^{b}V_{a}\}\\ &\quad +V^{b}\{-\psi_{,\lambda}T^{\lambda}_{a}-\psi h^{\lambda}_{a}V_{\lambda}+(K-T)\psi_{,a}\}+2\psi h^{b\lambda}V_aV_{\lambda}\\ &\quad +V_a\{-\psi_{,\lambda}T^{\lambda b}-\psi h^{b\lambda}V_{\lambda}+(K-T)\psi_{,}{}^{b}\}\\ &= \psi h^{b}_{\lambda}\left\{\left(T-\frac{n-1}{n}K\right)T^{\lambda}_{a}+V_aV^{\lambda}\right\}\\ &\quad -\psi h^{\lambda}_{a}\left\{\left(T-\frac{n-1}{n}K\right)T^{b}_{\lambda}+V^{b}V_{\lambda}\right\}\\ &\quad +2\psi_{,\lambda}V^{\lambda}\left(T^{b}_{a}-\frac{1}{n}K\delta^{b}_{a}\right)+\frac{1}{n}K(\psi_{,a}V^{b}+\psi_{,}{}^{b}V_a)\\ &\quad -V^{b}T^{\lambda}_{a}\psi_{,\lambda}-V_aT^{\lambda b}\psi_{,\lambda},\end{aligned}$$

that is

$$\text{(30)}\quad \frac{\partial}{\partial y}\tau^{b}_{a} = \psi(h^{b}_{\lambda}\tau^{\lambda}_{a}-h^{\lambda}_{a}\tau^{b}_{\lambda}) + \frac{1}{T-\frac{n-1}{n}K}\{2\psi_{,\lambda}V^{\lambda}\tau^{b}_{a}-V_a\psi_{,\lambda}\tau^{b}_{\mu}g^{\lambda\mu}-V^{b}\psi_{,\lambda}\tau^{\lambda}_{a}\}.$$

$$\begin{aligned}\frac{\partial}{\partial y}\eta_{ab} &= -\psi_{,\lambda b}T^{\lambda}_{a}-\psi_{,\lambda}T^{\lambda}_{a,b}-\psi_{,b}h^{\lambda}_{a}V_{\lambda}-\psi h^{\lambda}_{a,b}V_{\lambda}\\ &\quad -\psi h^{\lambda}_{a}V_{\lambda,b}-\psi_{,a}T_{,b}+(K-T)\psi_{,ab}\\ &\quad -V_{\lambda}\{\psi(h_{ab,}{}^{\lambda}-h^{\lambda}_{a,b}-h^{\lambda}_{b,a})+\psi_{,}{}^{\lambda}h_{ab}-\psi_{,b}h^{\lambda}_{a}-\psi_{,a}h^{\lambda}_{b}\}\end{aligned}$$

$$
\begin{aligned}
&+ h_{b\lambda}\{\psi(h^\lambda_\mu T^\lambda_a - h^\mu_a T^\lambda_\mu) + \psi_{,a}V^\lambda + \psi_,{}^\lambda V_a\}\\
&+ T^\lambda_a\{\psi(h h_{\lambda b} - 2h^\mu_\lambda h_{\mu b} - R_{\lambda b} + T_{\lambda b}) + \psi_{,\lambda b}\}\\
&- (K-T)\{\psi(h h_{ab} - 2h^\lambda_a h_{\lambda b} - R_{ab} + T_{ab}) + \psi_{,ab}\}\\
&+ 2h_{ab}\psi_{,\lambda}V^\lambda,
\end{aligned}
$$

that is

$$
\begin{aligned}
(31)\quad \frac{\partial}{\partial y}\eta_{ab} &= -\psi_{,\lambda}\xi^\lambda_{ab} - \psi_{,a}\xi_\lambda{}^\lambda{}_b - \psi h^\lambda_a \eta_{\lambda b}\\
&-\psi[V_\lambda(h_{ab},{}^\lambda - h^\lambda_{b,a}) - T^\lambda_a(Q_{\lambda b} + T_{\lambda b}) + (K-T)(Q_{ab} + T_{ab})].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\xi_a{}^b{}_c &= \psi_{,c}(h^b_\lambda T^\lambda_a - h^\lambda_a T^b_\lambda)\\
&+ \psi(h^b_{\lambda,c}T^\lambda_a + h^b_\lambda T^\lambda_{a,c} - h^\lambda_{a,c}T^b_\lambda - h^\lambda_a T^b_{\lambda,c})\\
&+ \psi_{,ac}V^b + \psi_{,a}V^b_{,c} + \psi_,{}^b{}_c V_a + \psi_,{}^b V_{a,c}\\
&+ T^\lambda_a\{\psi(h_{\lambda c},{}^b - h^b_{\lambda,c} - h^b_{c,\lambda}) + \psi_,{}^b h_{\lambda c} - \psi_{,c}h^b_\lambda - \psi_{,\lambda}h^b_c\}\\
&- T^b_\lambda\{\psi(h_{ac},{}^\lambda - h^\lambda_{a,c} - h^\lambda_{c,a}) + \psi_,{}^\lambda h_{ac} - \psi_{,c}h^\lambda_a - \psi_{,a}h^\lambda_c\}\\
&- V^b\{\psi(h h_{ac} - 2h^\lambda_a h_{\lambda c} - R_{ac} + T_{ac}) + \psi_{,ac}\} - 2\psi h_{ac}h^{b\lambda}V_\lambda\\
&- h_{ac}\{-\psi_{,\lambda}T^{b\lambda} - \psi h^{b\lambda}V_a + (K-T)\psi_,{}^b\}\\
&- V_a\{\psi(h h^b_c - R^b_c + T^b_c) + \psi_,{}^b{}_c\}\\
&- h^b_c\{-\psi_{,\lambda}T^\lambda_a - \psi h^\lambda_a V_\lambda + (K-T)\psi_{,a}\},
\end{aligned}
$$

that is

$$
\begin{aligned}
(32)\quad \frac{\partial}{\partial y}\xi_a{}^b{}_c &= \psi(h^b_\lambda \xi^\lambda_{ac} - h^\lambda_a \xi_\lambda{}^b{}_c) + g^{b\lambda}(\psi_{,a}\eta_{\lambda c} + \psi_{,\lambda}\eta_{ac})\\
&+ \psi[T^\lambda_a(h_{\lambda c},{}^b - h^b_{c,\lambda}) - T^b_\lambda(h_{ac},{}^\lambda - h^\lambda_{c,a})\\
&- V^b(Q_{ac} + T_{ac}) - V_a(Q^b_c + T^b_c)].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&T^\lambda_a(h_{\lambda c},{}^b - h^b_{c,\lambda}) - T^b_\lambda(h_{ac},{}^\lambda - h^\lambda_{c,a}) - V^b(Q_{ac} + T_{ac}) - V_a(Q^b_c + T^b_c)\\
&= \frac{1}{T - \frac{n-1}{n}K}\{\tau^\lambda_a(h_{\lambda c},{}^b - h^b_{c,\lambda}) - \tau^b_\lambda(h_{ac},{}^\lambda - h^\lambda_{c,a})\}\\
&- \frac{1}{T - \frac{n-1}{n}K}\Big\{V_a\Big[V^\lambda(h_{\lambda c},{}^b - h^b_{c,\lambda}) + \Big(T - \frac{n-1}{n}K\Big)(Q^b_c + T^b_c)\Big]\\
&\qquad + V^b\Big[V_\lambda(h^\lambda_{c,a} - h_{ac},{}^\lambda) + \Big(T - \frac{n-1}{n}K\Big)(Q_{ac} + T_{ac})\Big]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&V_\lambda(h_{ab},{}^\lambda - h^\lambda{}_{b,a}) - T^\lambda_a(Q_{\lambda b} + T_{\lambda b}) + (K-T)(Q_{ab} + T_{ab})\\
&= -\frac{1}{T - \frac{n-1}{n}K}\tau^\lambda_a(Q_{\lambda b} + T_{\lambda b})
\end{aligned}
$$

$$+\left[V_\lambda(h_{ab,}{}^\lambda - h^\lambda_{b,a}) - \left(T - \frac{n-1}{n}K\right)(Q_{ab} + T_{ab})\right]$$
$$+ \frac{1}{T - \frac{n-1}{n}K} V_a V^\lambda (Q_{\lambda b} + T_{\lambda b}).$$

Since we have analogously

$$V^\lambda T_{\lambda b} = \frac{1}{n} K V_b + \frac{1}{T - \frac{n-1}{u}K}(V_\lambda \tau^\lambda_b - V_b V_\lambda V^\lambda)$$
$$= \frac{1}{T - \frac{n-1}{n}K}(\tau^\lambda_b V_\lambda - \tau V_b) + \left(T - \frac{n-1}{n}K\right)V_b,$$

the right hand side of the second relation above becomes

$$= -\frac{1}{T - \frac{n-1}{n}K}\tau^\lambda_a (Q_{\lambda b} + T_{\lambda b}) + \frac{1}{\left(T - \frac{n-1}{n}K\right)^2} V_a(\tau^\lambda_b V_\lambda - \tau V_b)$$
$$+ \left[V_\lambda(h_{ab,}{}^\lambda - h^\lambda_{b,a}) - \left(T - \frac{n-1}{n}K\right)(Q_{ab} + T_{ab})\right.$$
$$\left. + V_a\left\{V_b + \frac{1}{T - \frac{n-1}{n}K} V^\lambda Q_{\lambda b}\right\}\right].$$

Accordingly, we see that if g_{ab}, h^b_a, T^b_a, V_a satisfy also the relation (β_{II}), then the following relation must hold good,

$$\varphi_{ab} \equiv V_\lambda(h^\lambda_{b,a} - h_{ab,}{}^\lambda) + \left(T - \frac{n-1}{n}K\right)(Q_{ab} + T_{ab})$$
$$- V_a\left(V_b + \frac{1}{T - \frac{n-1}{n}K} V^\lambda Q_{\lambda b}\right) = 0. \tag{33}$$

Now, making use of φ_{ab}, we have the relation

$$T^\lambda_a(h_{\lambda c,}{}^b - h^b_{c,\lambda}) - T^b_\lambda(h_{ac,}{}^\lambda - h^\lambda_{c,a}) - V^b(Q_{ac} + T_{ac}) - V_a(Q^b_c + T^b_c)$$
$$= \frac{1}{T - \frac{n-1}{n}K}\{\tau^\lambda_a(h_{\lambda c,}{}^b - h^b_{c,\lambda}) - \tau^b_\lambda(h_{ac,}{}^\lambda - h^\lambda_{c,a}) - V_a\varphi^b{}_c - V^b\varphi_{ac}\}$$
$$- \frac{2}{T - \frac{n-1}{n}K} V_a V^b\left[V_c + \frac{1}{T - \frac{n-1}{n}K} V^\lambda Q_{\lambda a}\right].$$

Accordingly, if g_{ab}, h^b_a, T^b_a, V_a satisfy also the relation (β_{II}), it must follow

$$\left(T - \frac{n-1}{n}K\right)V_a + V^\lambda Q_{\lambda a} = 0, \tag{34}$$

since $g_{\lambda\mu}V^\lambda V^\lambda \neq 0$. Hence, under this circumstance, we get from (33), (34) the relation

$$(35) \quad \theta_{ab} \equiv V_\lambda(h^\lambda_{a,b} - h_{ab},{}^\lambda) + \left(T - \frac{n-1}{n}K\right)(Q_{ab} + T_{ab}) = 0.$$

Furthermore

$$\theta_{ab} - \theta_{ba} \equiv V_\lambda(h^\lambda_{a,b} - h^\lambda_{b,a}) = 0,$$

and

$$\begin{aligned} V^\lambda\theta_{\lambda a} &\equiv V^\lambda V^\mu(h_{\lambda\mu,a} - h_{\lambda a,\mu}) \\ &+ \left(T - \frac{n-1}{n}K\right)\left(V_\lambda Q^\lambda_a + \frac{1}{n}KV_a\right) \\ &+ \tau^\lambda_a V_\lambda - V_a\left(\tau - (T-K)\left(T - \frac{n-1}{n}K\right)\right) \\ &= V^\mu(\theta_{\mu a} - \theta_{a\mu}) + \tau^\lambda_a V_\lambda - \tau V_a \\ &+ \left(T - \frac{n-1}{n}K\right)\left[\left(T - \frac{n-1}{n}K\right)V_a + V_\lambda Q^\lambda_a\right] \end{aligned}$$

which shows that (34) and (33) can be derived from (35) and (β_{II}).

Returning to the assumption in the beginning of this paragraph, in the following, we shall denote equalities mod τ^b_a, ρ_a, $\xi_a{}^b{}_c$, η_{ab}, τ, θ_{ab} and their covariant derivatives by "$\simeq$". Then, we get from the definition of η_{ab}

$$\begin{aligned} \eta_{a[b,c]} &= V_{a,[bc]} + T^\lambda_{a,[c}h_{b]\lambda} + T^\lambda_a h_{\lambda[b,c]} - h_{a[b,c]}(K-T) + h_{a[b}T_{,c]} \\ &\simeq -\frac{1}{2}R_a{}^\lambda{}_{bc}V_\lambda + \left(T - \frac{n-1}{n}K\right)h_{a[b,c]} + h_{a[b}h^\lambda_{c]}V_\lambda, \end{aligned}$$

hence

$$(33) \quad \left(T - \frac{n-1}{n}K\right)(h_{ab,c} - h_{ac,b}) + V_\lambda(h_{ab}h^\lambda_c - h_{ac}h^\lambda_b) - V_\lambda R_a{}^\lambda{}_{bc} \simeq 0.$$

Furthermore, covariantly differentiating (36), we get

$$\begin{aligned} &\left(T - \frac{n-1}{n}K\right)(h_{ab,ce} - h_{ac,be}) + 2V_\lambda h^\lambda_e((h_{ab,c} - h_{ac,b}) \\ &-\left(T - \frac{n-1}{n}K\right)h_{e\lambda}(h_{ab}h^\lambda_e - h_{ac}h^\lambda_b) \\ &+ \frac{1}{T - \frac{n-1}{n}K}V_\lambda V_\mu h^\mu_e(h_{ab}h^\lambda_c - h_{ac}h^\lambda_b) \\ &+ V_\lambda(h_{ab}h^\lambda_{c,e} + h_{ab,e}h^\lambda_c - h_{ac,e}h^\lambda_b - h_{ac}h^\lambda_{b,e}) \end{aligned}$$

$$+\left(T-\frac{n-1}{n}K\right)h_{e\lambda}R_{a}{}^{\lambda}{}_{bc}$$
$$-\frac{1}{T-\frac{n-1}{n}K}V_\lambda V_\mu h^\mu_e R_a{}^\lambda{}_{bc} - V_\lambda R_a{}^\lambda{}_{bc,e} \simeq 0,$$

hence by virtue of (36)

$$\begin{aligned}(37)\qquad &\left(T-\frac{n-1}{n}K\right)(h_{ab,ce}-h_{ac,be}+h_{e\lambda}F_a{}^\lambda{}_{bc})\\ &+\frac{1}{T-\frac{n-1}{n}K}V_\lambda V_\mu h^\mu_e F_a{}^\lambda{}_{bc} - V_\lambda R_a{}^\lambda{}_{bc,e}\\ &+V_\lambda(h_{ab}h^\lambda_{c,e}+h_{ab,e}h^\lambda_c - h_{ac,e}h^\lambda_b - h_{ac}h^\lambda_{b,e}) \simeq 0,\end{aligned}$$

where we put

$$(38)\qquad F_{abce} \equiv R_{abce} - h_{ac}h_{be} + h_{ae}h_{bc}.$$

Now we have by (α_{II})

$$\begin{aligned}\frac{\partial}{\partial y}Q_{ab} \simeq\ & 2\psi\{hhh_{ab} - hh^\lambda_a h_{b\lambda} + h^\lambda_a h_{\lambda\mu}h^\mu_b\}\\ &+\psi\{h_{ab}(T-R) - h^\lambda_a(T_{b\lambda}-R_{b\lambda}) - h^\lambda_b(T_{a\lambda}-R_{a\lambda}) + h(T_{ab}-R_{ab})\}\\ &-\psi_{,a}V_b - \psi_{,b}V_a + \psi_{,\lambda}\{(h^\lambda_{a,b} - h_{ab,}{}^\lambda) + (h^\lambda_{b,a} - h_{ab,}{}^\lambda)\}\\ &+\frac{\psi}{2}\{(h^\lambda_{a,\lambda b} - h^\lambda_{\lambda,ab}) + (h^\lambda_{b,\lambda a} - h^\lambda_{\lambda,ba})\\ &\quad + (h_{a\lambda,b}{}^\lambda - h_{ab,\lambda}{}^\lambda) + (h_{b\lambda,a}{}^\lambda - h_{ba,\lambda}{}^\lambda)\\ &\quad + R_\mu{}^\lambda{}_{b\lambda}h^\mu_a - R_a{}^\mu{}_b{}^\lambda h^\lambda_\mu + R_\mu{}^\lambda{}_{a\lambda}h^\mu_b - R_b{}^\mu{}_{a\lambda}h^\lambda_\mu\}.\end{aligned}$$

On the other hand, we get from (37), (38) the relations

$$\begin{aligned}&\left(T-\frac{n-1}{n}K\right)(h^\lambda_{b,\lambda e} - h^\lambda_{\lambda,be} + h_{e\lambda}Q^\lambda_b)\\ &\quad+\frac{1}{T-\frac{n-1}{n}K}V_\lambda V_\mu h^\mu_a Q^\lambda_b + V_\lambda R^\lambda_{b,e}\\ &\quad+V_\lambda(h^\mu_b h^\lambda_{\mu,e} + h^\mu_{b,e}h^\lambda_\mu - h_{,e}h^\lambda_b - hh^\lambda_{b;e}) \simeq 0,\\ &\left(T-\frac{n-1}{n}K\right)(h_{a\lambda,c}{}^\lambda - h_{ac,\lambda}{}^\lambda + h^\mu_\lambda F_a{}^\lambda{}_{\mu c})\\ &\quad+\frac{1}{T-\frac{n-1}{n}K}V_\lambda V_\mu h^{\mu\rho}F_a{}^\lambda{}_{\rho c} - V_\lambda R_a{}^{\lambda\mu}{}_{c,\mu}\\ &\quad+V_\lambda(h^\mu_a h^\lambda_{c,\mu} + h^\mu_{a,\mu}h^\lambda_c - h_{ac,\mu}h^{\lambda\mu} - h_{ac}h^{\lambda\mu}{}_{,\mu}) \simeq 0\end{aligned}$$

and

$$g^{\lambda\mu}F_{\lambda ab\mu} = Q_{ab}.$$

Making use of these relations and putting $M = 1/T - \frac{n-1}{n}K$, we get

$$\begin{aligned}
\frac{\partial}{\partial y}Q_{ab} \simeq\ & 2\psi\{hhh_{ab} - 2hh_a^\lambda h_{b\lambda} + h_a^\lambda h_{\lambda\mu}h_b^\mu\} \\
& + \psi\Big\{h_{ab}(T-R) - h_a^\lambda\Big(T_{b\lambda} - \frac{3}{2}R_{b\lambda}\Big) \\
& \qquad - h_b^\lambda\Big(T_{a\lambda} - \frac{3}{2}R_{a\lambda}\Big) + h(T_{ab} - R_{ab})\Big\} \\
& - \psi_{,a}V_b - \psi_{,b}V_a + \frac{\psi}{2}MV_\lambda(V_bh_a^\lambda + V_ah_b^\lambda) - \psi h^{\lambda\mu}R_{a\lambda b\mu} \\
& - \frac{\psi}{2}M^2V_\lambda h_\mu^\lambda V_\rho(F_b{}^{\rho\mu}{}_a + F_a{}^{\rho\mu}{}_b) + M\psi_{,\lambda}V_\mu(F_a{}^{\lambda\mu}{}_b + F_b{}^{\mu\lambda}{}_a) \\
& + \frac{\psi}{2}[- h_b^\lambda Q_{a\lambda} - h_a^\lambda Q_{b\lambda} - 2M^2V_\lambda V_\mu h_a^\mu Q_b^\lambda - MV_\lambda(R_{b,a}^\lambda + R_{a,b}^\lambda) \\
& \qquad - h_\lambda^\mu(F_a{}^\lambda{}_{\mu b} + F_b{}^\lambda{}_{\mu a}) - M^2V_\lambda V_\mu h_\rho^\mu(F_a{}^\lambda{}_{\rho b} + F_b{}^\lambda{}_{\rho a}) \\
& \qquad + MV_\lambda(R_a{}^{\lambda\mu}{}_{b,\mu} + R_b{}^{\lambda\mu}{}_{a,\mu}) \\
& \qquad - MV_\lambda\{h_b^\mu(h_{\mu,a}^\lambda + h_{a,\mu}^\lambda) + h_a^\mu(h_{\mu,b}^\lambda + h_{b,\mu}^\lambda) \\
& \qquad\qquad - h(h_{a,b}^\lambda + h_{b,a}^\lambda) - 2h_{ab}h^{\lambda\mu}{}_{,\mu}\}].
\end{aligned}$$

Now, we can easily see that the relation (36) is, in fact, an equality mod $\xi_\lambda{}^\mu{}_\nu$, $\eta_{\lambda\mu}$, τ_λ^μ, hence

$$\Big(T - \frac{n-1}{n}K\Big)V_\lambda(h_{a,b}^\lambda - h_{b,a}^\lambda) \simeq 0 \qquad (\text{mod } \xi_{\lambda\nu}^\mu,\ \eta_{\lambda\mu},\ \tau_\lambda^\mu).$$

Accordingly, *we may put anew*

$$\text{(39)} \qquad \theta_{ab} \equiv \frac{1}{2}V_\lambda(h_{a,b}^\lambda + h_{b,a}^\lambda - 2h_{ab,}{}^\lambda) + \Big(T - \frac{n-1}{n}K\Big)(Q_{ab} + T_{ab}).$$

Thus we obtain the following relations

$$\text{(31')} \qquad \frac{\partial}{\partial y}\eta_{ab} \simeq 0,$$

$$\text{(32')} \qquad \frac{\partial}{\partial y}\xi_a{}^b{}_c \simeq 0.$$

Now, by means of (2), (α_{II}), the definitions of τ_a^b, ρ_a, $\xi_a{}^b{}_a$, η_{ab}, ζ, θ_{ab}, we get

$$\frac{\partial}{\partial y}\theta_{ab} = \frac{1}{2}V_\lambda\Big[\Big(\frac{\partial}{\partial y}h_a^\lambda\Big)_{,b} + \Big(\frac{\partial}{\partial y}h_b^\lambda\Big)_{,a}$$

$$
\begin{aligned}
& \qquad - 2\Big(\frac{\partial}{\partial y} h_{ab}\Big)_{,}{}^{\lambda} - 4\psi h^{\lambda\mu} h_{ab,\mu} \\
& \qquad + h_a^\mu \frac{\partial}{\partial y} \Gamma^\lambda_{\mu b} + h_b^\mu \frac{\partial}{\partial y} \Gamma^\lambda_{\mu a} - 2 h^\lambda_\mu \frac{\partial}{\partial y} \Gamma^\mu_{ab} \\
& \qquad + 2 g^{\lambda\rho} h_{a\mu} \frac{\partial}{\partial y} \Gamma^\mu_{b\rho} + 2 g^{\lambda\rho} h_{b\mu} \frac{\partial}{\partial y} \Gamma^\mu_{a\rho}\Big] \\
& + \frac{1}{2}(h^\lambda_{a,b} + h^\lambda_{b,a} - 2h_{ab,}{}^{\lambda}) \frac{\partial}{\partial y} V_\lambda + 2\psi_{,\lambda} V^\lambda (Q_{ab} + T_{ab}) \\
& + \Big(T - \frac{n-1}{n} K\Big)\Big(\frac{\partial}{\partial y} T_{ab} + \frac{\partial}{\partial y} Q_{ab}\Big) \\
\simeq\ & \frac{1}{2} V_\lambda [\psi_{,a}(Q^\lambda_b + T^\lambda_b) + \psi_{,b}(Q^\lambda_a + T^\lambda_a) \\
& \quad + \psi(h_{,a} h^\lambda_b + h_{,b} h^\lambda_a - R^\lambda_{a,b} - R^\lambda_{b,a}) \\
& \quad + \psi(V_a h^\lambda_b + V_b h^\lambda_a + 2V^\lambda h_{ab}) - \psi_{,\mu}(R_a{}^{\mu\lambda}{}_b + R_b{}^{\mu\lambda}{}_a) \\
& \quad - 2\psi_{,}{}^{\lambda}(Q_{ab} + T_{ab}) - 2\psi(h_{,}{}^{\lambda} h_{ab} - R_{ab,}{}^{\lambda} + V_a h^\lambda_b + V_b h^\lambda_a) \\
& \quad + \psi_{,\mu}(h^\mu_a h^\lambda_b + h^\mu_b h^\lambda_a) - 2\psi_{,}{}^{\mu} h^\lambda_\mu h_{ab}] \\
& - \psi h\Big(T - \frac{n-1}{n}K\Big)(Q_{ab} + T_{ab}) \\
& + \frac{3\psi}{2}\Big(T - \frac{n-1}{n}K\Big)\{h^\mu_a(Q_{\mu b} + T_{\mu b}) + h^\mu_b(Q_{\mu a} + T_{\mu a})\} \\
& - M\psi V_\lambda h^\lambda_\mu (V_\rho F_a{}^\rho{}_b{}^\mu + V_\rho F_b{}^\rho{}_a{}^\mu) \\
& + \frac{\psi}{2} V_\lambda (h^\lambda_a h^\lambda_{b,\mu} + h^\mu_b h^\lambda_{a,\mu} - 2h^\lambda_\mu h_{ab,\mu}) \\
& + \frac{M}{2}\Big[-\Big(T - \frac{n-1}{n}K\Big)\psi_{,\lambda} - \psi h^\mu_\lambda V_\mu \\
& \quad + M\psi_{,\mu} V^\mu V_\lambda\Big](V_\rho F_a{}^{\rho\lambda}{}_b + V_\rho F_b{}^{\rho\lambda}{}_a) \\
& + 2\psi_{,\lambda} V^\lambda (Q_{ab} + T_{ab}) \\
& + \Big(T - \frac{n-1}{n}K\Big)[-\psi(h^\lambda_a T_{b\lambda} + h^\lambda_b T_{a\lambda}) + \psi_{,a} V_b + \psi_{,b} V_a] \\
& + \Big(T - \frac{n-1}{n}K\Big)\frac{\partial}{\partial y} Q_{ab} \\
\simeq\ & \frac{1}{2} V_\lambda [\psi_{,a}(Q^\lambda_b + T^\lambda_b) + \psi_{,b}(Q^\lambda_a + T^\lambda_a) + \psi(h_{,a} h^\lambda_b + h_{,b} h^b_\lambda) \\
& \quad - \psi(2R^\lambda_{a,b} + 2R^\lambda_{b,a} - 2R_{ab,}{}^{\lambda} - R_a{}^{\lambda\mu}{}_{b,\mu} - R_b{}^{\lambda\mu}{}_{a,\mu}) \\
& \quad - 2\psi h_{,}{}^{\lambda} h_{ab} - \psi M h^\lambda_\mu V_\rho (F_a{}^{\rho\mu}{}_b + F_b{}^{\rho\mu}{}_a) \\
& \quad + M^2 \psi_{,\mu} V^\mu (V_\rho F_a{}^{\rho\lambda}{}_b + V_\rho F_b{}^{\rho\lambda}{}_a) \\
& \quad + 2\psi_{,}{}^{\lambda}(Q_{ab} + T_{ab}) - \psi M V_\mu (h^\mu_a Q^\lambda_b + h^\mu_b Q^\lambda_a) \\
& \quad + \psi\{h(h^\lambda_{a,b} + h^\lambda_{b,a}) + 2h_{ab} h^{\lambda\mu}{}_{,\mu} - 2h^\lambda_\mu h_{ab,}{}^{\mu} \\
& \qquad - h^\mu_a h^\lambda_{\mu,b} - h^\mu_b h^\lambda_{\mu,a}\}]
\end{aligned}
$$

$$+\psi(K-T)\Big(T-\frac{n-1}{n}K\Big)h_{ab}-\psi\Big(T-\frac{n-1}{n}K\Big)hQ_{ab}$$
$$+\psi\Big(T-\frac{n-1}{n}K\Big)\Big[h^{\mu}_{a}\Big(Q_{\mu b}-\frac{1}{2}T_{\mu b}\Big)+h^{\mu}_{b}\Big(Q_{\mu a}-\frac{1}{2}T_{\mu a}\Big)\Big]$$
$$+2\psi\Big(T-\frac{n-1}{n}K\Big)(hhh_{ab}-2hh^{\lambda}_{a}h_{b\lambda}+h^{\lambda}_{a}h_{\lambda\mu}h^{\mu}_{b})$$
$$+\psi\Big(T-\frac{n-1}{n}K\Big)\Big[h_{ab}(T-R)+\frac{3}{2}h^{\lambda}_{a}R_{b\lambda}+\frac{3}{2}h^{\lambda}_{b}R_{a\lambda}-hR_{ab}\Big]$$
$$-\psi\Big(T-\frac{n-1}{n}K\Big)h^{\lambda\mu}[R_{a\lambda\mu b}-h_{a\mu}h_{\lambda b}+h_{ab}h_{\lambda\mu}+R_{a\lambda b\mu}].$$

Making use of the identities $R_{a}{}^{\lambda\mu}{}_{b,\mu}=R^{\lambda}_{b,a}-R_{ba,}{}^{\lambda}$ derived from the Bianchi identities $R_{abcd,e}+R_{abde,c}+R_{abec,d}=0$, and $h^{\mu}_{a,b}-h_{ab,}{}^{\mu}\simeq MV_{\lambda}F_{a}{}^{\lambda\mu}{}_{b}$, the last side of the relation above becomes

$$\frac{\partial}{\partial y}\theta_{ab}\simeq\frac{1}{2}V_{\lambda}[\psi_{,a}(Q^{\lambda}_{b}+T^{\lambda}_{a})+\psi_{,b}(Q^{\lambda}_{a}+T^{\lambda}_{a})+2\psi_{,}{}^{\lambda}(Q_{ab}+T_{ab})$$
$$+M^{2}\psi_{,\mu}V^{\mu}V_{\rho}(F_{a}{}^{\rho\lambda}{}_{b}+F_{b}{}^{\rho\lambda}{}_{a})$$
$$-\psi MV_{\mu}(h^{\mu}_{a}Q^{\lambda}_{b}+h^{\mu}_{b}Q^{\lambda}_{a})+\psi(Q^{\lambda}_{a,b}+Q^{\lambda}_{b,a})]$$
$$-\frac{\psi}{2}\Big(T-\frac{n-1}{n}K\Big)[h^{\lambda}_{a}(Q_{b\lambda}+T_{b\lambda})+h^{\lambda}_{b}(Q_{a\lambda}+T_{a\lambda})$$
$$-2(K-T)h_{ab}].$$

On the other hand, we get from (34)

$$Q^{\lambda}_{a,b}V_{\lambda}+Q^{\lambda}_{a}V_{\lambda,b}+\Big(T-\frac{n-1}{n}K\Big)V_{a,b}+V_{a}T_{,b}\simeq 0,$$

hence

$$Q^{\lambda}_{a,b}V_{\lambda}+(K-T)h^{\lambda}_{b}Q_{a\lambda}-T^{\mu}_{\lambda}h_{\mu b}Q^{\lambda}_{a}+2V_{a}V_{\lambda}h^{\lambda}_{b}$$
$$+\Big(T-\frac{n-1}{n}K\Big)\{(K-T)h_{ab}-T^{\lambda}_{a}h_{\lambda b}\}\simeq 0.$$

Accordingly, by the relation above and (34) we get

$$\frac{\partial}{\partial y}\theta_{ab}\simeq\frac{1}{2}V_{\lambda}[\psi_{,a}(Q^{\lambda}_{b}+T^{\lambda}_{b})+\psi_{,b}(Q^{\lambda}_{a}+T^{\lambda}_{a})+2\psi_{,}{}^{\lambda}(Q_{ab}+T_{ab})$$
$$+M^{2}\psi_{,\mu}V^{\mu}V_{\rho}(F_{a}{}^{\rho\lambda}{}_{b}+F_{b}{}^{\rho\lambda}{}_{a})-\psi MV_{\mu}(h^{\mu}_{a}Q^{\lambda}_{b}+h^{\mu}_{b}Q^{\lambda}_{a})]$$
$$(40)\qquad -\frac{\psi}{2}MV_{\lambda}V^{\mu}(h_{\mu b}Q^{\lambda}_{a}+h_{\mu a}Q^{\lambda}_{b})-\psi V_{\lambda}(V_{a}h^{\lambda}_{b}+V_{b}h^{\lambda}_{a})$$
$$\simeq-\frac{1}{2}\Big(T-\frac{n-1}{n}K\Big)(\psi_{,a}V_{b}+\psi_{,b}V_{a})+\psi_{,\lambda}V^{\lambda}(Q_{ab}+T_{ab})$$
$$+M^{2}\psi_{,\mu}V^{\mu}V_{\lambda}V_{\rho}F_{a}{}^{\rho\lambda}{}_{b}+\frac{1}{2}(\psi_{,a}V_{\lambda}T^{\lambda}_{b}+\psi_{,b}V_{\lambda}T^{\lambda}_{a}).$$

We get easily

$$V_\lambda T_a^\lambda \simeq \left(T - \frac{n-1}{n}K\right)V_a$$

and by (36)

$$V_\rho(h^\rho_{a,b} - h^\rho_{b,a}) \simeq M V_\rho V_\lambda F_a{}^{\lambda\rho}{}_b .$$

Hence by (35) we get

$$-\left(T - \frac{n-1}{n}K\right)(Q_{ab} + T_{ab}) \simeq M V_\rho V_\lambda F_a{}^{\lambda\rho}{}_b .$$

Substituting this relation into (40), we get finally.

$$\frac{\partial}{\partial y}\theta_{ab} \simeq 0. \tag{41}$$

Thus, by virtue of (28), (29), (30), (31′), (32′) (41), we obtain a theorem.

Theorem 5. *In order that a given n-dimensional Riemann space V_n with line element $ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu$ can be imbedded in an $(n+1)$-dimensional Riemann space V_{n+1} with non zero scalar curvature K satisfying the conditions* (a), (b), *as a hypersurface whose normal direction at each point to it is not the null direction of the Ricci tensor of V_{n+1}, it is necessary and sufficient that a system of equations with unknown quantities h_a^b, T_a^b, $V_a (g_{\lambda\mu}V^\lambda V^\mu \neq 0)$:*

$$\left(T_a^b - \frac{1}{n}K\delta_a^b\right)\left(T - \frac{n-1}{n}K\right) + V_a V^b = 0, \tag{27}$$

$$V_a - h_{,a} + h^\lambda_{a;\lambda} = 0,$$

$$T^b_{a,c} - h_{ac}V^a - h_c^b V_a = 0, \tag{23}$$

$$V_{a,b} + T_a^\lambda h_{\lambda b} - h_{ab}(K - T) = 0, \tag{24}$$

$$hh - h_\lambda^\mu h_\mu^\lambda - R - K + 2T = 0,$$

$$\frac{1}{2}V_\lambda(h^\lambda_{a,b} + h^\lambda_{b,a} - 2h_{ab,}{}^\lambda) \tag{35′}$$

$$+\left(T - \frac{n-1}{n}K\right)(hh_{ab} - h_a^\lambda h_{b\lambda} - R_{ab} + T_{ab}) = 0,$$

has a solution.

Remark 1. The system of equations are reduced to a system of differential equations including only h_{ab} and its covariant derivatives of order 2 at most.

Remark 2. If $T \neq \frac{2n-1}{2n}K$, (23) is derived from the others. For, we get from the integrability condition of (24)

$$\left(T - \frac{n-1}{n}K\right)(h_{ab,c} - h_{ac,b}) - V_\lambda F_a{}^\lambda{}_{bc} = 0,$$

hence

$$\left(T - \frac{n-1}{n}K\right)V_\lambda(h^\lambda_{a,b} - h^\lambda_{b,a}) - V_\lambda V_\mu F_a{}^{\lambda\mu}{}_b = 0.$$

Accordingly, (35′) can be replaced with

$$V_\lambda V_\mu F_a{}^{\lambda\mu}{}_b + \left(T - \frac{n-1}{n}K\right)^2(Q_{ab} + T_{ab}) = 0.$$

We get from (27)

$$(42) \quad \left(T^b_a - \frac{1}{n}K\delta^b_a\right)T_{,c} + \left(T - \frac{n-1}{n}K\right)T^b_{a,c}$$
$$+ \{-T^\lambda_a h_{\lambda c} + h_{ac}(K-T)\}V^b + V_a\{-T^{b\lambda}h_{\lambda c} + h^b_c(K-T)\} = 0,$$

hence it follows that

$$(T-K)T_{,c} + \left(T - \frac{n-1}{n}K\right)T_{,c} + 2(K-T)h^\lambda_c V_\lambda - 2h^\lambda_c T^\mu_\lambda V_\mu = 0.$$

We can easily prove the relation

$$T^\lambda_a V - \left(T - \frac{n-1}{n}K\right)V_a = 0.$$

Hence, we get from the above relation

$$\left(T - \frac{2n-1}{2n}K\right)(T_{,c} - 2h^\lambda_c V_\lambda) = 0.$$

that is $T_{,c} - 2h^\lambda_c V_\lambda = 0$ by the assumption $T - \frac{2n-1}{2n}K \neq 0$. Substituting this relation into (42), we obtain the relation

$$\left(T - \frac{n-1}{n}K\right)(T^b_{a,c} - h_{ac}V^b - h^b_c V_a) = 0.$$

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