ON THE INDUCED CHARACTERS OF GROUPS OF FINITE ORDER

MASARU OSIMA

In this paper we shall study some properties of induced characters of groups[1]. Let $\mathfrak{G}$ be a group of finite order, and let K be an algebraic number field which contains the absolutely irreducible characters of $\mathfrak{G}$ as well as those of the subgroups of $\mathfrak{G}$. We consider the representations of $\mathfrak{G}$ in K. The distinct irreducible characters of $\mathfrak{G}$ will be denoted by $\chi_1, \chi_2, \cdots\cdots, \chi_n$, where in particular χ_1 means, as usual, the 1-character: $\chi_1(G) = 1$ for all G in $\mathfrak{G}$. Here n is equal to the number of classes of conjugate elements in $\mathfrak{G}$. Let $\mathfrak{Q}$ be a fixed Sylow-subgroup of $\mathfrak{G}$ belonging to a prime q. We denote by $\vartheta_1, \vartheta_2, \cdots\cdots, \vartheta_m$ the distinct irreducible characters of $\mathfrak{Q}$. We assume also that ϑ_1 is the 1-character, If we denote by ϑ_ν^* the character of $\mathfrak{G}$ induced by the character ϑ_ν, then we have by Frobenius' theorem,

$$(*)\qquad \begin{cases} \chi_\mu(Q) = \sum_{\nu=1}^{m} r_{\mu\nu}\vartheta_\nu(Q) & \text{(for } Q \text{ in } \mathfrak{Q}) \\ \vartheta_\nu^*(G) = \sum_{\mu=1}^{n} r_{\mu\nu}\chi_\mu(G) & \text{(for } G \text{ in } \mathfrak{G}) \end{cases}$$

where the $r_{\mu\nu}$ are rational integers, $r_{\mu\nu} \geq 0$. Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots\cdots, \mathfrak{K}_n$ be the classes of conjugate elements in $\mathfrak{G}$, and let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots\cdots, \mathfrak{K}_h$ be those which contain the elements of $\mathfrak{Q}$. As is well known, the number of linearly independent characters ϑ_ν^* is h. In §1, we shall construct the generalized characters $\vartheta_1', \vartheta_2', \cdots\cdots, \vartheta_h'$ of $\mathfrak{Q}$ which satisfy the following conditions:

(i) $\vartheta_\lambda'(Q) = \vartheta_\lambda(Q) + \sum_{\kappa=1}^{m-h} b_{\lambda, h+\kappa}\vartheta_{h+\kappa}(Q)$, where the $b_{\lambda, h+\kappa}$ are rational numbers with the denominators prime to q;

(ii) $\vartheta_1'(Q) = \vartheta_1(Q)$;

(iii) $\vartheta_1'(Q), \vartheta_2'(Q), \cdots\cdots, \vartheta_h'(Q)$ are linearly independent;

(iv) $\vartheta_\lambda'(Q) = \vartheta_\lambda'(Q')$, if two elements Q and Q' of $\mathfrak{Q}$ are conjugate in $\mathfrak{G}$;

(v) $\vartheta_1^*(G), \vartheta_2^*(G), \cdots\cdots, \vartheta_h^*(G)$ are linearly independent;

(vi) $\chi_\mu(Q) = \sum_{\lambda=1}^{h} r_{\mu\lambda}\vartheta_\lambda'(Q)$ with the same $r_{\mu\lambda}$ $(\lambda = 1, 2, \cdots\cdots, h)$ as in $(*)$.

---

1) A summary of the results obtained herein appeared in [8].

In §2 we study the connection between the group characters and these generalized characters. Let G be an element of $\mathfrak{G}$ which does not belong to $\mathfrak{K}_i$ $(i=1, 2, \cdots\cdots, h)$. Then G is of form $G = AQ = QA$, where the order of A is prime to q and the order of Q is a power $q^\nu \geq 1$ of q. We denote by $\mathfrak{N}(A)$ the normalizer of A in $\mathfrak{G}$, and by $\mathfrak{N}(A)_q$ a q-Sylow-subgroup of $\mathfrak{N}(A)$. We may construct the generalized characters of $\mathfrak{N}(A)_q$ which have the same meaning for $\mathfrak{N}(A)$ as the ϑ'_λ have for $\mathfrak{G}$. Then the value $\chi_\mu(AQ)$ is expressed by these generalized characters of $\mathfrak{N}(A)_q$. The coefficients are not necessarily rational, but they are algebraic integers. As an application, we shall prove a group theoretical theorem due to Brauer ([3], Theorem 1) which played a fundamental role to prove the conjectures of Artin and Schur (see [3], [4]). Our proof may be considered as an improvement of Brauer's original one[1].

In §3 we shall apply our method to the theory of modular characters of $\mathfrak{G}$ for a prime $p \neq q$. In particular we shall prove Brauer's theorem concerning the determinant of Cartan invariants of $\mathfrak{G}$ ([1], Theorem 1).

1. We consider a group $\mathfrak{G}$ of finite order $g = q^a g'$ where q is a prime number and $(g', q) = 1$. Let $Q_1, Q_2, \cdots\cdots, Q_h$ be representatives for the h classes $\mathfrak{K}_1, \mathfrak{K}_2, \cdots\cdots, \mathfrak{K}_h$, as described in the introduction, and let $\mathfrak{Q}^{(i)}$ be a q-Sylow-subgroup of the normalizer $\mathfrak{N}(Q_i)$ of Q_i in $\mathfrak{G}$. Replacing Q_i by a suitable conjugate $G^{-1}Q_iG$, we may assume, in virtue of Sylow's theorem, that

$$\mathfrak{Q}^{(i)} \subset \mathfrak{Q} \qquad (i = 1, 2, \cdots\cdots, h). \tag{1.1}$$

Denote by n_i the order of $\mathfrak{N}(Q_i)$. We set $n_i = q_i n'_i$ where q_i is a power of q and $(n'_i, q) = 1$. Then we see by (1.1) that the order of the normalizer of Q_i in $\mathfrak{Q}$ is q_i.

In $\mathfrak{Q}$, the elements $Q_1, Q_2, \cdots\cdots, Q_h$ need not form a complete system of representatives for the classes of conjugate elements. However, we may construct such a system by adding further elements Q to the set $Q_1, Q_2, \cdots\cdots, Q_h$. Each Q will be conjugate in $\mathfrak{G}$ to a certain Q_i, i being uniquely determined by Q. We denote the elements Q belonging to Q_i by $Q_i = Q_i^{(0)}, Q_i^{(1)}, \cdots\cdots, Q_i^{(l_i)}$, $(l_i \geq 0)$. Then

$$m = h + \sum_{i=1}^{h} l_i, \tag{1.2}$$

1) Recently new simpler proofs for this theorem were obtained from the properties of the character ring of $\mathfrak{G}$. See [5], [9] and [10].

where m denotes the number of irreducible characters of $\mathfrak{Q}$. Let $q_i^{(\kappa)}$ be the order of the normalizer of $Q_i^{(\kappa)}$ in $\mathfrak{Q}$. We have

$$q_i^{(0)} = q_i . \tag{1.3}$$

From (*), we have

$$(\chi_\mu(Q_i)) = (r_{\mu\nu})(\vartheta_\nu(Q_i)) \tag{1.4}$$

($\mu = 1, 2, \cdots\cdots, n$; $\nu = 1, 2, \cdots\cdots, m$; $i = 1, 2, \cdots\cdots, h$). Here $r_{11} = 1$, $r_{1\nu} = 0$ for $\nu \neq 1$. The rank of the matrix $(r_{\mu\nu})$ is h. Since Q_i and $Q_i^{(\kappa)}$ are conjugate in $\mathfrak{G}$, $\chi_\mu(Q_i) = \chi_\mu(Q_i^{(\kappa)})$ and hence

$$\sum_\nu r_{\mu\nu}\vartheta_\nu(Q_i) = \sum_\nu r_{\mu\nu}\vartheta_\nu(Q_i^{(\kappa)}). \tag{1.5}$$

We denote by $\bar{\vartheta}_\nu$ the character conjugate complex to ϑ_ν. Then

$$\vartheta_\nu(Q^{-1}) = \overline{\vartheta_\nu(Q)}.$$

We have from (1.5)

$$\sum_\nu r_{\mu\nu}\overline{\vartheta_\nu(Q_i)} = \sum_\nu r_{\mu\nu}\overline{\vartheta_\nu(Q_i^{(\kappa)})}. \tag{1.6}$$

We arrange $\vartheta_\nu(Q_i^{(\kappa)})$ in matrix form

$$\Theta = (\vartheta_\nu(Q_i^{(\kappa)})) \qquad \nu \text{ row index; } i, \kappa \text{ column indices.} \tag{1.7}$$

We arrange the columns so that first the h columns with $\kappa = 0$ appear and then the l_1 columns with $i = 1$ and so on. Thus

$$\Theta = (\Theta_0 \ \ \Theta_1), \tag{1.8}$$

where Θ_0 is of type (m, h) and Θ_1 of type $(m, m-h)$. If we set

$$(\overline{\vartheta_\nu(Q_i^{(\kappa)})}) = \bar{\Theta} = (\bar{\Theta}_0 \ \ \bar{\Theta}_1), \tag{1.9}$$

then

$$|\Theta| = \pm |\bar{\Theta}| . \tag{1.10}$$

We denote by M' the transpose of a matrix M. By the orthogonality relations for the characters of $\mathfrak{Q}$

$$\bar{\Theta}'\Theta = \begin{pmatrix} q_1 & & & & & 0 \\ & \ddots & & & & \\ & & q_h & & & \\ & & & q_1^{(1)} & & \\ & & & & \ddots & \\ 0 & & & & & q_h^{(l_h)} \end{pmatrix}. \tag{1.11}$$

Hence

(1.12) $$|\bar{\Theta}'\Theta| = \pm|\Theta|^2 = \prod_i \prod_\kappa q_i^{(\kappa)}.$$

We subtract every column $(i, 0)$ from all the columns (i, κ) with $\kappa > 0$, and with the same first index. Then we obtain a new matrix $(\Theta_0 \;\; \Theta_2)$ which may be written as

(1.13) $$(\Theta_0 \;\; \Theta_2) = (\Theta_0 \;\; \Theta_1)P = \Theta P,$$

where P is a unimodular matrix with $|P| = 1$. From (1.13) we have

(1.14) $$|(\Theta_0 \;\; \Theta_2)| = |\Theta||P| = |\Theta|.$$

Since $(\bar{\Theta}_0 \;\; \bar{\Theta}_2) = \bar{\Theta}P$, it follows that

$$\begin{pmatrix} \bar{\Theta}_0' \\ \bar{\Theta}_2' \end{pmatrix}(\Theta_0 \;\; \Theta_2) = P'(\bar{\Theta}'\Theta)P.$$

We then obtain from the form of P and (1.11)

(1.15) $$\bar{\Theta}_2'\Theta_2 = \begin{pmatrix} \Omega_1 & & & 0 \\ & \Omega_2 & & \\ & & \ddots & \\ 0 & & & \Omega_h \end{pmatrix},$$

where $\Omega_i = (\rho_{\kappa\lambda}^{(i)})$, $(1 \leqq \kappa, \lambda)$ is of type (l_i, l_i) and

$$\rho_{\kappa\lambda}^{(i)} = \begin{cases} q_i + q_i^{(\kappa)} & \text{for } \kappa = \lambda \\ q_i & \text{for } \kappa \neq \lambda. \end{cases}$$

We see easily that

$$|\Omega_i| = q_i^{(1)}q_i^{(2)} \cdots\cdots q_i^{(l_i)}\left(1 + \frac{q_i}{q_i^{(1)}} + \cdots\cdots + \frac{q_i}{q_i^{(l_i)}}\right).$$

Since $q_i^{(\kappa)}$ $(\kappa = 0, 1, 2, \cdots\cdots, l_i)$ are divisors of q_i and moreover the number d_i of $q_i^{(\kappa)}$ such that $q_i^{(\kappa)} = q_i$, is prime to q (see [1], Lemma), we have

(1.16) $$|\bar{\Theta}_2'\Theta_2| \equiv 0 \qquad (\text{mod } \prod_i \prod_{0<\kappa} q_i^{(\kappa)}),$$

and

(1.17) $$|\bar{\Theta}_2'\Theta_2| \not\equiv 0 \qquad (\text{mod } q(\prod_i \prod_{0<\kappa} q_i^{(\kappa)})).$$

Since there exists a minor $|\Delta|$ of degree h of $(\chi_\mu(Q_i))$ $(\mu = 1, 2, \cdots\cdots, n;\ i = 1, 2, \cdots\cdots, h)$ with $|\Delta| \neq 0$, we may assume that

$$\Delta = \begin{pmatrix} \chi_\rho(Q_1) & \chi_\rho(Q_2) & \cdots\cdots & \chi_\rho(Q_h) \\ \chi_\sigma(Q_1) & \chi_\sigma(Q_2) & \cdots\cdots & \chi_\sigma(Q_h) \\ \cdots & \cdots & \cdots & \cdots \\ \chi_\tau(Q_1) & \chi_\tau(Q_2) & \cdots\cdots & \chi_\tau(Q_h) \end{pmatrix}$$

and $|\Delta| \neq 0$. If we set

$$Z = \begin{pmatrix} r_{\rho 1} & r_{\rho 2} & \cdots\cdots & r_{\rho m} \\ r_{\sigma 1} & r_{\sigma 2} & \cdots\cdots & r_{\sigma m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{\tau 1} & r_{\tau 2} & \cdots\cdots & r_{\tau m} \end{pmatrix}$$

then we have

$$\Delta = Z\theta_0 .$$

We see by (1.5) and (1.6) that $Z\theta_2 = 0$ and $Z\bar{\theta}_2 = 0$. If we set

$$U = \begin{pmatrix} Z \\ \bar{\theta}_2' \end{pmatrix}$$

then U is of type (m, m) and

$$U(\theta_0\ \ \theta_2) = \begin{pmatrix} Z \\ \bar{\theta}_2' \end{pmatrix}(\theta_0\ \ \theta_2) = \begin{pmatrix} \Delta & 0 \\ * & \bar{\theta}_2'\theta_2 \end{pmatrix}. \tag{1.18}$$

It follows from $|\bar{\theta}_2'\theta_2| \neq 0$ that $|U| \neq 0$. Now we set

$$V = \begin{pmatrix} Z \\ \theta_2' \end{pmatrix}.$$

Then $|U| = \pm|V|$, and

$$UV' = \begin{pmatrix} ZZ' & 0 \\ 0 & \bar{\theta}_2'\theta_2 \end{pmatrix}.$$

Hence

$$|U|^2 = \pm|ZZ'|\,|\bar{\theta}_2'\theta_2| \equiv 0 \qquad (\mathrm{mod}\ \prod_i \prod_{0<\kappa} q_i^{(\kappa)}). \tag{1.19}$$

On combining (1.14), (1.16), (1.18) and (1.19), we have for every minor of degree h of $(\chi_\mu(Q_i))$

(1.20) $$|\varDelta|^2 \equiv 0 \qquad (\mathrm{mod}\ \prod_i q_i).$$

Let $\mathfrak{q}$ be a prime ideal of K which divides the prime q, and let $\mathfrak{q}^*$ be the highest power of the prime ideal $\mathfrak{q}$ which divides $(q_1 q_2 \cdots\cdots q_h)^{\frac{1}{2}}$. Then there exists at least one minor $|\varDelta|$ of degree h of $(\chi_\mu(Q_i))$, such that $|\varDelta| \not\equiv 0$ (mod $\mathfrak{q}\mathfrak{q}^*$) (see [3], p. 508). If we choose such minor $|\varDelta|$, then it follows from (1.19) that $|ZZ'| \not\equiv 0$ (mod q). This implies that there exists at least one minor of degree h of Z which is not divisible by q. Further we may assume that this minor contains the coefficients in the first row of $(r_{\mu\nu})$, since $r_{11} = 1$, $r_{1\nu} = 0$ $(\nu \neq 1)$, and the rank of $(r_{\mu\nu})$ is h. Hence if the notation is suitably chosen, we have

(1.21) $$|Z_1| = \begin{vmatrix} r_{11} & r_{12} & \cdots\cdots & r_{1h} \\ r_{21} & r_{22} & \cdots\cdots & r_{2h} \\ \cdots & \cdots & \cdots & \cdots \\ r_{h1} & r_{h2} & \cdots\cdots & r_{hh} \end{vmatrix} \not\equiv 0 \qquad (\mathrm{mod}\ q),$$

where $r_{11} = 1$, $r_{1\nu} = 0$ $(\nu \neq 1)$. Set

(1.22) $$\varDelta = (\chi_\mu(Q_i)) \qquad (\mu, i = 1, 2, \cdots\cdots, h).$$

We then have

(1.23) $$|\varDelta| \equiv 0 \quad (\mathrm{mod}\ \mathfrak{q}^*), \qquad |\varDelta| \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{q}\mathfrak{q}^*).$$

Thus we have the following

Lemma 1 *Let n_i be the order of the normalizer $\mathfrak{N}(Q_i)$ of Q_i in $\mathfrak{G}$, and let $\mathfrak{q}^*$ be the highest power of the prime ideal $\mathfrak{q}$ which divides $(n_1 n_2 \cdots\cdots n_h)^{\frac{1}{2}}$. Then there exists a minor $|\varDelta|$ of degree h of $(\chi_\mu(Q_i))$, such that the matrix $\varDelta$ contains the coefficients in the first row of $(\chi_\mu(Q_i))$ and*

$$|\varDelta| \equiv 0 \quad (\mathrm{mod}\ \mathfrak{q}^*) \qquad |\varDelta| \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{q}\mathfrak{q}^*).$$

If the notation is chosen so as $\varDelta$ appears in the first h rows of $(\chi_\mu(Q_i))$, then the first h rows of $(r_{\mu\nu})$ contain a minor of degree h which is not divisible by q. The rank of $(r_{\mu\nu})$ (mod q) is h.

Evidently this lemma may be considered as a special case of Brauer's result in [3] (see p. 507).

We denote by R the matrix of the first h columns of $(r_{\mu\nu})$:

$$R = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \qquad |Z_1| \not\equiv 0 \qquad (\text{mod } q).$$

Since

$$(r_{\mu\nu}) = \begin{pmatrix} Z_1 & Z_3 \\ Z_2 & Z_4 \end{pmatrix} = R(I\ \ B),$$

we see that the coefficients of B are rational numbers with the denominators $|Z_1|$. Moreover all the coefficients in the first row of B are zero. Now we set

(1.24) $$(I\ \ B)(\vartheta_\nu(Q_i)) = (\vartheta'_\lambda(Q_i))$$

$(\nu = 1, 2, \cdots\cdots, m;\ i, \lambda = 1, 2, \cdots\cdots, h)$, then

(1.25) $$\vartheta'_1 = \vartheta_1.$$

It follows from $(\chi_\mu(Q_i)) = R(I\ \ B)(\vartheta_\nu(Q_i)) = R(\vartheta'_\nu(Q))$ that

(1.26) $$\chi_\mu(Q) = \sum_{\lambda=1}^{h} r_{\mu\lambda}\vartheta'_\lambda(Q) \qquad (\text{for } Q \text{ in } \mathfrak{Q}).$$

If we set $\Theta^* = (\vartheta'_\lambda(Q_i))$, then $|\Delta| = |Z_1||\Theta^*|$ and from (1.21), (1.23) we obtain

(1.27) $$|\Theta^*| \equiv 0 \quad (\text{mod } \mathfrak{q}^*), \qquad |\Theta^*| \not\equiv 0 \quad (\text{mod } \mathfrak{q}\mathfrak{q}^*).$$

Hence we have

Lemma 2. $\vartheta'_1(Q), \vartheta'_2(Q), \cdots\cdots, \vartheta'_h(Q)$ *are linearly independent.*

Combination of (1.26) with $|Z_1| \neq 0$, yields

(1.28) $$\vartheta'_\lambda(Q_i) = \vartheta'_\lambda(Q_i^{(\kappa)}).$$

If we set

(1.29) $$B = (b_{\lambda, h+\kappa}) \qquad \lambda = 1, 2, \cdots\cdots, h;\ \kappa = 1, 2, \cdots\cdots, m-h,$$

where $b_{1, h+\kappa} = 0$ $(\kappa = 1, 2, \cdots\cdots, m-h)$, then (1.24) shows

(1.30) $$\vartheta'_\lambda(Q) = \vartheta_\lambda(Q) + \sum_{\kappa=1}^{m-h} b_{\lambda, h+\kappa}\vartheta_{h+\kappa}(Q) \qquad (\lambda = 1, 2, \cdots\cdots, h).$$

We have from (*)

$$\vartheta^*_\lambda(G) = \sum_\mu r_{\mu\nu}\chi_\mu(G) \qquad (\lambda = 1, 2, \cdots\cdots, h),$$

or in matrix form

(1.31) $$\Psi = (\vartheta_\lambda^*(Q_i)) = R'(\chi_\mu(Q_i)).$$

On the other hand we find

$$\vartheta_{h+\kappa}^*(G) = \sum_\mu r_{\mu, h+\kappa}\chi_\mu(G) = \sum_\mu (\sum_{\lambda=1}^h r_{\mu\lambda} b_{\lambda, h+\kappa})\chi_\mu(G)$$
$$= \sum_{\lambda=1}^h b_{\lambda, h+\kappa}(\sum_\mu r_{\mu\nu}\chi_\mu(G)) = \sum_{\lambda=1}^h b_{\lambda, h+\kappa}\vartheta_\lambda^*(G).$$

As is well known, we have

$$\sum_{\nu=1}^m \vartheta_\nu^*(Q_i)\vartheta_\nu(Q_j^{-1}) = \delta_{ij} n_i,$$

and hence, on replacing $\vartheta_{h+\kappa}^*$ by $\sum_\lambda b_{\lambda, h+\kappa}\vartheta_\lambda^*$, we obtain

(1.32) $$\sum_{\lambda=1}^h \vartheta_\lambda^*(Q_i)\vartheta_\lambda'(Q_j^{-1}) = \delta_{ij} n_i.$$

Then (1.32) yields

(1.33) $$\sum_i g_i \vartheta_\kappa^*(Q_i)\vartheta_\lambda'(Q_i^{-1}) = \delta_{\kappa\lambda} g \qquad (\kappa, \lambda = 1, 2, \cdots\cdots, h),$$

where $g_i = g/n_i$. Further (1.32) implies that $\vartheta_1^*(G), \vartheta_2^*(G), \cdots\cdots, \vartheta_h^*(G)$ are linearly independent.

If we set

(1.34) $$W = R'R = (w_{\kappa\lambda}),$$

then, since $\Psi = R'(\chi_\mu(Q_i)) = R'R\Theta^*$,

(1.35) $$\vartheta_\kappa^*(Q) = \sum_\lambda w_{\kappa\lambda}\vartheta_\lambda'(Q) \qquad \text{(for } Q \text{ in } \mathfrak{Q}).$$

We have $|\bar{\Theta}^*| = \pm |\Theta^*|$, where $\bar{\Theta}^* = (\overline{\vartheta_\lambda'(Q_i)})$. Hence it follows from (1.32) that

$$|\bar{\Theta}^*||\Psi| = \pm|\Theta^*|^2|W| = n_1 n_2 \cdots\cdots n_h.$$

This implies that $|\Theta^*|^2$ is a rational number and

(1.36) $$|W| \not\equiv 0 \qquad (\text{mod } q).$$

It we set $W^{-1} = (\sigma_{\kappa\lambda})$, then (1.33) yields

(1.37) $$\sum_i g_i \vartheta_\kappa'(Q_i)\vartheta_\lambda'(Q_i^{-1}) = \sigma_{\kappa\lambda} g \qquad (\kappa, \lambda = 1, 2, \cdots\cdots, h),$$

(1.38) $$\sum_i g_i \vartheta_\kappa^*(Q_i)\vartheta_\lambda^*(Q_i^{-1}) = w_{\kappa\lambda} g \qquad (\kappa, \lambda = 1. 2, \cdots\cdots, h).$$

Theorem 1. *If we set* $(\vartheta'_\lambda(Q_i)) = \theta^*$, *then*

$$|\theta^*|^2 = q_1 q_2 \cdots\cdots q_s / v,$$

where v is a rational integer prime to q.

Proof. Since $(I \ B)\theta_2 = 0$ by (1.28), we have

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}(\theta_0, \ \theta_2) = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} \theta_{0,1} & \theta_{2,1} \\ \theta_{0,2} & \theta_{2,2} \end{pmatrix} = \begin{pmatrix} \theta^* & 0 \\ \theta_{0,2} & \theta_{2,2} \end{pmatrix}$$

and hence $|\theta| = |\theta^*||\theta_{2,2}|$, where $|\theta_{2,2}|$ is an algebraic integer. (1.10) and (1.11) show that $|\theta|^2$ is a power of q. Hence $|\theta^*|^2$ is not divisible by any prime number $p \neq q$. Further, since $|\theta^*|^2$ is rational, we see from (1.27) that our theorem is valid.

We shall consider a special case when $\mathfrak{G}$ contains a normal q-Sylow-subgroup $\mathfrak{Q}$. The irreducible characters $\vartheta_1, \vartheta_2, \cdots\cdots, \vartheta_m$ of $\mathfrak{Q}$ are distributed into classes of characters which are associated with regard to $\mathfrak{G}$; two characters ϑ_μ and ϑ_ν being associated if

$$\vartheta_\mu(Q) = \vartheta_\nu(G^{-1}QG),$$

where Q is a variable element of $\mathfrak{Q}$ and G is a fixed element of $\mathfrak{G}$. The number of such classes is equal to h. Let $\vartheta_1, \vartheta_2, \cdots\cdots, \vartheta_h$ be a complete system of representatives for those classes. Further let $\vartheta_\lambda = \vartheta_\lambda^{(0)}, \vartheta_\lambda^{(1)}, \cdots\cdots, \vartheta_\lambda^{(t_\lambda)}$ be mutually associated characters. It is easy to see that

(1.39) $$\vartheta'_\lambda(Q) = \sum_\rho \vartheta_\lambda^{(\rho)}(Q).$$

Hence we see that $v = 1$ in Theorem 1. If $r_{\mu\nu} \neq 0$ for some λ in (1.25), then $r_{\mu\kappa} = 0$ for $\kappa \neq \lambda$, that is, $\chi_\mu(Q) = r_{\mu\lambda}\vartheta'_\lambda(Q)$. We say that χ_μ corresponds to the character ϑ'_λ. Let $\chi_{\lambda_1}, \chi_{\lambda_2}, \cdots\cdots, \chi_{\lambda_s}$ be the characters corresponding to ϑ'_λ, then

(1.40) $$w_{\kappa\lambda} = \begin{cases} 0 & \text{for } \kappa \neq \lambda \\ \sum_{i=1}^{s} r_{\lambda_i\lambda}^2 & \text{for } \kappa = \lambda. \end{cases}$$

We see from (1.36) that there exists at least one $r_{\lambda_i\lambda}$ which is prime to q for each λ.

2. We call an element G of $\mathfrak{G}$ q-regular if its order is prime to q. Let $A_1 = 1, A_2, \cdots\cdots, A_t$ be a maximal system of elements of $\mathfrak{G}$ such that A_k, A_l are not conjugate for $k \neq l$ and the order of each A_k is prime to q. Let $\mathfrak{N}_k$ be the normalizer of A_k in $\mathfrak{G}$ and let $\mathfrak{Q}_k$ be a q-Sylow-subgroup of $\mathfrak{N}_k$. A full system $\sum$ of elements of $\mathfrak{G}$ representing the different classes of conjugate elements can be obtained in the following manner: Let $Q_1^{(k)}, Q_2^{(k)}, \cdots\cdots, Q_{h(k)}^{(k)}$ ($Q_i^{(k)} \in \mathfrak{Q}_k$) represent the different classes of conjugate elements in $\mathfrak{N}_k$, in which the orders of the elements are powers of q. Then $\sum$ consists of the elements $A_k Q_i^{(k)}$ ($k = 1, 2, \cdots\cdots, t$; $i = 1, 2, \cdots\cdots, h(k)$). Thus we have

$$n = \sum_{k=1}^{t} h(k), \qquad (h(1) = h). \tag{2.1}$$

Let us denote by $n_i^{(k)}$ the order of the normalizer $\mathfrak{N}(A_k Q_i^{(k)})$ of $A_k Q_i^{(k)}$ in $\mathfrak{G}$. Then the order of the normalizer of $Q_i^{(k)}$ in $\mathfrak{N}_k$ is equal to $n_i^{(k)}$. We set $n_i^{(k)} = q_i^{(k)} n_i^{(k)\prime}$, where $(n_i^{(k)\prime}, q) = 1$ and $q_i^{(k)}$ is a power of q. We denote by $\chi_{k,1}, \chi_{k,2}, \cdots\cdots, \chi_{k,n(k)}$ the irreducible characters of $\mathfrak{N}_k$, and by $\vartheta_{k,1}, \vartheta_{k,2}, \cdots\cdots, \vartheta_{k,m(k)}$ those of $\mathfrak{Q}_k$. If we apply the argument in §1 to $\mathfrak{N}_k$, we have for $Q^{(k)}$ in $\mathfrak{Q}_k$

$$\chi_{k,\mu}(Q^{(k)}) = \sum_{\nu=1}^{m(k)} r_{k,\mu\nu} \vartheta_{k,\nu}(Q^{(k)}) = \sum_{\lambda=1}^{h(k)} r_{k,\mu\lambda} \vartheta'_{k,\lambda}(Q^{(k)}), \tag{2.2}$$

where the $\vartheta'_{k,\lambda}$ have the same meaning for $\mathfrak{N}_k$ as the ϑ'_λ have for $\mathfrak{G}$. We have from (2.2) (see similar argument in [2], p. 928.)

$$\chi_\mu(A_k Q^{(k)}) = \sum_{\lambda=1}^{h(k)} r^k_{\mu\lambda} \vartheta'_{k,\lambda}(Q^{(k)}) \qquad (\text{for } Q^{(k)} \text{ in } \mathfrak{Q}_k). \tag{2.3}$$

Here the $r^k_{\mu\lambda}$ are integers of the field of the ρ_kth roots of unity and ρ_k means the order of A_k. For $k = 1$, we have $A_1 = 1$, $\mathfrak{N}_1 = \mathfrak{G}$. Hence $r^1_{\mu\lambda} = r_{\mu\lambda}$. We arrange these numbers $r^k_{\mu\lambda}$ for a fixed k in form of a matrix $R^k = (r^k_{\mu\lambda})$ with μ as row index and λ as column index, and set

$$\mathsf{R} = (R^1, R^2, \cdots\cdots, R^t). \qquad R^1 = R. \tag{2.4}$$

We see from (2.1) that R is a square matrix of the same degree n as the matrix X of the group characters χ_μ of $\mathfrak{G}$. (2.3) yields

$$X = (\chi_\mu(A_k Q_i^{(k)})) = \mathsf{R}\Gamma \tag{2.5}$$

($\mu = 1, 2, \cdots\cdots, n$; $k = 1, 2, \cdots\cdots, t$; $i = 1, 2, \cdots\cdots, h(k)$). We see from

(2.5) that R is non-singular. Moreover the matrix Γ breaks up completely into the matrices $\Theta_k^* = (\vartheta'_{k,\lambda})$ $(k = 1, 2, \cdots\cdots, t)$:

$$\Gamma = \begin{pmatrix} \Theta_1^* & & & 0 \\ & \Theta_2^* & & \\ & & \ddots & \\ 0 & & & \Theta_t^* \end{pmatrix}. \tag{2.6}$$

Theorem 1 implies

$$| \Gamma |^2 = \prod_k | \Theta_k^* |^2 = \prod_k \left(\prod_{i=1}^{h(k)} q_i^{(k)} / v_k \right), \tag{2.7}$$

where $(v_k, q) = 1$. This implies

$$| \mathsf{R} | \not\equiv 0 \pmod{\mathfrak{q}}. \tag{2.8}$$

If we denote by $\vartheta_{k,\lambda}^*$ the character of $\mathfrak{N}_k$ induced by the character $\vartheta_{k,\lambda}$ of $\mathfrak{Q}_k$, then

$$\sum_{\lambda=1}^{h(k)} \vartheta_{k,\lambda}^*(Q_i^{(k)}) \overline{\vartheta'_{k,\lambda}(Q_j^{(k)})} = \delta_{ij} n_i^{(k)}. \tag{2.9}$$

On the other hand we have

$$\sum_\mu \chi_\mu(A_k Q_i^{(k)}) \overline{\chi_\mu(A_k Q_j^{(k)})} = \delta_{ij} n_i^{(k)},$$

and hence (2.3) yields

$$\sum_{\lambda=1}^{h(k)} \left(\sum_\mu \bar{r}_{\mu\lambda}^k \chi_\mu(A_k Q_i^{(k)}) \right) \overline{\vartheta'_{k,\lambda}(Q_j^{(k)})} = \delta_{ij} n_i^{(k)}. \tag{2.10}$$

Since $\vartheta'_{k,1}, \vartheta'_{k,2}, \cdots\cdots, \vartheta'_{k,h(k)}$ are linearly independent, we obtain from (2.9), (2.10)

$$\vartheta_{k,\lambda}^*(Q^{(k)}) = \sum_\mu \bar{r}_{\mu\lambda}^k \chi_\mu(A_k Q^{(k)}). \tag{2.11}$$

(2.11), combined with (2.3), yields

$$\begin{aligned} \vartheta_{k,\lambda}^*(Q^{(k)}) &= \sum_\kappa w_{\kappa\lambda}^k \vartheta'_{k,\kappa}(Q^{(k)}) \\ &= \sum_\kappa \left(\sum_\mu \bar{r}_{\mu\lambda}^k r_{\mu\kappa}^k \right) \vartheta'_{k,\kappa}(Q^{(k)}), \end{aligned}$$

and hence

$$w_{\kappa\lambda}^k = \sum_\mu r_{\mu\kappa}^k \bar{r}_{\mu\lambda}^k, \tag{2.12}$$

where the $w^k_{\kappa\lambda}$ have the same meaning for $\mathfrak{N}_k$ as the $w_{\kappa\lambda}$ have for $\mathfrak{G}$. Further from

$$\sum_\mu \chi_\mu(A_k Q^{(k)}_i)\overline{\chi_\mu(A_l Q^{(l)}_j)} = 0 \qquad (k \neq l),$$

we have $\sum_\mu \bar{r}^l_{\mu\lambda}\chi_\mu(A_k Q^{(k)}_i) = 0$, and hence

$$\sum_\mu r^k_{\mu\kappa}\bar{r}^l_{\mu\lambda} = 0 \qquad (k \neq l). \tag{2.13}$$

The group $\mathfrak{H}_k = \{A_k, \mathfrak{Q}_k\}$ generated by A_k and $\mathfrak{Q}_k$ is a direct product: $\mathfrak{H}_k = \{A_k\} \times \mathfrak{Q}_k$. An irreducible character $\psi^{(k)}_\sigma$ of $\mathfrak{G}_k$ is the product of an irreducible character $\xi^{(k)}_\alpha$ of the cyclic group $\{A_k\}$ and an irreducible character $\vartheta_{k,\nu}$ of $\mathfrak{Q}_k$:

$$\psi^{(k)}_\sigma(A_k Q^{(k)}_i) = \xi^{(k)}_\alpha(A_k)\,\vartheta_{k,\nu}(Q^{(k)}_i). \tag{2.14}$$

Let us denote by $(\xi^{(k)}_\alpha \vartheta_{k,\nu})^*$ the character of $\mathfrak{G}$ induced by the character $\xi^{(k)}_\alpha \vartheta_{k,\nu}$. Then we have, by Frobenius' theorem,

$$\begin{cases} \chi_\mu(A_k Q^{(k)}) = \sum_\nu \sum_\alpha r^k_{\alpha\mu\nu}\xi^{(k)}_\alpha(A_k)\,\vartheta_{k,\nu}(Q^{(k)}) \\ (\xi^{(k)}_\alpha \vartheta_{k,\nu})^*(G) = \sum_\mu r^k_{\alpha\mu\nu}\chi_\mu(G) \end{cases} \tag{2.15}$$

where the $r^k_{\alpha\mu\nu}$ are rational integers, $r^k_{\alpha\mu\nu} \geqq 0$. Then (2.3) and (2.15) yield

$$\sum_\lambda r^k_{\mu\lambda}\vartheta'_{k,\lambda}(Q^{(k)}) = \sum_\nu \Big(\sum_\alpha r^k_{\alpha\mu\nu}\xi^{(k)}_\alpha(A_k)\Big)\vartheta_{k,\nu}(Q^{(k)}).$$

Since $\vartheta_{k,1}, \vartheta_{k,2}, \cdots\cdots, \vartheta_{k,m(k)}$ are linearly independent, it follows from (1.30) that

$$r^k_{\mu\lambda} = \sum_\alpha r^k_{\alpha\mu\lambda}\xi^{(k)}_\alpha(A_k) \qquad (\lambda = 1, 2, \cdots\cdots, h(k)). \tag{2.16}$$

Observe that we have formulas analogous to (1.30) for $\vartheta'_{k,\lambda}$. We obtain from (2.16)

$$(r^k_{1\lambda}, r^k_{2\lambda}, \cdots\cdots, r^k_{n\lambda}) = (\xi^{(k)}_1(A_k), \cdots\cdots, \xi^{(k)}_\rho(A_k))\,L^{(k)}_\lambda, \tag{2.17}$$

where

$$L^{(k)}_\lambda = (r^k_{\alpha\mu\lambda}) \qquad \alpha \text{ row index; } \mu \text{ column index}$$

$(\alpha = 1, 2, \cdots\cdots, \rho;\ \mu = 1, 2, \cdots\cdots, n)$. Here $\rho = \rho_k$ is the order of A_k. We set

$$M_k = (\xi^{(k)}_1(A_k), \xi^{(k)}_2(A_k), \cdots\cdots, \xi^{(k)}_\rho(A_k)),$$

and

$$M_k^* = \begin{pmatrix} M_k & & & 0 \\ & M_k & & \\ & & \ddots & \\ 0 & & & M_k \end{pmatrix}, \tag{2.18}$$

where M_k appears in the main diagonal with multiplicity $h(k)$. Hence M_k^* is of type $(h(k), h(k)\rho_k)$. Further we set

$$L_k^* = \begin{pmatrix} L_1^{(k)} \\ L_2^{(k)} \\ \vdots \\ L_{h(k)}^{(k)} \end{pmatrix}. \tag{2.19}$$

Then L_k^* is of type $(h(k)\rho_k, n)$ and we have by (2.17)

$$(R^k)' = M_k^* L_k^*, \tag{2.20}$$

where $(R^k)'$ is the transpose of R^k. Hence if we set

$$M = \begin{pmatrix} M_1^* & & & 0 \\ & M_2^* & & \\ & & \ddots & \\ 0 & & & M_t^* \end{pmatrix}, \qquad L = \begin{pmatrix} L_1^* \\ L_2^* \\ \vdots \\ L_t^* \end{pmatrix}, \tag{2.21}$$

then M is of type $(n, \sum h(k)\rho_k)$ and L of type $(\sum h(k)\rho_k, n)$, and

$$\mathrm{R}' = ML. \tag{2.22}$$

We see from (2.8) and (2.22) that there exists at least one minor $|D|$ of degree n of L such that $|D| \not\equiv 0 \pmod q$. Moreover we may assume from the form of M that the matrix D contains exact one row of every $L_\lambda^{(k)}$ $(k = 1, 2, \cdots\cdots, t;\ \lambda = 1, 2, \cdots\cdots. h(k))$. Suppose that D contains a row $(r_{\alpha_\lambda 1\lambda}^k, r_{\alpha_\lambda 2\lambda}^k, \cdots\cdots, r_{\alpha_\lambda n\lambda}^k)$ of $L_\lambda^{(k)}$. We set

$$Y = ((\xi_{\alpha_\lambda}^{(k)} \vartheta_{k,\lambda})^* (A_l Q_i^{(l)})), \tag{2.23}$$

(k, λ row indices, l, i column indices). The matrix Y is of type (n, n) and it follows from (2.15) that $Y = DX$. This implies

$$D^{-1} Y = X. \tag{2.24}$$

Since $|D| \not\equiv 0 \pmod q$, (2.24) shows that the irreducible character

χ_μ is expressed as a linear combination of $(\xi_{\alpha_\lambda}^{(k)}\vartheta_{k,\lambda})^*$, where the coefficients are rational numbers with the denominator $|D|$. Thus we have

Lemma 3. *If we choose $h(k)$ irreducible characters $\xi_{\alpha_\lambda}^{(k)}\vartheta_{k,\lambda}$ $(\lambda = 1, 2, \cdots\cdots, h(k))$ of each subgroup $\mathfrak{H}_k = \{A_k\} \times \mathfrak{Q}_k$ $(k = 1, 2, \cdots\cdots, t)$ suitably and if we denote by $(\xi_{\alpha_\lambda}^{(k)}\vartheta_{k,\lambda})^*$ the character of $\mathfrak{G}$ induced by the character $\xi_{\alpha_\lambda}^{(k)}\vartheta_{k,\lambda}$, then every character of $\mathfrak{G}$ is expressed as a linear combination of $(\xi_{\alpha_\lambda}^{(k)}\vartheta_{k,\lambda})^*$, where the coefficients are rational numbers with the denominators prime to q.*

As a special case of Lemma 3, we have

Lemma 4. *Let q be a prime such that $(q, g) = 1$. If we choose an irreducible character $\xi_\alpha^{(k)}$ of each cyclic subgroup $\{A_k\}$ $(k = 1, 2, \cdots\cdots, n)$ suitably and if we denote by $(\xi_\alpha^{(k)})^*$ the character of $\mathfrak{G}$ induced by the character $\xi_\alpha^{(k)}$, then every character of $\mathfrak{G}$ is expressed as a linear combination of $(\xi_\alpha^{(k)})^*$ $(k = 1, 2, \cdots\cdots, n)$, where the coefficients are rational numbers with the denominators prime to q.*

We have from Lemma 4

Lemma 5 (Artin). *Every character of $\mathfrak{G}$ is expressed as a linear combination of characters of $\mathfrak{G}$ induced by irreducible characters of cyclic subgroups, where the coefficients are rational numbers whose denominators are divisors of g.*

We call a group elementary, if it is a direct product $\{A\} \times \mathfrak{B}$ of a cyclic group $\{A\}$ and a group $\mathfrak{B}$ of prime power order ([4]). Then groups $\mathfrak{H}_k$ in Lemma 3 are elementary. By Brauer (see [3], Lemma 4), every irreducible character of $\mathfrak{H}_k$ is induced by a linear character of a subgroup $\{A_k\} \times \mathfrak{C}_k$, $\mathfrak{C}_k \subseteq \mathfrak{Q}_k$. Evidently $\{A_k\} \times \mathfrak{C}_k$ is elementary. Hence we have from Lemmas 3, 5

Theorem 2 (Brauer). *Every character of $\mathfrak{G}$ is expressed as a linear combination $\sum c_\rho w_\rho^*$, where the c_ρ are rational integers and where w_ρ^* are characters of $\mathfrak{G}$ induced by linear characters w_ρ of elementary subgroups of $\mathfrak{G}$.*

3. The arguments in §§1 and 2 are also applicable to the theory of modular characters of $\mathfrak{G}$ for a prime $p \neq q$. The distinct irreducible modular characters of $\mathfrak{G}$ will be denoted by $\varphi_1, \varphi_2, \cdots\cdots, \varphi_l$, where φ_1 is the 1-character. Here l is equal to the number of conjugate classes in $\mathfrak{G}$ which contain the p-regular elements ([6]). Let us denote by $\eta_1, \eta_3, \cdots\cdots, \eta_l$ the characters of indecomposable con-

stituents of the modular regular representation of $\mathfrak{G}$ (mod p). Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots\cdots, \mathfrak{K}_l$ be the classes of conjugate elements in $\mathfrak{G}$ which contain the p-regular elements, and let H_j be a representative element of $\mathfrak{K}_j$ $(j = 1, 2, \cdots\cdots, l)$. Since $p \neq q$, we may assume that $H_i = Q_i$ $(i = 1, 2, \cdots\cdots, h)$. We assume that $\mathfrak{Q}$, $\mathfrak{Q}^{(i)}$, ϑ_ν and ϑ'_λ have the same meaning as in §1. We have by Nakayama's theorem ([6], [7])

$$(3.1) \qquad \varphi_\kappa(Q) = \sum_{\nu=1}^{m} s_{\kappa\nu}\vartheta_\nu(Q) \qquad \text{(for } Q \text{ in } \mathfrak{Q})$$

$$(3.2) \qquad \vartheta^*_\nu(H) = \sum_{\kappa=1}^{l} s_{\kappa\nu}\eta_\kappa(H) \qquad \text{(for } p\text{-regular elements } H \text{ in } \mathfrak{G})$$

where the $s_{\kappa\nu}$ are rational integers, $s_{\kappa\nu} \geqq 0$. The combination of (1.32) and (3.2) yields

$$(3.3) \qquad \sum_{\lambda=1}^{h}\Big(\sum_{\kappa=1}^{l} s_{\kappa\lambda}\eta_\kappa(H_j)\Big)\vartheta'_\lambda(Q_i^{-1}) = \sum_{\kappa=1}^{l}\Big(\sum_{\lambda=1}^{h} s_{\kappa\lambda}\vartheta'_\lambda(Q_i^{-1})\Big)\eta_\kappa(H_j) = \begin{cases} n_i & (Q_i = H_j) \\ 0 & (Q_i \neq H_j). \end{cases}$$

On the other hand we have

$$(3.4) \qquad \sum_{\kappa=1}^{l} \varphi_\kappa(Q_i^{-1})\eta_\kappa(H_j) = \begin{cases} n_i & (Q_i = H_j) \\ 0 & (Q_i \neq H_j). \end{cases}$$

Since $\eta_1(H), \eta_2(H), \cdots\cdots, \eta_l(H)$ are linearly independent, it follows from (3.3) and (3.4)

$$(3.5) \qquad \varphi_\kappa(Q) = \sum_{\lambda=1}^{h} s_{\kappa\lambda}\vartheta'_\lambda(Q) \qquad \text{(for } Q \text{ in } \mathfrak{Q}).$$

We denote by $d_{\mu\kappa}$ the decomposition numbers of $\mathfrak{G}$ for p:

$$(3.6) \qquad \chi_\mu(H) = \sum_\kappa d_{\mu\kappa}\varphi_\kappa(H).$$

We have from (1.26), (3.5) and (3.6)

$$(3.7) \qquad r_{\mu\lambda} = \sum_\kappa d_{\mu\kappa}s_{\kappa\lambda},$$

or in matrix form

$$(3.8) \qquad R = DS,$$

where $D = (d_{\mu\kappa})$ and $S = (s_{\kappa\lambda})$. Let C be the matrix of Cartan invariants of $\mathfrak{G}$ for p. Since $C = D'D$, we obtain from (1.34) and (3.8)

(3.9) $$W = R'R = S'D'DS = S'CS.$$

Let $A_1, A_2, \cdots\cdots, A_t$ have the same meaning as in §2. We may assume that $A_1, A_2, \cdots\cdots, A_r$ are a maximal system of elements of $\mathfrak{G}$ such that A_i, A_j are not conjugate for $i \neq j$ and the order of each A_i is prime to p and q. We obtain by the similar way as in §2

(3.10) $$\varphi_\kappa(A_i Q_j^{(i)}) = \sum_{\lambda=1}^{h(i)} s_{\kappa\lambda}^i \vartheta'_{i,\lambda}(Q_j^{(i)}),$$

where the $s_{\kappa\lambda}^i$ are algebraic integers. We set $S^i = (s_{\kappa\lambda}^i)$ and

(3.11) $$\mathbf{S} = (S^1, S^2, \cdots\cdots, S^r), \qquad S^1 = S.$$

Then $\mathbf{S}$ is a square matrix of the same degree $l = \sum h(i)$ as the matrix Φ of the modular group characters φ_κ of $\mathfrak{G}$. (3.10) yields

(3.12) $$\Phi = (\varphi_\kappa(A_i Q_j^{(i)})) = \mathbf{S}\Lambda,$$

where the matrix Λ breaks up completely into the matrices $\Theta_i^* = (\vartheta'_{i,\lambda}(Q_j^{(i)}))$ $(i = 1, 2, \cdots\cdots, r)$. Hence, by Theorem 1

(3.13) $$|\Lambda|^2 = \prod_{i=1}^{r}(\prod_{j=1}^{h(i)} q_j^{(i)}/v_i), \qquad (v_i, q) = 1.$$

We see from (3.12) and (3.13)

(3.14) $$|\Phi|^2 \equiv 0 \qquad (\mathrm{mod}\ \prod_{i=1}^{r}\prod_{j=1}^{h(i)} q_j^{(i)}).$$

(3.14), combined with $|\Phi|^2 |C| = \prod_{i=1}^{r}\prod_{j=1}^{h(i)} n_j^{(i)}$, yields

(3.15) $$|C| \not\equiv 0 \qquad (\mathrm{mod}\ q).$$

Since (3.14) and (3.15) hold for arbitrary prime divisor $q \neq p$ of the order g of $\mathfrak{G}$ and $(|\Phi|, p) = 1$, we have

Theorem 3 (Brauer). *The determinant $|C|$ of the matrix of Cartan invariants of $\mathfrak{G}$ for p is equal to the highest power of p which divides $\prod_{i=1}^{r}\prod_{j=1}^{h(i)} n_j^{(i)}$.*

Further we have from (3.12), (3.13) and (3.14)

(3.16) $$|\mathbf{S}| \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{q}), \qquad |\mathbf{S}| \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{p}),$$

where $\mathfrak{p}$ is a prime ideal which divides the prime p

Let $\mathfrak{H}_i$ have the same meaning as in §2 : $\mathfrak{H}_i = \{A_i\} \times \mathfrak{Q}_i$. We consider only $\mathfrak{H}_i$ such that the order of A_i is prime to p. Let

$$\varphi_\kappa(A_i Q_j^{(i)}) = \sum_\nu \sum_\alpha s^i_{\alpha\kappa\nu} \xi^{(i)}_\alpha(A_i) \vartheta_{i,\nu}(Q_j^{(i)}), \tag{3.17}$$

where the $s^i_{\alpha\kappa\nu}$ are rational integers, $s^i_{\alpha\kappa\nu} \geqq 0$. Then by Nakayama's theorem we have

$$(\xi^{(i)}_\alpha \vartheta_{i,\nu})^*(H) = \sum_\kappa s^i_{\alpha\kappa\nu} \eta_\kappa(H) \tag{3.18}$$

for p-regular elements H in $\mathfrak{G}$. The combination of (3.10) with (3.17) yields

$$\sum_{\lambda=1}^{h(i)} s^i_{\kappa\lambda} \vartheta'_{i,\lambda}(Q_j^{(i)}) = \sum_\nu \Big(\sum_\alpha s^i_{\alpha\kappa\nu} \xi^{(i)}_\alpha(A_i)\Big) \vartheta_{i,\nu}(Q_j^{(i)}),$$

and hence we have

$$s^i_{\kappa\nu} = \sum_\alpha s^i_{\alpha\kappa\lambda} \xi^{(i)}_\alpha(A_i) \qquad (\lambda = 1, 2, \cdots\cdots, h(i)). \tag{3.19}$$

From (3.16), (3.18) and (3.19), we have by the similar way as in Lemma 3

Lemma 6. *If we choose suitably $h(i)$ irreducible characters $\xi^{(i)}_{\alpha_\lambda} \vartheta_{i,\lambda}$ $(\lambda = 1, 2, \cdots\cdots, h(i))$ from the irreducible characters of each subgroup $\mathfrak{H}_i = \{A_i\} \times \mathfrak{Q}_i$ $(i = 1, 2, \cdots\cdots, r)$, then every character η_κ is expressed as a linear combination of characters $(\xi^{(i)}_{\alpha_\lambda} \vartheta_{i,\lambda})^*$, where the coefficients are rational numbers with denominators prime to q.*

Further from $(|\,\Phi\,|, p) = 1$ we have

Lemma 7. *Every character η_κ of $\mathfrak{G}$ is expressed as a linear combination of characters of $\mathfrak{G}$ which are induced by irreducible characters of cyclic subgroups $\{H_i\}$ $(i = 1, 2, \cdots\cdots, l)$ of orders prime to p, where the coefficients are rational numbers with denominators prime to p.*

Now let q be a prime such that $(g, q) = 1$. Since $(|\,\Phi\,|, q) = 1$, Lemma 7 is also valid if we replace p by q, that is, η_κ is expressed as a linear combination of characters of $\mathfrak{G}$ which are induced by irreducible characters of cyclic subgroups $\{H_i\}$, where the coefficients are rational numbers with denominators prime to q. We set $g = p^b g^*$, where $(g^*, p) = 1$. We then have

Lemma 8. *Every character η_κ of $\mathfrak{G}$ is expressed as a linear combination of characters of $\mathfrak{G}$ which are induced by irreducible characters of cyclic subgroups $\{H_i\}$ $(i = 1, 2, \cdots\cdots, l)$, where the coefficients are rational numbers whose denominators are divisors of g^*.*

Consequently we have by Lemmas 6 and 8

Theorem 4 (Brauer). *Every character η_κ of $\mathfrak{G}$ is expressed as a linear combination $\sum d_\sigma \omega^*_\sigma$, where the d_σ are rational integers and the*

ω_σ^* are characters of $\mathfrak{G}$ induced by linear characters ω_σ of elementary subgroups of order prime to p.

REFERENCES

[1] R. BRAUER, On the Cartan invariants of groups of finite order, Ann. of Math., **42** (1941), pp. 53 - 61.

[2] ————, On the connection between the ordinary and the modular characters of groups of finite order, Ann. of Math., **42** (1941), pp. 926 - 935.

[3] ————, On Artin's L-series with general group characters, Ann. of Math., **48** (1947), pp. 502 - 514.

[4] ————, Applications of induced characters, Amer. J. Math., **69** (1947), pp. 709 - 716.

[5] ————, A characterization of the characters of groups of finite order, Ann. of Math., **57** (1953), pp. 357 - 377.

[6] ———— and C. Nesbitt, On the modular characters of groups, Ann. of Math., **42** (1941), pp. 556 - 590.

[7] T. NAKAYAMA, Some studies on regular representations, induced representations and modular representations, Ann. of Math., **39** (1938), pp. 361 - 369.

[8] M. OSIMA, On the induced characters of a group, Proc. Japan Acad., **28** (1952), pp. 243 - 248.

[9] P. ROQUETTE, Arithmetische Untersuchung des Charakterrings einer endlichen Gruppe, J. Reine Angew. Math., **190** (1952), pp. 148 - 168.

[10] E. WITT, Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einer Zahlkörper, J. Reine Angew. Math., **190** (1952), pp. 231 - 245.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(Received June 4, 1953)