ON PRIMARY IDEAL DECOMPOSITIONS IN NON-COMMUTATIVE RINGS[1]

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It is the purpose of this note to present the condition that every ideal in a (non-commutative) ring is represented as the intersection of a finite number of *s*-primary ideals. Although the fact that the well-known results of E. Noether hold in non-commutative case for those ideals which can be represented as the intersection of a finite number of right primary ideals has been shown under maximum condition [3][2], a necessary and sufficient condition that such a representation exists for every ideal is still unknown.

Throughout this note, the term "ideals" will mean "two-sided ideals" and $\mathfrak{R}$ will be a ring considered.

1. The *right* [*left*] *quotient* $\mathfrak{a}\mathfrak{b}^{-1}[\mathfrak{b}^{-1}\mathfrak{a}]$ of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by $\mathfrak{a}\mathfrak{b}^{-1}=\{x\in\mathfrak{R}\mid x\mathfrak{b}\subseteq\mathfrak{a}\}$ $[\mathfrak{b}^{-1}\mathfrak{a}=\{x\in\mathfrak{R}\mid \mathfrak{b}x\subseteq\mathfrak{a}\}]$. The following properties of quotients are easily verified:

1) $(\mathfrak{a}\mathfrak{b}^{-1})\mathfrak{c}^{-1}=\mathfrak{a}(\mathfrak{c}\mathfrak{b})^{-1}$,

2) $(\bigcap_\lambda \mathfrak{a}_\lambda)\mathfrak{b}^{-1}=\bigcap_\lambda \mathfrak{a}_\lambda\mathfrak{b}^{-1}$,

3) $\mathfrak{a}(\sum_\lambda \mathfrak{b}_\lambda)^{-1}=\bigcap_\lambda \mathfrak{a}\mathfrak{b}_\lambda^{-1}$, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}_\lambda$ and $\mathfrak{b}_\lambda$ are ideals.

Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals, if $\mathfrak{a}\mathfrak{b}^{-1}\supset\mathfrak{a}$, we say that $\mathfrak{b}$ is *non-prime to* $\mathfrak{a}$. If, for some positive integer k, $\mathfrak{a}\mathfrak{b}^{-k}=\mathfrak{a}\mathfrak{b}^{-(k+1)}$[3], then we say that $\mathfrak{a}\mathfrak{b}^{-k}$ is the *right limit ideal of* $\mathfrak{a}$ *by* $\mathfrak{b}$. Clearly, the right limit ideal $\mathfrak{a}\mathfrak{b}^{-k}=\bigcup_{i=1}^{\infty}\mathfrak{a}\mathfrak{b}^{-i}$. The *left limit ideal* is defined in the obvious way. In the case where the right limit ideal of $\mathfrak{a}$ by $\mathfrak{b}$ coincides with the left one, we call it the *limit ideal of* $\mathfrak{a}$ *by* $\mathfrak{b}$.

An ideal $\mathfrak{p}$ is said to be *prime* [1] if $\mathfrak{a}^{-1}\mathfrak{p}=\mathfrak{p}\mathfrak{a}^{-1}=\mathfrak{p}$ for any ideal $\mathfrak{a}\not\subseteq\mathfrak{p}$. As well-known, for every prime divisor $\mathfrak{p}$ of any ideal $\mathfrak{a}$, there exists a minimal prime divisor of $\mathfrak{a}$ which are contained in $\mathfrak{p}$ [1]. The intersection of all the minimal prime divisors of $\mathfrak{a}$ is called the *radical* of $\mathfrak{a}$ and denoted by $\bar{\mathfrak{a}}$.

An ideal $\mathfrak{q}$ is called *right* [*left*] *primary* if $\mathfrak{q}\mathfrak{a}^{-1}=\mathfrak{q}[\mathfrak{a}^{-1}\mathfrak{q}=\mathfrak{q}]$ for any ideal $\mathfrak{a}\not\subseteq\bar{\mathfrak{q}}$ [3]. An ideal called *primary* if it is both right and

1) This note has been completed by the encouragement of Prof. M. Moriya. I express him my hearty thanks.

2) Numbers in brackets refer to the bibliography at the end of the note.

3) $\mathfrak{a}\mathfrak{b}^{-k}$ is defined as $(\mathfrak{a}\mathfrak{b}^{-(k-1)})\mathfrak{b}^{-1}$ inductively.

left primary, and a [right, left] primary ideal is called *s-[right, left] primary* if its radical is nilpotent modulo the ideal.

Theorem 1. *The radical of an s-right primary ideal* $\mathfrak{q}$ *is prime.*

In fact, let $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$, then $(\mathfrak{a}\mathfrak{b})^n \subseteq \mathfrak{q}$ for a sufficiently large n. Let n be the least with this property. If $\mathfrak{a}, \mathfrak{b} \not\subseteq \bar{\mathfrak{q}}$, then $(\mathfrak{a}\mathfrak{b})^n = (\mathfrak{a}\mathfrak{b})^{n-1}\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$ implies $(\mathfrak{a}\mathfrak{b})^{n-1} \subseteq \mathfrak{q}$ because $\mathfrak{q}$ is right primary. (We set $(\mathfrak{a}\mathfrak{b})^{n-1} = \mathfrak{R}$ if $n = 1$.) But this contradicts with the minimality of n.

If a prime ideal is the radical of an s-primary ideal $\mathfrak{q}$, we say that $\mathfrak{q}$ is an s-primary ideal *belonging to* the prime ideal. And a prime ideal $\mathfrak{p}$ is called *a prime ideal associated with* an ideal $\mathfrak{a}$ [2] if there exists an s-primary ideal $\mathfrak{q}$ belonging to $\mathfrak{p}$ such that $\mathfrak{q} = \mathfrak{a}\mathfrak{r}^{-1}$, where $\mathfrak{r}$ is an ideal not contained in $\mathfrak{a}$.

2. In this section, we assume that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$, where $\mathfrak{q}_i$ are s-primary and the representation is irredundant[1]. As easily verified, a prime ideal $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{a}$ if and only if $\mathfrak{p}$ is minimal in the set $\{\bar{\mathfrak{q}}_i\}$. If $\bar{\mathfrak{q}}_i = \mathfrak{p}'$ for all i, then $\mathfrak{a}$ is also a primary ideal with the radical $\mathfrak{p}'$. In fact, $\mathfrak{p}'$ is the unique minimal prime divisor of $\mathfrak{a}$ and the rest of the proof is easy.

Theorem 2. *Let* $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$, *where* $\mathfrak{q}_i$ *are s-primary, then* $\bar{\mathfrak{a}}$ *is nilpotent modulo* $\mathfrak{a}$.

In fact, let k_i be the nilpotency index of $\bar{\mathfrak{q}}_i$ modulo $\mathfrak{q}_i$, then $(\bar{\mathfrak{a}})^{k_1+\cdots\cdots+k_n} \subseteq \mathfrak{a}$.

From the preceding, we can assume, without loss of generality, that $\bar{\mathfrak{q}}_i$ does not coincide with any $\bar{\mathfrak{q}}_j$ $(i \neq j)$ and so that the representation $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cup \mathfrak{q}_n$ is a short representation of $\mathfrak{a}$[2].

If $n > 1$, *then* $\mathfrak{a}$ *is not primary.* In fact, let $\bar{\mathfrak{q}}_1$ be minimal in the set $\{\bar{\mathfrak{q}}_i\}$. Then there exist elements a_i $(i = 2, \cdots\cdots, n)$ such that $a_i \in \bar{\mathfrak{q}}_i \setminus \bar{\mathfrak{q}}_1$, where $\bar{\mathfrak{q}}_i \setminus \bar{\mathfrak{q}}_1$ means the complement of $\bar{\mathfrak{q}}_1$ in $\bar{\mathfrak{q}}_i$. And so, for some positive integer m, $(a_i)^m \subseteq \mathfrak{q}_i$ $(i = 2, \cdots\cdots, n)$, where (a_i) means the two-sided ideal generated by a_i. As clearly $\mathfrak{q}_1 \supset \mathfrak{a}$, there exists an element $q_1 \in \mathfrak{q}_1 \setminus \mathfrak{a}$, and $(q_1)\prod_{i=2}^{n}(a_i)^m \subseteq \mathfrak{a}$. Suppose now that $\mathfrak{a}$ is primary, then $(q_1) \not\subseteq \mathfrak{a}$ implies $\prod_{i=2}^{n}(a_i)^m \subseteq \bar{\mathfrak{a}} \subseteq \bar{\mathfrak{q}}_1$, but it is impossible.

In the rest of this section, *we assume that* $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$ *be*

1) A representation $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$ is called irredundant if none of the $\mathfrak{q}_i$ contains the intersection of the remainings.

2) The term "short representations" will be used for representations as the intersection of a *finite* number of s-primary ideals.

a short representation of $\mathfrak{a}$. By using the same argument as in p. 35 of [4], we can prove, for every short representation of $\mathfrak{a}$, the uniqueness of the number of primary components and the radicals of the primary components. We state here the proof of the uniqueness of the isolated components[1].

If $\mathfrak{p}\subset\mathfrak{R}$ is any prime ideal, then we denote by $\mathfrak{a}'(\mathfrak{p})$, where $\mathfrak{a}'$ is an ideal, the set of all the elements b of $\mathfrak{R}$ such that $b\mathfrak{r}\subseteq\mathfrak{a}'$ for some ideal $\mathfrak{r}\nsubseteq\mathfrak{p}$. (If $\mathfrak{p}=\mathfrak{R}$, $\mathfrak{a}'(\mathfrak{r})=\mathfrak{a}'$, by definition.) As easily verified, $\mathfrak{a}'(\mathfrak{p})$ is a two-sided ideal containing $\mathfrak{a}'$.

Lemma 1. *Let* $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_n$ *be a short representation. If* $\mathfrak{p}$ *is a prime ideal containing* $\bar{\mathfrak{q}}_1,\cdots\cdots,\bar{\mathfrak{q}}_r$ $(1\leq r\leq n)$ *but not containing* $\bar{\mathfrak{q}}_{r+1},\cdots\cdots,\bar{\mathfrak{q}}_n$, *then* $\mathfrak{a}(\mathfrak{p})=\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_r$. *If* $\mathfrak{p}$ *contains none of the* $\bar{\mathfrak{q}}_i$, *then* $\mathfrak{a}(\mathfrak{p})=\mathfrak{R}$.

We first assume that $\mathfrak{p}$ contains $\bar{\mathfrak{q}}_1,\cdots\cdots,\bar{\mathfrak{q}}_r$. Let b be any element of $\mathfrak{a}(\mathfrak{p})$, then for some ideal $\mathfrak{s}\nsubseteq\mathfrak{p}$, $b\mathfrak{s}\subseteq\mathfrak{a}$ and so $(b)\mathfrak{s}\subseteq\mathfrak{q}_i$ $(i=1,\cdots\cdots,n)$. As $\mathfrak{s}\nsubseteq\bar{\mathfrak{q}}_i$ $(i=1,\cdots\cdots,r)$ and $\mathfrak{q}_i$ are s-primary, $(b)\subseteq\mathfrak{q}_i$ $(i=1,\cdots\cdots,r)$. Hence $\mathfrak{a}(\mathfrak{p})\subseteq\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_r$. The converse inclusion is proved as following. If $n=r$, it is trivial. Therefore, we assume that $r<n$ and let c be any element of $\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_r$. For $i=r+1,\cdots\cdots,n$, we choose elements $p_i\in\bar{\mathfrak{q}}_i\setminus\mathfrak{p}$, then for some positive integer h, $(p_i)^h\subseteq\mathfrak{q}_i$ $(i=r+1,\cdots\cdots,n)$. If we set $\mathfrak{r}'=(p_{r+1})^h\cdots\cdots(p_n)^h$, then $\mathfrak{r}'\nsubseteq\mathfrak{p}$ and $\subseteq\mathfrak{q}_{r+1}\cap\cdots\cdots\cap\mathfrak{q}_n$. Since $c\in\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_r$ it follows that $c\mathfrak{r}'\subseteq\mathfrak{a}$, and hence $c\in\mathfrak{a}(\mathfrak{p})$.

If $\mathfrak{p}$ contains none of the $\bar{\mathfrak{q}}_i$, then the last part of the above proof shows that there is an ideal $\mathfrak{r}''=(p_1)^h\cdots\cdots(p_n)^h$ which is not contained in $\mathfrak{p}$, where p_i is in $\bar{\mathfrak{q}}_i\setminus\mathfrak{p}$ and $(p_i)^h\subseteq\mathfrak{q}_i$. Hence, $y\mathfrak{r}''\subseteq\mathfrak{a}$ for all elements $y\in\mathfrak{R}$, that is, $\mathfrak{a}(\mathfrak{p})=\mathfrak{R}$.

Corollary[2]. *Let* $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_n$ *be a short representation. Then* $\mathfrak{a}(\mathfrak{p})$ *is s-primary if* $\mathfrak{p}$ *is a minimal prime divisor of* $\mathfrak{a}$.

This is the direct consequence of Lemma 1.

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1) Let $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_n$ be a short representation. Consider a subset S of the set $\{\bar{\mathfrak{q}}_i\}$ having the property that if $\bar{\mathfrak{q}}_i\in S$, then $\bar{\mathfrak{q}}_j\subseteq\bar{\mathfrak{q}}_i$ implies $\bar{\mathfrak{q}}_j\in S$. The intersection of the s-primary components belonging to the prime ideals in S is called an isolated component of $\mathfrak{a}$.

2) D. C. Murdoch proved that if $\mathfrak{a}$ is represented as the intersection of a finite number of right primary ideals, then $\mathfrak{u}(\mathfrak{a},\mathfrak{p})$ is right primary for every minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$, where the maximum condition for ideals is assumed (Corollary 2 to Theorem 17 of [3]). In our case, we obtain that $\mathfrak{u}(\mathfrak{a},\mathfrak{p})=\mathfrak{l}(\mathfrak{a},\mathfrak{p})=\mathfrak{a}(\mathfrak{p})$. Thus our corollary corresponds to the Murdoch's result stated above.

Let $\mathfrak{b} = \mathfrak{q}_{t_1} \cap \cdots\cdots \cap \mathfrak{q}_{t_m}$ be an isolated component of $\mathfrak{a}$, then $\mathfrak{a}(\bar{\mathfrak{q}}_{t_s})$ $(s = 1, \cdots\cdots, m)$ is represented as the intersection of the primary ideals belonging to a subset of $\{\mathfrak{q}_{t_j}\}$ $(j = 1, \cdots\cdots, m)$ containing $\mathfrak{q}_{t_s}$. It follows that $\mathfrak{b} = \mathfrak{a}(\bar{\mathfrak{q}}_{s_1}) \cap \cdots\cdots \cap \mathfrak{a}(\bar{\mathfrak{q}}_{t_m})$.

We summarize here the uniqueness theorems.

Theorem 3 *If an ideal* $\mathfrak{a}$ *is represented as the intersection of a finite number of s-primary ideals, then there exists a short representation of* $\mathfrak{a}$. *And,*

(1) *The number of s-primary components in every short representation of* $\mathfrak{a}$ *and the radicals of them are uniquely determined.*

(2) *The radicals of primary ideals belonging to an isolated component of* $\mathfrak{a}$ *uniquely determine the isolated component, and so the isolated components of* $\mathfrak{a}$ *coincide in all the short representations.*

Let $\mathfrak{b}$ be an arbitrary ideal contained in $\bar{\mathfrak{q}}_i$, then $\mathfrak{R} = \mathfrak{q}_i\mathfrak{b}^{-k} = \mathfrak{q}_i\mathfrak{b}^{-(k+1)} = \mathfrak{b}^{-(k+1)}\mathfrak{q}_i = \mathfrak{b}^{-k}\mathfrak{q}_i$ for a sufficiently large k. On the other hand, if $\mathfrak{b} \not\subseteq \bar{\mathfrak{q}}_i$, then $\mathfrak{q}_i = \mathfrak{q}_i\mathfrak{b}^{-h} = \mathfrak{b}^{-h}\mathfrak{q}_i$ for every positive integer h. Thus, for any ideal $\mathfrak{c}$, there exists a positive integer t such that $\mathfrak{a}\mathfrak{c}^{-t} = \mathfrak{a}\mathfrak{c}^{-(t+1)} = \mathfrak{c}^{-(t+1)}\mathfrak{a} = \mathfrak{q}_{i_1} \cap \cdots\cdots \cap \mathfrak{q}_{i_m}$, where $\{\mathfrak{q}_{i_1}, \cdots\cdots, \mathfrak{q}_{i_m}\}$ is a subset of $\{\mathfrak{q}_1, \cdots\cdots, \mathfrak{q}_n\}$. This proves the next

Theorem 4. *Let* $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$ *be a short representation. Then, for any ideal* $\mathfrak{b}$, *there exists the limit ideal of* $\mathfrak{a}$ *by* $\mathfrak{b}$. *And the number of ideals which, starting from* $\mathfrak{a}$, *are obtained by repeating the procedures to make limit ideals successively is finite, and is uniquely determined by* $\mathfrak{a}$.

From the existence of the limit ideal of $\mathfrak{a}$, we see readily the following

Corollary.[1] $\mathfrak{a}\mathfrak{b}^{-1} \supset \mathfrak{a}$ *if and only if* $\mathfrak{b}^{-1}\mathfrak{a} \supset \mathfrak{a}$, *accordingly,* $\mathfrak{a}$ *is primary if and only if* $\mathfrak{a}$ *is right* (*or left*) *primary.*

Let $\mathfrak{p}$ be a minimal prime divisor of $\mathfrak{a}$ $(\subset \mathfrak{R})$ and let h be a positive integer such that $\mathfrak{p}^h \subseteq \mathfrak{q}_i$ for all $\mathfrak{q}_i$ with $\bar{\mathfrak{q}}_i$ containing $\mathfrak{p}$, then clearly $\mathfrak{a}\mathfrak{p}^{-h} \supset \mathfrak{a}$. Hence we have

Theorem 5. *Let* $\mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n = \mathfrak{a}$ *be a short representation of* $\mathfrak{a} \subset \mathfrak{R}$. *If* $\mathfrak{p}$ *is a minimal prime divisor of* $\mathfrak{a}$, *then* $\mathfrak{p}$ *is non-prime to* $\mathfrak{a}$.

Lemma 2. *If* $\mathfrak{q}$ *is s-primary, then for any ideal* $\mathfrak{r} \not\subseteq \mathfrak{q}$, $\mathfrak{q}\mathfrak{r}^{-1}$ *is s-left primary.*

Clearly, $\overline{\mathfrak{q}\mathfrak{r}^{-1}} = \bar{\mathfrak{q}}$. $\mathfrak{u}\mathfrak{v} \subseteq \mathfrak{q}\mathfrak{r}^{-1}$ implies $\mathfrak{u}\mathfrak{v}\mathfrak{r} \subseteq \mathfrak{q}$, where $\mathfrak{u}$, $\mathfrak{v}$ are ideals. If $\mathfrak{u} \not\subseteq \overline{\mathfrak{q}\mathfrak{r}^{-1}}$, then we have $\mathfrak{v}\mathfrak{r} \subseteq \mathfrak{q}$. Thus $\mathfrak{v} \subseteq \mathfrak{q}\mathfrak{r}^{-1}$.

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1) This corollary is derived from only the fact that there exists the limit ideal of $\mathfrak{a}$.

Theorem 6. *If every ideal in $\mathfrak{R}$ is represented as the intersection of a finite number of s-primary ideals, then a prime divisor $\mathfrak{p}$ of an arbitrary ideal $\mathfrak{a}$ is a prime ideal associated with $\mathfrak{a}$ if and only if $\mathfrak{p}$ coincides with one of the radicals $\bar{\mathfrak{q}}_i$ in a short representation $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$. And every primary component $\mathfrak{q}_j$ $(j = 1, \cdots\cdots, n)$ has the following property: For any ideal $\mathfrak{c} \subseteq \mathfrak{q}_j$, $\nsubseteq \mathfrak{a}$, $\mathfrak{a}\mathfrak{c}^{-1}$ is not an s-primary ideal belonging to $\bar{\mathfrak{q}}_j$.*

The second part of this theorem follows from $\mathfrak{a}\mathfrak{c}^{-1} = \mathfrak{q}_1\mathfrak{c}^{-1} \cap \cdots\cdots \cap \mathfrak{q}_{j-1}\mathfrak{c}^{-1} \cap \mathfrak{q}_{j+1}\mathfrak{c}^{-1} \cap \cdots\cdots \cap \mathfrak{q}_n\mathfrak{c}^{-1}$.

Now we shall prove the first part. As $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots\cdots \cap \mathfrak{q}_n$ is a short representation, $\bigcap_{j \neq i} \mathfrak{q}_j \nsubseteq \mathfrak{q}_i$. Clearly $\mathfrak{a}(\bigcap_{j \neq i} \mathfrak{q}_j)^{-1} = \mathfrak{q}_i(\bigcap_{j \neq i} \mathfrak{q}_j)^{-1}$. By Lemma 2 and Corollary to Theorem 4, $\mathfrak{a}(\bigcap_{j \neq i} \mathfrak{q}_j)^{-1}$ is an s-primary ideal belonging to $\bar{\mathfrak{q}}_i$. (If $n = 1$, we set $\bigcap_{j \neq i} \mathfrak{q}_j = \mathfrak{R}$.) Conversely, let $\mathfrak{p}$ be a prime ideal associated with $\mathfrak{a}$, that is, $\mathfrak{q} = \mathfrak{a}\mathfrak{r}^{-1}(\mathfrak{r} \nsubseteq \mathfrak{a})$ be an s-primary ideal belonging to $\mathfrak{p}$. If $\mathfrak{r} \subseteq \mathfrak{q}_1, \cdots\cdots, \mathfrak{q}_r$ but $\nsubseteq \mathfrak{q}_{r+1}, \cdots\cdots, \mathfrak{q}_n$, then $\mathfrak{a}\mathfrak{r}^{-1} = \mathfrak{q}_{r+1}\mathfrak{r}^{-1} \cap \cdots\cdots \cap \mathfrak{q}_n\mathfrak{r}^{-1}$. Again by Lemma 2 and Corollary to Theorem 4, the ideals $\mathfrak{q}_{r+1}\mathfrak{r}^{-1}, \cdots\cdots, \mathfrak{q}_n\mathfrak{r}^{-1}$ are s-primary. By Theorem 3. (1), we have $\mathfrak{a}\mathfrak{r}^{-1} = \mathfrak{q}_i\mathfrak{r}^{-1}$ for some i $(r + 1 \leq i \leq n)$. Hence $\mathfrak{p}$ coincides with $\bar{\mathfrak{q}}_i$.

Summarizing the above-mentioned results, we obtain

Theorem 7. *In order that every ideal in $\mathfrak{R}$ is represented as the intersection of a finite number of s-primary ideals, the following conditions are necessary:*

(A) *The radical of any ideal $\mathfrak{a}$ is nilpotent modulo $\mathfrak{a}$*[1].

(B) *For any ideals $\mathfrak{a}$, $\mathfrak{b}$, there exists the limit ideal of $\mathfrak{a}$ by $\mathfrak{b}$ and there exists a finite number $n(\mathfrak{a})$ of ideals which, starting from $\mathfrak{a}$, are obtained by repeating the procedures to make limit ideals successively. The number $n(\mathfrak{a})$ is uniquely determined by $\mathfrak{a}$.*

(C) *Each minimal prime divisor of any ideal $\mathfrak{a} \subset \mathfrak{R}$ is non-prime to $\mathfrak{a}$.*

(D) *If $\mathfrak{p}$ is an arbitrary prime ideal associated with an ideal $\mathfrak{a}$, there exists an s-primary ideal $\mathfrak{q} \supseteq \mathfrak{a}$ belonging to $\mathfrak{p}$ such that, for any ideal $\mathfrak{b} \subseteq \mathfrak{q}$, $\nsubseteq \mathfrak{a}$, $\mathfrak{a}\mathfrak{b}^{-1}$ is no primary ideal belonging to $\mathfrak{p}$.*

3. In this section, we assume first the condition (B) in Theorem 7. Let $\mathfrak{a}$ be an ideal and let $\mathfrak{p} \subset \mathfrak{R}$ be a minimal prime divisor of $\mathfrak{a}$.

1) In any ring with maximum condition for ideals, the condition (A) is satisfied. See, for example, Theorem 10 of [3].

Then we consider the set M of the limit ideals of $\mathfrak{a}$ by $\mathfrak{r}'$, where $\mathfrak{r}'$ runs over all ideals not contained in $\mathfrak{p}$. Clearly, by (B), M is a finite set. Therefore, there exists a maximal ideal $\mathfrak{a}_0$ in M and, for some ideal $\mathfrak{r}\not\subseteq\mathfrak{p}$, $\mathfrak{a}_0 = \mathfrak{a}\mathfrak{r}^{-k} = \mathfrak{a}\mathfrak{r}^{-(k+1)} = \mathfrak{r}^{-(k+1)}\mathfrak{a} = \mathfrak{r}^{-k}\mathfrak{a}$, where k is a sufficiently large positive integer. Being $\mathfrak{r}^k\not\subseteq\mathfrak{p}$, $\mathfrak{p}$ is obviously a minimal prime divisor of $\mathfrak{a}_0$. By the definition, $\mathfrak{a}_0\subseteq\mathfrak{a}(\mathfrak{p})$. Let now b be any element in $\mathfrak{a}(\mathfrak{p})$. Then there exists an ideal $\mathfrak{s}\not\subseteq\mathfrak{p}$ such that $b\mathfrak{s}\subseteq\mathfrak{a}\subseteq\mathfrak{a}_0$. Hence, $b\in\mathfrak{a}_0\mathfrak{s}^{-1} = \mathfrak{a}(\mathfrak{s}\mathfrak{r}^k)^{-1}\subseteq\bigcup_{i=1}^{\infty}\mathfrak{a}(\mathfrak{s}\mathfrak{r}^k)^{-i}$. Since $\mathfrak{a}_0\subseteq\bigcup_{i=1}^{\infty}\mathfrak{a}(\mathfrak{s}\mathfrak{r}^k)^{-i}\in M$ and $\mathfrak{a}_0$ is maximal in M, then $\bigcup_{i=1}^{\infty}\mathfrak{a}(\mathfrak{s}\mathfrak{r}^k)^{-i} = \mathfrak{a}_0$, whence $b\in\mathfrak{a}_0$. Thus, we have $\mathfrak{a}(\mathfrak{p}) = \mathfrak{a}_0$ and, furthermore, $\mathfrak{a}_0 = \mathfrak{a}_0\mathfrak{s}^{-1}$ for every $\mathfrak{s}\not\subseteq\mathfrak{p}$.

Let us assume next the condition (C) in Theorem 7, in addition to (B). If $\overline{\mathfrak{a}(\mathfrak{p})}\neq\mathfrak{p}$, then there exists a minimal prime divisor $\mathfrak{p}'\neq\mathfrak{p}$ of $\mathfrak{a}(\mathfrak{p})$. Since, by (C), $\mathfrak{p}'$ is non-prime to $\mathfrak{a}(\mathfrak{p})$, for some ideal $\mathfrak{b}\not\subseteq\mathfrak{a}(\mathfrak{p})$, $\mathfrak{b}\mathfrak{p}'\subseteq\mathfrak{a}(\mathfrak{p})$. But $\mathfrak{b}\subseteq\mathfrak{a}(\mathfrak{p})\mathfrak{p}'^{-1} = \mathfrak{a}(\mathfrak{p})$. This contradiction shows $\overline{\mathfrak{a}(\mathfrak{p})} = \mathfrak{p}$. Hence we have proved the following

Lemma 3. *Let $\mathfrak{R}$ satisfy the conditions* (B) *and* (C) *in Theorem 7. Then $\mathfrak{a}(\mathfrak{p})$ is primary, where $\mathfrak{p}\subset\mathfrak{R}$ is a minimal prime divisor of $\mathfrak{a}$, and there exists an ideal $\mathfrak{r}_0\not\subseteq\mathfrak{p}$ such that $\mathfrak{a}(\mathfrak{p}) = \mathfrak{a}\mathfrak{r}_0^{-1} = \mathfrak{r}_0^{-1}\mathfrak{a}$.*

We prove next the following

Lemma 4. *If $\mathfrak{R}$ satisfies the conditions* (A) *and* (B) *in Theorem 7, then the number of prime ideals associated with a non-primary ideal $\mathfrak{a}$ is finite.*

Let $\{\mathfrak{p}_\sigma\}$ be the set of all prime ideals associated with $\mathfrak{a}$ and let $\mathfrak{q}_\sigma = \mathfrak{a}\mathfrak{r}_\sigma^{-1}$ ($\mathfrak{r}_\sigma\not\subseteq\mathfrak{a}$) be a primary ideal belonging to $\mathfrak{p}_\sigma$. By Lemma 3 and the condition (A), the set $\{\mathfrak{p}_\sigma\}$ is not empty.

At first, let $\mathfrak{p}_1\supset\mathfrak{p}_2\supset\cdots\cdots\supset\mathfrak{p}_k$ be a chain in $\{\mathfrak{p}_\sigma\}$, then $k\leq n(\mathfrak{a})$. If not, we define the ideals $\mathfrak{r}'_i$ ($i = 1, \cdots\cdots, k$) by setting $\mathfrak{r}'_1 =$ the limit ideal of $\mathfrak{a}$ by $\mathfrak{p}_1$, $\mathfrak{r}'_i =$ the limit ideal of $\mathfrak{r}'_{i-1}$ by $\mathfrak{p}_i$ ($i>1$). Then, for some positive integer h, $\mathfrak{p}_1^h\cdots\cdots\mathfrak{p}_j^h\mathfrak{r}'_j\subseteq\mathfrak{a}$, where, by (A), we assume $\mathfrak{p}_m^h\subseteq\mathfrak{q}_m$ for every m ($1\leq m\leq j$). As $\mathfrak{q}_m\mathfrak{r}_m\subseteq\mathfrak{a}$ and $\mathfrak{p}_1^h\cdots\cdots\mathfrak{p}_j^h\subseteq\mathfrak{p}_m^h$, we have $\mathfrak{r}_m\subseteq\mathfrak{r}'_j$. But, if $t>j$, $\mathfrak{r}_t\not\subseteq\mathfrak{r}'_j$, because $\mathfrak{p}_1^h\cdots\cdots\mathfrak{p}_j^h\mathfrak{r}_t\subseteq\mathfrak{a}$ implies $\mathfrak{p}_1^h\cdots\cdots\mathfrak{p}_j^h\subseteq\mathfrak{q}_t\subseteq\mathfrak{p}_t$. Hence, we have an ascending chain $\mathfrak{a}\subset\mathfrak{r}'_1\subset\cdots\cdots\subset\mathfrak{r}'_k$ in which each term is the limit ideal of the preceding one and whose length is k. But it contradicts with the condition (B). From this, we can easily see that there is a maximal one in $\{\mathfrak{p}_\sigma\}$.

Next, let $\{\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_k\}$ be a finite subset of $\{\mathfrak{p}_\sigma\}$ and let every $\mathfrak{p}_i$ be not contained in any remaining one. Then $k\leq n(\mathfrak{a})$. If not, by using the above argument, we can construct an ascending chain $\mathfrak{a}\subset\mathfrak{r}_1''\subset\cdots\cdots\subset\mathfrak{r}_k''$ in which each term is the limit ideal of the preceding one and whose length is k.

By the facts proved above, we see that there exists a finite number of maximal elements in $\{\mathfrak{p}_\sigma\}$. Now, we omit from the set $\{\mathfrak{p}_\sigma\}=M_1$ all the maximal elements $\mathfrak{p}_{1,1},\cdots\cdots,\mathfrak{p}_{1,s_1}$, and denote by M_2 the set of the remaining ideals. Since M_2 has obviously a finite number of maximal elements $\mathfrak{p}_{2,1},\cdots\cdots,\mathfrak{p}_{2,s_2}$, we obtain the set M_3 by omitting $\mathfrak{p}_{2,1},\cdots\cdots,\mathfrak{p}_{2,s_2}$ from M_2. Clearly, each ideal of $\mathfrak{p}_{2,1},\cdots\cdots,\mathfrak{p}_{2,s_2}$ is contained in some of $\mathfrak{p}_{1,i}$'s. Repeating this procedures, we obtain a descending chain $M_1\supseteq M_2\supseteq\cdots\cdots$, but $M_{n(\mathfrak{a})+1}$ is the empty set, because, otherwise, there exists a descending chain of prime ideals from $\{\mathfrak{p}_\sigma\}$ whose length exceeds $n(\mathfrak{a})-1$. q.e.d.

We assume here the condition (D) besides (A), (B) and (C) in Theorem 7.

Let $\mathfrak{p}_1,\cdots\cdots,\mathfrak{p}_n$ be all the prime ideals associated with an non primary ideal $\mathfrak{a}$ (by Lemma 4), and let $\mathfrak{q}_1,\cdots\cdots,\mathfrak{q}_n$ be the primary divisors of $\mathfrak{a}$ belonging to $\mathfrak{p}_1,\cdots\cdots,\mathfrak{p}_n$ respectively which possess the property in (D). We set $\mathfrak{d}=\mathfrak{q}_1\cap\cdots\cdots\cap\mathfrak{q}_n\ (\supseteq\mathfrak{a})$. By Lemma 3, every minimal prime divisor of $\mathfrak{a}$ is a prime ideal associated with $\mathfrak{a}$. Hence $\mathfrak{d}\subseteq\bar{\mathfrak{a}}$. As $\bar{\mathfrak{a}}$ is nilpotent modulo $\mathfrak{a}$ (by (A)), $\mathfrak{a}\mathfrak{d}^{-1}\supset\mathfrak{a}$. We suppose now that $\mathfrak{d}\supset\mathfrak{a}$. If $\mathfrak{a}\mathfrak{d}^{-1}$ is non primary, by Lemma 3, we have, for some ideal $\mathfrak{r}_0\not\subseteq\mathfrak{a}\mathfrak{d}^{-1}$, a primary ideal $\mathfrak{a}\mathfrak{d}^{-1}\mathfrak{r}_0^{-1}\subset\mathfrak{R}$ and set $\mathfrak{r}=\mathfrak{r}_0\mathfrak{d}$. On the other hand, if $\mathfrak{a}\mathfrak{d}^{-1}$ is primary, we set $\mathfrak{d}=\mathfrak{r}$. Hence, in either case, we have a primary ideal $\mathfrak{q}=\mathfrak{a}\mathfrak{r}^{-1}$, where $\mathfrak{r}\not\subseteq\mathfrak{a}$, $\mathfrak{r}\subseteq\mathfrak{d}$, and $\bar{\mathfrak{q}}$ is a prime ideal associated with $\mathfrak{a}$. Therefore, for some i, $\bar{\mathfrak{q}}=\mathfrak{p}_i$. Since $\mathfrak{r}\subseteq\mathfrak{d}\subseteq\mathfrak{q}_i$, the ideal $\mathfrak{q}=\mathfrak{a}\mathfrak{r}^{-1}$ is not primary (by (D)), but this is a contradiction. Hence, we have $\mathfrak{a}=\mathfrak{d}$. This proves the sufficiency part of the next principal theorem.

Theorem 8. *Every ideal in* $\mathfrak{R}$ *is represented as the intersection of a finite number of s-primary ideals if and only if the conditions* (A), (B), (C) *and* (D) *are satisfied.*

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