# INVERSES IN EUCLIDEAN MOBS*

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In this note we show, among other things, that if a clan is contained in $R^n$, $n \geq 2$, then all elements with inverses lie in the boundary. This is preceded by results on acts.

It is pleasant to be able to record my obligations to members of the Tulane Topology Seminar, and in particular to Dr. R. J. Koch.

We recall that a *mob* is a Hausdorff space together with a continuous associative multiplication. In what follows $S$ will always denote a mob.

If $t \in S$ let, for $n \geq 1$,

$$\Gamma_n(t) = \{t^m \mid m \geq n\}^*,$$

We write $\Gamma(t) = \Gamma_1(t)$ and

$$K(t) = \cap \{\Gamma_n(t) \mid n \geq 1\}.$$

If $\Gamma(t)$ is compact then $K(t)$ is a compact (topological) group and indeed, $K(t)$ is the minimal closed ideal of $\Gamma(t)$, see Koch [1], Numakura [2] and Peck [3]. We refer to this result as (A).

An *act* is a map

$$\pi : S \times X \longrightarrow X$$

such that (i) $S$ is a mob (ii) $X$ is a Hausdorff space (iii) $\pi(t_1, \pi(t_2, x)) = \pi(t_1 t_2, x)$ for any $t_1, t_2 \in S$ and $x \in X$. We shall say that $S$ *acts on* $X$. If $T \subset S$ and $A \subset X$ we write $TA$ for $\pi(T \times A)$.

Let $S$ act on $X$, let $t \in S$ and let $\Gamma(t)$ be compact. If $A$ is a compact part of $X$ such that $tA \subset A$ then

$$t'A = \cap \{t^n A \mid n \geq 1\}$$

for any $t' \in K(t)$. This result is due to Koch [1]. We refer to it as (B).

Let $f : X \rightarrow Y$ be a function on $X$ to $Y$ and let $a$ be a *descending family* of closed subsets of $X$, i.e., if $A_1, A_2 \in a$ then $A_3 \subset A_1 \cap A_2$ for some $A_3 \in a$. Suppose that either (i) some $A \in a$ is compact and $f^{-1}(y)$

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is closed for each $y \in Y$ or (ii) $f^{-1}(y)$ is compact for each $y \in Y$. Then

$$f(\cap \{A \mid A \in a\}) = \cap \{f(A) \mid A \in a\}.$$

If, in addition, for each $A_1 \in a$ there is an $A_2 \in a$ such that $A_2 \subset f(A_1)$ then (with $X = Y$)

$$\cap \{A \mid A \in a\} \subset f(\cap \{A \mid A \in a\}).$$

We refer to this result as (C).

**Theorem 1.** *Let the mob $S$ act on the space $X$, let $t \in S$ such that $\Gamma(t)$ is compact and let $A$ be such a compact part of $X$ that $tA \supset A$. Then $t_1 A = A$ for each $t_1 \in \Gamma(t)$, and each such $t_1$ acts as a homeomorphism on $A$. In particular the unit of $K(t)$ acts as the identity on $A$.*

*Proof.* The space $\Gamma(t)A$ is compact so that the compact mob $\Gamma(t)$ acts on the compact space $\Gamma(t)A$ by restriction. For notational simplicity let $S = \Gamma(t)$ and $X = \Gamma(t)A$. Let $e$ be the unit of $K(t)$. For each $n \geq 1$ we have

$$A \subset tA \subset \cdots\cdots \subset t^n A \subset t^n X \subset \cdots\cdots \subset tX \subset X.$$

From (B) it follows that

$$A \subset tA \subset eX.$$

Since $e^2 = e$ we get

$$A = eA \subset etA.$$

From (A) we know that $et$ has an inverse $v$ in the group $K(t)$. Thus

$$vA \subset vetA = eA = A.$$

We now apply (A) and (B) using $v$ in place of $t$, noting that $\Gamma(v) \subset \Gamma(t)$ and thus $\Gamma(v)$ is compact. If $f$ is the unit of $K(v)$ then

$$fA = \cap \{v^n A \mid n \geq 1\}.$$

Now $v \in K(t)$ so that $f \in K(v) \subset K(t)$. Consequently $f = e$ since $f^2 = f$ and $K(t)$ is a group. Hence

$$A = eA = fA \subset vA.$$

From this .

$$tA = teA \subset etvA = eA = A.$$

It follows that $tA = A$. Since also $t^n A = A$ for each $n \geq 1$ we have $t_1 A = A$ for any $t_1 \in \Gamma(t)$ because $\{t, t^2, \cdots\cdots\}$ is dense in $\Gamma(t)$. Since $e^2 = e$ and $eA = A$ we clearly have $ex = x$ for $x \in A$. Now let $t_1 \in \Gamma(t)$ and let $x, y \in A$ with $x \neq y$ and $t_1 x = t_1 y$. Let $U, V$ be disjoint open sets about $x, y$ respectively. Since $ex = x$ and $ey = y$ there is an open set $W$ including $e$ such that $Wx \subset U$ and $Wy \subset V$. Now $K(t_1)$ is a sub*group* of $\Gamma(t)$ and so $K(t_1) \subset K(t)$ because $K(t)$ is *the* maximal sub*group* of $\Gamma(t)$, see [4] and [5]. Thus $e$ is the unit of

$$K(t_1) = \cap \{\Gamma_n(t_1) \mid n \geq 1\}.$$

Since $W$ is open we have $t_1^m \in W$ for some $m \geq 1$. Thus $t_1^m x \in U$ and $t_1^m y \in V$. But $t_1 x = t_1 y$ and hence $t_1^m x = t_1^m y$. This contradiction completes the proof.

**Corollary 1.** *Let the compact mob $S$ act on the compact space $X$ and let $T$ be a subset of $S$; suppose $a$ is a descending family of subsets of $X$ such that if $t \in T$ and if $A \in a$ then $A_1 \subset tA$ for some $A_1 \in a$. If*

$$B = \cap \{A^* \mid A \in a\}$$

*then*

$$tB = B$$

*for each $t \in M$, the smallest closed submob of $S$ including the set $T$.*

*Proof.* If $t \in T$ we easily see that $B \subset tB$, using (C). We then apply Theorem 1 to get $tB = B$ for $t \in T$. It easily follows that $tB = B$ for each $t \in M$.

If $S$ is a mob we let $S * S$ denote the mob obtained from the set $S \times S$ with the multiplication

$$(a, b) * (x, y) = (ax, yb).$$

**Corollary 2.** *Let $A$ be a compact subset of the mob $S$ and let $a, b \in S$ be such that $\Gamma(a), \Gamma(b)$ are compact. If $A \subset aAb$ then $A = aAb$.*

*Proof.* It is clear that $S * S$ acts on $S$ using

$$\pi((a, b), x) = axb.$$

Moreover $\Gamma(a, b)$ in $S * S$ is compact since $\Gamma(a) \times \Gamma(b)$ is compact. The result follows by Theorem 1.

We can apply Theorem 1 to improve (A), see [3].

**Corollary 3.** *Let $S$ be a compact mob, let $J$ be a descending family of subsets of $S$, let $R = \cup \{T \mid T \in J\}$ and let*

$$S_0 = \cap \{T^* \mid T \in J\}.$$

*Assume further that, if $t \in R$ and $T \in J$, then $T_1 \subset tT$ and $T_2 \subset Tt$ for some $T_1, T_2 \in J$. Then $S_0$ is a compact (topological) group and if $M$ is the smallest closed submob of $S$ including $R$ then $S_0$ is the minimal closed ideal of $M$.*

*Proof.* If $t \in R$ we apply (C) to get $tS_0 \supset S_0 \subset S_0 t$. Theorem 1 then gives $tS_0 = S_0 = S_0 t$ for each $t \in R$ and hence $tS_0 = S_0 = S_0 t$ for each $t \in M$. Thus $S_0$ is a group and a closed ideal of $M$. Hence $S_0$ is the minimal closed ideal of $M$. That $S_0$ is topological is known, see for example [4].

It is possible to reformulate Corollary 3 in various ways. In particular let $T$ be a submob of $S$ such that $\{tT \mid t \in T\}$ is a descending family and assume that if $t_1, t_2 \in T$ then $t_3 T \subset t_1 T t_2$ for some $t_3 \in T$. Then $\cap \{(tT)^* \mid t \in T\}$ is a group and the minimal closed ideal of $T^*$. We assume that $S$ is compact.

In what follows $H^p(X, A)$ will denote the Alexander-Kolmogoroff cohomology group of the pair $(X, A)$ over a random (and concealed) coefficient group, see Spanier [7].

**Lemma 1.** *Let the connected mob $S$ act on the compact space $X$, let $A$ be a closed set in $X$ such that $SA \subset A$ and let $t \in S$ be such that $tX \subset A$. If, for some $z \in S$, with $\Gamma(z)$ compact we have $zX = X$, then $H^p(X, A) = 0$, $p \geq 0$.*

*Proof.* If $e$ is the unit of $\Gamma(z)$ then $e$ acts as the identity on $X$, see (A) and Theorem 1. For $a \in S$ define

$$\mu_a : (X, A) \longrightarrow (S \times X, S \times A)$$

by $\mu_a(x) = (a, x)$. Using Spanier's proof of the homotopy theorem [7] we get

$$\mu_a^* = \mu_b^* : H^p(S \times X, S \times A) \longrightarrow H^p(X, A),$$

Thus $\mu_t^* = \mu_e^*$. Now $tX \subset A$ implies $\mu_t^* \pi^* = \mu_e^* \pi^*$ where $\pi$ from $(S \times X, S \times A) \rightarrow (X, A)$ is defined by $\pi(a, x) = ax$. Now $\mu_t^* \pi^* = (\pi \mu_t)^*$ and since $\pi \mu_t : (X, A) \rightarrow (X, A)$ satisfies $\pi \mu_t(X) \subset A$ we get $(\pi \mu_t)^* = 0$.

On the other hand $\mu_e^* \pi^* = (\pi \mu_e)^*$ and $\pi \mu_e(x) = x$ for all $x \in X$. Hence $(\pi \mu_e)^*$ is the identity. Thus $H^p(X, A) = 0$.

**Lemma 2.** *Let $X$ be a compact set in $R^n$, let $F$ be the boundary of $X$ and let $A$ be a closed part of $X$ satisfying $F \subset A \neq X$. If $i_1 : A \subset X$ then $i_1^* ; H^{n-1}(X) \to H^{n-1}(A)$ is not onto, $n \geq 2$.*

*Proof.* We may assume that

$$X \subset \operatorname{Int} E^n, E^n = \{y \mid \|y\| \leq 1\}.$$

We may also assume that

$$0 = (0, 0, \cdots\cdots) \in X \setminus A.$$

We have $E^n = B \cup X$ where $B \subset (E^n \setminus X) \cup A$ is closed because $F \subset A$. Further $S^{n-1}$, the boundary of $E^n$, is contained in $B \subset E^n \setminus \{0\}$. Now the map $f : B \to S^{n-1}$ defined by $f(x) = \dfrac{x}{\|x\|}$ is a retraction and thus $H^{n-1}(B) \neq 0$. Let $i_2 : A \subset B$. Since $H^{n-1}(E^n) = 0$ we get an isomorphism

$$J^* : H^{n-1}(X) \times H^{n-1}(B) \longrightarrow H^{n-1}(A)$$

with $J^*(h_1, h_2) = i_1^* h_1 - i_2^* h_2$ (the Mayer-Vietoris Theorem, see [8]). Take $h_2 \neq 0$ in $H^{n-1}(B)$ so that $J^*(0, h_2) = -i_2^* h_2 \neq 0$. If $h_1 \in H^{n-1}(X)$ and $i_1^* h_1 = i_2^* h_2$ then $J^*(h_1, h_2) = 0$. Thus $(h_1, h_2) = 0$ since $J^*$ is an isomorphism. Hence $h_2 = 0$. This is a contradiction. Thus $i_1^*$ is not onto.

**Theorem 2.** *Let the mob $S$ act on the compact set $X \subset R^n$, $n \geq 2$. Let $t \in S$ be such that $\Gamma(t)$ is compact. If $F$ is the boundary of $X$ and if $tF \supset F$ then $t_1 X = X$ for each $t_1 \in \Gamma(t)$, the unit of $K(t)$ acting as the identity on $X$.*

*Proof.* From (A) and Theorem 1, $eF = F$, $e$ the unit of $K(t)$. Thus $F \subset eX$. Now $eX$ is a retract of $X$ since $e^2 = e$. Hence $i_1^* : H^{n-1}(X) \to H^{n-1}(eX)$ is onto, $i_1 : eX \subset X$. By Lemma 2, $eX = X$. By (B), $eX = \cap \{t^m X \mid m \geq 1\}$ so that $X = tX$. By Theorem 1 the remaining results follow.

We recall that a *clan* is a compact connected mob with unit.

**Theorem 3.** *Let the clan $S$ be contained in $R^n$, $n \geq 2$. If the closed ideal $I$ of $S$ includes the boundary of $S$ then $I = S$.*

*Proof.* Let $F$ be the boundary of $S$. Then $F \subset I$ implies $F \subset SFS \subset I$.

Then $i_1^*: H^{n-1}(S) \approx H^{n-1}(I)$ since, by Lemma 1, $H^{n-1}(S, I) = 0 = H^n(S, I)$. Hence $I = S$ by Lemma 2.

**Corollary.** *Let the clan S with unit u be contained in* $R^n$, $n \geq 2$. *If H(u) is the subgroup of those elements of S with inverses and if F is the boundary of S then*

$$H(u) = \{t \mid tF \supset F\} \subset F.$$

*Proof.* If $t \in H(u)$ then $t$ acts as a homeomorphism on $S$ and thus, as well known, $t$ preserves boundary points so that $tF \supset F$. If $tF \supset F$ then by Theorem 2 we have $tS = S$. It follows easily that $t$ has a right inverse, $\bar{t}$. Thus $t\bar{t} = u$. Hence $St\bar{t} = S$. By a result of Koch's [1] we have $S = St$ so that $u = t_1 t$ for some $t_1$. A simple argument shows that $t_1 = \bar{t}$ is the unique inverse of $t$. Thus $t \in H(u)$. As we noted earlier we have $H(u)FH(u) \subset F$ and thus [6] $SFS \cap H(u) = F \cap H(u)$. Now $F \subset SFS$ so that, by Theorem 3, $SFS = S$. Hence $H(u) = F \cap H(u)$. This completes the proof.

We note that this corollary extends a result in [6].

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