# ON THE HOMOTOPY GROUPS OF ROTATION GROUPS

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G. W. Whitehead [10][1)] and B. Eckmann [3] determined the $r$-th homotopy groups $\pi_r(R_n)$ of the rotation groups $R_n$ of the $n$-dimensional euclidean space $E^n$ for some values of $r$; and N. E. Steenrod [7, §§22 - 24] summarized them and further results for the values of $r$ ranging from 1 to 5. Their calculations are based on the homotopy groups $\pi_r(S^n)$ of the $n$-dimensional spheres $S^n$; and, in recent years, J. Serre [5, 6] and H. Toda [9] have independently determined the groups $\pi_r(S^n)$ for $r$ equal to $n+3$, $n+4$ and $n+5$. Therefore, we can calculate the groups $\pi_r(R_n)$ for $r$ equal to 6, 7 and 8, by the analogous processes developed in [7, §§22 - 24]. The results are stated as follows:

**Theorem 1.** i) $\pi_6(R_2)=0$, $\pi_6(R_3)=12$, $\pi_6(R_4)=12+12$ *and* $\pi_6(R_n)=0$ *for* $n \geqslant 5$.

ii) $\pi_7(R_2)=0$, $\pi_7(R_3)=2$, $\pi_7(R_4)=2+2$ *and* $\pi_7(R_5)=\infty$; *and* $\pi_7(R_6)=\infty+2$, $\pi_7(R_7)=\infty+4$, $\pi_7(R_8)=\infty+4+\infty$ *and* $\pi_7(R_n)=\infty+8$ *for* $n \geqslant 9$, *or* $\pi_7(R_6)=\infty$, $\pi_7(R_7)=\infty$, $\pi_7(R_8)=\infty+\infty$ *and* $\pi_7(R_n)=\infty$ *for* $n \geqslant 9$.

iii) $\pi_8(R_2)=0$, $\pi_8(R_3)=2$, $\pi_8(R_4)=2+2$, $\pi_8(R_5)=0$, $\pi_8(R_6)=24$, $\pi_8(R_7)=2+2$, $\pi_8(R_8)=2+2+2$, $\pi_8(R_9)=2+2$ *and* $\pi_8(R_n)=2$ *for* $n \geqslant 10$.

As the corollary, some results for the determinations of the groups $\pi_r(S^n)$ having a non-zero element are obtained by using the map $J: \pi_r(R_n) \to \pi_{r+n}(S^n)$ of G. W. Whitehead [12]:

**Theorem 2.** $\pi_r(S^n) \neq 0$ *for the following values of* $r$ *and* $n$:

| $r$ | 12 | 13 | 14 | 15 | 15 | 16 | 16 | 17 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 5 | 6 | 6 | 7 | 4 | 7 | 4 | 8 | 8 |

1) Numbers in brackets refer to the references cited at the end of this note.

2) We adopt the conventions that equating a group $\infty$ or $p$ means it is cyclic of order infinite or $p$, respectively.

## 1. Preliminaries

**1.1.**[1] We shall use notations analogous to those of [3]. Let $R_{n+1}$ be the rotation group of the $(n+1)$-dimensional euclidean space $E^{n+1}$ and $S^n$ the unit sphere of $E^{n+1}$. Then $R_{n+1}$ is the bundle space over the base space $S^n$ with the fibre and group $R_n$ and the natural projection $p: R_{n+1} \to S^n$ and, therefore, we can consider the exact homotopy sequence of this fibre bundle $\{R_{n+1}, p, S^n, R_n, R_n\}$:

$$\cdots\cdots \to \pi_{r+1}(S^n) \xrightarrow{\Delta} \pi_r(R_n) \xrightarrow{i_*} \pi_r(R_{n+1}) \xrightarrow{p_*} \pi_r(S^n) \to \cdots\cdots,$$

where $i_*$ and $p_*$ are the induced homomorphisms of $i$, the inclusion map of $R_n$ into $R_{n+1}$, and $p$ respectively, and $\Delta$ is the composed homomorphism $p_*^{-1}\partial$ of the isomorphism $p_*^{-1}: \pi_{r+1}(S^n) \to \pi_{r+1}(R_{n+1}, R_n)$ and the boundary homomorphism $\partial: \pi_{r+1}(R_{n+1}, R_n) \to \pi_r(R_n)$. The kernel of $i_*$ is $T_{n+1*}\pi_r(S^{n-1})$ for $r \leqslant 2n-3$, where $T_{n+1}: S^{n-1} \to R_n$ is the characteristic map of this bundle; and $i_*: \pi_r(R_n) \to \pi_r(R_{n+1})$ is isomorphic onto for $n \geqslant r+2$, and onto for $n = r+1$ and moreover $n = r$ if $r$ is even.

Representing $S^3$ by the group of quaternions $q$ of absolute value 1, and let $\rho: S^3 \to R_3$ and $\sigma: S^3 \to R_4$ be the map such that

$$\rho(q)\cdot q' = qq'q^{-1} \qquad \text{and} \qquad \sigma(q)\cdot q' = qq',$$

respectively, where $q' + \bar{q}' = 0$, that is $q' \in S^2$, in the former case. Then, the induced homomorphism $\rho_*: \pi_r(S^3) \to \pi_r(R_3)$ is isomorphic onto for $r \geqslant 2$; and $\pi_r(R_4) \approx \pi_r(S^3) + \pi_r(R_3)$ for $r \geqslant 1$, where the isomorphism of $\pi_r(S^3)$ into $\pi_r(R_4)$ is given by $\sigma_*$. For $r = 1, 2, 3$ and 4, the groups $\pi_r(R_n)$ and their generators are known as follows.

i) $\pi_1(R_2) = 0$, $\pi_1(R_n) = 2$ *for* $r \geqslant 3$, *and* $\pi_r(R_2) = 0$ *for* $r \geqslant 2$. $\pi_2(R_n) = 0$ *for all* $n$.

ii) $\pi_3(R_3) = \infty = \{\alpha_3\}$[2], $\pi_3(R_4) = \infty + \infty = \{\alpha_3\} + \{\beta_3\}$[3], *and* $\pi_3(R_n) = \infty = \{\beta_3\}$ *for* $n \geqslant 5$, *where* $\alpha_3$ *and* $\beta_3$ *are the elements represented by* $\rho$ *and* $\sigma$, *respectively.*

iii) $\pi_4(R_3) = 2 = \{\alpha_4\}$, $\pi_4(R_4) = 2 + 2 = \{\alpha_4\} + \{\beta_4\}$, $\pi_4(R_5) = 2 = \{\beta_4\}$, *and* $\pi_4(R_n) = 0$ *for* $n \geqslant 5$, *where* $\alpha_4 = \alpha_3 \circ \eta_3$[4] *and* $\beta_4 = \beta_3 \circ \eta_3$.

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1) For the properties of this section, cf. [7], §§7, 17, 22 - 24.

2) We denote by $\{\alpha\}$ the cyclic group generated by the element $\alpha$.

3) In $\pi_3(R_4)$, the term $\alpha_3$ must be written $i_*\alpha_3$ precisely, where $i$ is the inclusion map of $R_3$ into $R_4$. From now on, if $i_*$ maps a subgroup $\{\alpha\}$ of $\pi_r(R_n)$ isomorphically onto a subgroup $\{i_*\alpha\}$ of $\pi_r(R_{n+1})$, we shall omit the letter $i_*$.

4) $\eta_3$ is the generator of $\pi_4(S^3)$, cf. 1.2, i), below.

**1.2.** The following groups $\pi_r(S^n)$ are known explicitely[1].

i) $\pi_n(S^n) = \infty = \{\iota_n\}$ *for* $n \geqslant 1$. $\pi_3(S^2) = \infty = \{\eta_2\}$ *and* $\pi_{n+1}(S^n) = 2 = \{\eta_n\}$, *for* $n \geqslant 3$, *where* $\eta_n = E^{n-2}\eta_2$. $\pi_{n+2}(S^n) = 2 = \{\eta_n \circ \eta_{n+1}\}$ *for* $n \geqslant 2$.

ii) $\pi_5(S^2) = 2 = \{\eta_2 \circ \eta_3 \circ \eta_4\}$, $\pi_6(S^3) = 12 = \{\mu_3\}$, $\pi_7(S^4) = \infty + 12 = \{\nu_4\} + \{\mu_4\}$, *and* $\pi_{n+3}(S^n) = 24 = \{\nu_n\}$ *for* $n \geqslant 5$, *where* $\mu_n = E^{n-3}\mu_3$ *and* $\nu_n = E^{n-4}\nu_4$.

iii) $\pi_6(S^2) = 12 = \{\eta_2 \circ \mu_3\}$, $\pi_7(S^3) = 2 = \{\eta_3 \circ \nu_4\}$, $\pi_8(S^4) = 2 + 2 = \{\eta_4 \circ \nu_5\} + \{\nu_4 \circ \eta_7\}$, $\pi_9(S^5) = 2 = \{\nu_5 \circ \eta_8\}$ *and* $\pi_{n+4}(S^n) = 0$ *for* $n \geqslant 6$.

iv) $\pi_7(S^2) = 2 = \{\eta_2 \circ \eta_3 \circ \nu_4\}$, $\pi_8(S^3) = 2 = \{\eta_3 \circ \nu_4 \circ \eta_7\}$, $\pi_9(S^4) = 2 + 2 = \{\eta_4 \circ \nu_5 \circ \eta_8\} + \{\nu_4 \circ \eta_7 \circ \eta_8\}$, $\pi_{10}(S^5) = 2 = \{\nu_5 \circ \eta_8 \circ \eta_9\}$, $\pi_{11}(S^6) = \infty$ *and* $\pi_{n+5}(S^n) = 0$ *for* $n \geqslant 7$.

**1.3.** The groups $\pi_5(R_n)$ are calculated without proofs in [7, 24.11]. Now, we shall determine their generators for the use of later.

**Proposition.** $\pi_5(R_3) = 2 = \{\alpha_5\}$ *and* $\pi_5(R_4) = 2 + 2 = \{\alpha_5\} + \{\beta_5\}$, *where* $\alpha_5 = \alpha_3 \circ \eta_3 \circ \eta_4$ *and* $\beta_5 = \beta_3 \circ \eta_3 \circ \eta_4$. $\pi_5(R_5) = 2 = \{\beta_5\}$, $\pi_5(R_6) = \infty = \{\delta_5\}$, *and* $\pi_5(R_n) = 0$ *for* $n \geqslant 7$, *where* $\delta_5$ *is transformed into* $2\iota_5$ *of* $\pi_5(S^5)$ *by the map* $p_* : \pi_5(R_6) \to \pi_5(S^5)$.

$\pi_5(R_3)$ and $\pi_5(R_4)$ are followed immediately from 1.1.

Consider the bundle $\{R_5, p, S^4, R_4, R_4\}$ and its homotopy sequence:

$$\pi_5(R_4) \xrightarrow{i^5_*} \pi_5(R_5) \xrightarrow{p_*} \pi_5(S^4) \xrightarrow{\Delta} \pi_4(R_4) \xrightarrow{i^4_*} \pi_4(R_5).$$

As image $\Delta$ = kernel $i^4_*$ is cyclic subgroup of $\pi_4(R_4)$ of order 2 [7, 23.9. Theorem] and $\pi_5(S^4) = 2$, $\Delta$ is isomorphic onto and hence $i^5_*$ is onto by exactness. The kernel of $i^5_*$ is $T_{5*}\pi_5(S^3)$ and its generator is $T_{5*}(\eta_3 \circ \eta_4) = (-\alpha_3 + 2\beta_3) \circ \eta_3 \circ \eta_4$[2] $= \alpha_3 \circ \eta_3 \circ \eta_4 = \alpha_5$, and therefore $\pi_5(R_5)$ is cyclic of order 2 generated by the image of $\beta_5$.

In the case $\pi_5(R_6)$, if we consider the sequence: $\pi_5(R_5) \xrightarrow{i_*} \pi_5(R_6) \xrightarrow{p_*} \pi_5(S^5) \xrightarrow{\Delta} \pi_4(R_5) \to \pi_4(R_6)$, then image $p_*$ = kernel $\Delta$ is the infinite cyclic subgroup of $\pi_5(S^5) = \infty$ consisting of all even elements, because $\pi_4(R_6) = 0$ and $\pi_4(R_5) = 2$. On the other hand, kernel $i_* = T_{6*}\pi_5(S^4) = \{(\beta_3 \circ \eta_3) \circ \eta_4\}$[2] $= \{\beta_6\} = \pi_5(R_5)$, and hence $\pi_5(R_6) = \infty$.

The homomorphism $i_* : \pi_5(R_6) \to \pi_5(R_7)$ is onto and its kernel is $T_{7*}\pi_5(S^5)$. It is known that $pT_7$ maps $S^5$ onto $S^5$ with degree 2 [7, 23.4. Theorem], and hence $pT_7$ represents $2\iota_5$ of $\pi_5(S^5)$. This shows

1) Cf. [9], Appendix 2, a) - f).

2) The maps $T_5$ and $T_6$ represent the elements $-\alpha_3 + 2\beta_3$ and $\beta_3 \circ \eta_3$, respectively, cf. [7], 23.6. Theorem, and proofs of 24.6. Theorem.

that $T_7$ *represents the generator* $\delta_5$ *of* $\pi_5(R_6)$, and therefore $T_{7*}\pi_5(S^5) = \pi_5(R_6)$. Thus we have $\pi_5(R_7) = 0$ and so $\pi_5(R_n) = 0$ for $n \geqslant 7$.

1.4. For $R_3$ and $R_4$, it follows immediately, from 1.1 and 1.2.

**Proposition.** $\pi_6(R_3) = 12 = \{\alpha_6\}$, $\pi_7(R_3) = 2 = \{\alpha_7\}$ *and* $\pi_8(R_3) = 2 = \{\alpha_8\}$, *where* $\alpha_6 = \alpha_3 \circ \mu_3$, $\alpha_7 = \alpha_3 \circ \eta_3 \circ \nu_4$ *and* $\alpha_8 = \alpha_3 \circ \eta_3 \circ \nu_4 \circ \eta_7$. $\pi_6(R_4) = 12 + 12 = \{\alpha_6\} + \{\beta_6\}$, $\pi_7(R_4) = 2 + 2 = \{\alpha_7\} + \{\beta_7\}$ *and* $\pi_8(R_4) = 2 + 2 = \{\alpha_8\} + \{\beta_8\}$, *where* $\beta_6 = \beta_3 \circ \mu_3$, $\beta_7 = \beta_3 \circ \eta_3 \circ \nu_4$ *and* $\beta_8 = \beta_3 \circ \eta_3 \circ \nu_4 \circ \eta_7$.

## 2. The groups $\pi_r(R_5)$

2.1. To determine $\pi_r(R_5)$, we must calculate the kernel of $i_*: \pi_r(R_4) \to \pi_r(R_5)$. For this purpose, we first consider a principal bundle $\mathfrak{B} = \{B, p, S^n, G, G\}$ over $S^n$. Let $S^{n-1}$ be a great $(n-1)$-sphere on $S^n$ determined by setting the last real coordinate to zero, and $E^n_+$, $E^n_-$ the closed hemi-spheres of $S^n$ determined by $S^{n-1}$. Moreover, let $a_1$ in $E^n_+$ and $a_2$ in $E^n_-$ be the poles of $S^{n-1}$, and $V_1$ and $V_2$ be open cells on $S^n$ bounded by $(n-1)$-spheres parallel to $S^{n-1}$ and containing $E^n_+$ and $E^n_-$ respectively. If the bundle $\mathfrak{B}$ is in normal form, that is, its coordinate neighborhoods are $V_1$ and $V_2$, and $g_{12}(a_0) = e$ the identity of $G$, where $a_0$ is the reference point on $S^{n-1}$ and $g_{12}: V_1 \cap V_2 \to G$ is the coordinate transformation, then the map $T = g_{12} \mid S^{n-1}: S^{n-1} \to G$ is known as the characteristic map of $\mathfrak{B}$; and, if $r \leqslant 2n-3$, the image of the homomorphism $\Delta: \pi_{r+1}(S^n) \to \pi_r(G_1)$ is the group $\xi_* T_* \pi_r(S^{n-1})$, where $G_1$ is the fibre over $a_1$ and $\xi = \phi_{1,a_1} = \phi_1 \mid a_1 \times G$ and $\phi_i: V_i \times G \to p^{-1}(V_i)$ is the coordinate function[1].

To prove this property, we consider the diagram:

$$\begin{array}{ccccc}
\pi_{r+1}(S^n) & \xleftarrow{p_*} & \pi_{r+1}(B, G_1) & \xrightarrow{\partial} & \pi_r(G_1) \\
\downarrow k_* & \nwarrow h_* & \uparrow h'_* & & \uparrow \xi_* T_* \\
\pi_{r+1}(S^n, E^n_+) & \xleftarrow{l_*} & \pi_{r-1}(E^n_-, S^{n-1}) & \xrightarrow{\partial'} & \pi_r(S^{n-1})
\end{array}$$

where $k$ and $l$ are inclusion maps, $\partial$ and $\partial'$ the boundary homomorphisms and $h: (E^n_-, S^{n-1}) \to (S^n, a_1)$ the map such that, for $x \in E^n_-$, $h(x)$ lies in the great circle arc $C(x) = a_2 x a_1$ and its arc length from $a_2$ is twice that of $x$; and, finally, if $k(x)$ is the point $C(x) \cap S^{n-1}$, $h': (E^n_-, S^{n-1}) \to (B, G_1)$ is the map defined by

1) Cf. [7], 23.2. Theorem.

$$h'(x) = \begin{cases} \phi_1(h(x), \ Tk(x)), & \text{when } h(x) \in E_1, \\ \phi_2(h(x), \ e), & \text{when } h(x) \in E_2. \end{cases}$$

Then, $k_*$, $p_*$ and $\partial'$ are isomorphic onto and commutative relations hold in the square and triangles, Moreover, as the composed map $k_*^{-1} l_* \partial'^{-1} : \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ is the suspension $E$, $\xi_* T_* \pi_r(S^{n-1}) = \partial p_*^{-1} E \pi_r(S^{n-1}) = \Delta E \pi_r(S^{n-1})$. If $r \leqslant 2n-3$, this shows the above property, as $E: \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ is onto. The last property is stated as follows.

**Theorem.** *In the above notations, the image group* $\xi_* T_* \pi_r(S^{n-1})$ *is the subgroup* $\Delta E \pi_r(S^{n-1})$ *of* $\pi_r(G_1)$, *and hence, if* $E: \pi_r(S^{n-1}) \to \pi_{r+1}(S^n)$ *is onto, the group* $\xi_* T_* \pi_r(S^{n-1})$ *is equal to the image of* $\Delta: \pi_{r+1}(S^n) \to \pi_r(G_1)$.

2.2. Now, we consider the principal bundle $\{R_{n+1}, p, S^n, R_n, R_n\}$. If $\alpha$ is a element of $\pi_{r+1}(S^n)$ such that $\Delta\alpha = 0$, then there is a element $\beta \in \pi_{r+1}(R_{n+1})$ such that $p_* \beta = \alpha$, by exactness of the homotopy sequence stated in 1.1. Let $f: S^{r+1} \to R_{n+1}$ be a map representing $\beta$, and $\bar{f}: S^{r+1} \times S^n \to S^n$ be the map determined by the formula: $\bar{f}(x, y) = f(x) \cdot y$, where $x \in S^{r+1}$ and $y \in S^n$. Then $\bar{f} \mid S^{r+1} \times y_0$ represents $p_* \beta = \alpha$ and $\bar{f} \mid x_0 \times S^n$ represents $\iota_n$ for base points $x_0 \in S^{r+1}$ and $y_0 \in S^n$, and hence $f$ has type $(\alpha, \iota_n)$. It is known that the existence of a map of type $(\alpha, \beta)$ is equivalent to $[\alpha, \beta] = 0$, where $[\alpha, \beta]$ is the Whitehead product of $\alpha$ and $\beta$ [11, Corollary (3.5)], and therefore we have

**Theorem.** *If* $\alpha$ *is a element of* $\pi_{r+1}(S^n)$ *such that* $\Delta\alpha = 0$, *then* $[\alpha, \iota_n] = 0$.

2.3. By using above two theorems, we can prove

**Lemma.** *The map* $\Delta: \pi_7(S^4) \to \pi_6(R_4)$ *is onto and its kernel is the infinite cyclic subgroup of* $\pi_7(S^4)$ *generated by* $12\nu_4$.

As the subgroup $\{\mu_4\} \subset \pi_7(S^4)$ is equal to $E\pi_6(S^3)$, by Theorem of 2.1, $\Delta\{\mu_4\} = T_{\xi*}\pi_6(S^3)$ and its generator is

$$\begin{aligned} T_{\xi*}\mu_3 &= (-\alpha_3 + 2\beta_3) \circ \mu_3 \\ &= (-\alpha_3) \circ \mu_3 + (2\beta_3) \circ \mu_3 + [-\alpha_3, 2\beta_3] \circ H(\mu_3)^{1)} \end{aligned}$$

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1) In [12], Theorem 5.15, G. W. Whitehead proved that, if $\alpha \in \pi_r(S^n)$ and $\beta_1, \beta_2 \in \pi_n(X)$, and if $n < 3r-3$, then $(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha)$, where $H(\alpha) \in \pi_r(S^{2n-1})$ is the generalized Hopf invariant; and Blakers and Massey [2, (5.5)] showed that the definition of the generalized Hopf invariant $H(\alpha)$ can be extended by one dimension, and the above relation also holds for the case $n = 3r-3$.

$$= -\alpha_3\circ\mu_3 + 2\beta_3\circ\mu_3 + [\alpha_3, \alpha_3]\circ H(\mu_3) + [\beta_3\circ\beta_3]\circ H(\mu_3) - [\alpha_3, \beta_3]\circ 2H(\mu_3)$$
$$= -\alpha_6 + 2\beta_6.$$

The last equality follows from the fact that

$$[\alpha_3, \alpha_3] = [\rho_*\iota_3, \rho_*\iota_3] = \rho_*[\iota_3, \iota_3] = 0,$$
$$[\beta_3, \beta_3] = [\sigma_*\iota_3, \sigma_*\iota_3] = \sigma_*[\iota_3, \iota_3] = 0,$$

because $E([\iota_3, \iota_3]) = 0$ [11, Theorem 3.11] and the fact that $E: \pi_5(S^3) \rightarrow \pi_6(S^4)$ is isomorphic onto imply $[\iota_3, \iota_3] = 0$. The above calculation shows that $\Delta\{\mu_4\} = \{-\alpha_6 + 2\beta_6\} = 12$.

On the other hand, it is known that $[\nu_4, \iota_4] = 2\nu_4\circ\nu_7$[1] and $\nu_4\circ\nu_7$ has order 24[2]; and hence, by Theorem of 2.2, $\Delta\{\nu_4\}$ is cyclic subgroup of $\pi_6(R_4)$ of order 12. Set $\Delta\nu_4 = \alpha$ and $\Delta\mu_4 = \beta$, then $\{\alpha\}$ and $\{\beta\}$ are both cyclic subgroups of order 12. If $m\alpha = n\beta$ for some integers $m$ and $n$ such that $0 \leqslant m, n < 12$, then $\Delta(m\nu_4 - n\mu_4) = 0$, and so $[m\nu_4 - n\mu_4, \iota_4] = 0$ by Theorem of 2.2. As $[\mu_4, \iota_4] = [\iota_4\circ E\mu_3, \iota_4\circ E\iota_3] = [\iota_4\circ\iota_4]\circ(\mu_3 * \iota_3)$[3] $= (2\nu_4 - \mu_4)\circ E^4\mu_3 = (2\nu_4 - \mu_4)\circ 2\nu_7$[4] $= (4\nu_4 - 2\mu_4)\circ\nu_7$, it follows that $((2m - 4n)\nu_4 - 2n\mu_4)\circ\nu_7 = 0$, and hence $2m - 4n \equiv 0 \mod 24$[5]. This shows that $m - 2n = 0$ or $-12$, and so $m + 12a = 2n$, where $a = 0$ or 1. As $\alpha$ has order 12, this shows that $2n\alpha = (m + 12a)\alpha = m\alpha$ and hence $2n\alpha = n\beta$. If $n \neq 0$, this relation contradicts with the fact that both $\alpha$ and $\beta$ have order 12; and therefore $n = 0$, and $m = 0$. Thus the intersection of $\{\alpha\}$ and $\{\beta\}$ contains the zero element only, and hence $\{\alpha\} + \{\beta\} = \pi_6(R_4)$, as $\pi_6(R_4) = 12 + 12$ has $12 \times 12$ elements.

The above results show that $\Delta\pi_7(S^4) = \Delta(\{\mu_4\} + \{\nu_4\}) = \pi_6(R_4)$, and the above lemma is completed.

2.4. To determine the image of $\Delta: \pi_{r+1}(S^4) \rightarrow \pi_r(R_4)$ for $r > 7$ and 8, we consider the homomorphism $Y_r: \pi_r(X) \rightarrow \pi_{r+1}(X)$, for $r > 2$, defined by

$$Y_r(\alpha) = \alpha\circ\eta_r, \qquad \alpha \in \pi_r(X).$$

1) This is a consequence of the fact that $[\iota_4, \nu_4] = 2\nu_4\circ\nu_7$ [9, Lemma (4.6)], and $[\alpha, \beta] = (-1)^{pq}[\alpha, \beta]$ for $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ [13, (3.3)].

2) If we apply to $\nu_4\circ\nu_7$ the Hopf homomorphism $H_0: \pi_r(S^n) \rightarrow \pi_{r+1}(S^{2n})$ of [9, (3.2)] for $r = 10$ and $n = 4$, and use [9, (3.4)] or [8, (2.7)], then $H_0(\nu_4\circ\nu_7) = H_0(\nu_4)\circ E^2\nu_6 = \iota_8\circ\nu_8 = \nu_8$, and so $\nu_4\circ\nu_7$ has order 24.

3) Cf. [12], (3.58), where $\mu_3 * \iota_3$ is the join of $\mu_3$ and $\iota_3$.

4) $[\iota_4, \iota_4] = 2\nu_4 - \mu_4$ and $E^{n-3}\mu_3 = 2\nu_n$, cf. [9], Lemma (4.3), ii).

5) As in the footnote 2), $H_0((a\nu_4 - b\mu_4)\circ\nu_7) = H_0(a\nu_4\circ\nu_7) - bH_0(E(\mu_3\circ\nu_6)) = a\nu_8$, by [9, (3.3)] or [8, (2.6)], and hence, if $(a\nu_4 - b\mu_4)\circ\nu_7 = 0$, $a$ is a multiple of 24.

As $\eta_r$ is the 'suspension of $\eta_{r-1}$, $Y_r$ is clearly a homomorphism. Let $\mathfrak{B} = \{B, p, X, Y, G\}$ be a fibre bundle, $Y_0$ the fibre over $x_0 \in X$ and $\Delta_r : \pi_{r+1}(X) \to \pi_r(Y_0)$ the boundary homomorphism of the homotopy sequence of this fibre bundle, then we have

**Lemma.** $\Delta_r Y_r = Y_{r-1}\Delta_{r-1}$.

Let $E^n_+$ and $E^n_-$ be as in 2.1, and $h : (E^r_+, S^{r-1}) \to (S^r, a_0)$ be a map representing $\iota_r \in \pi_r(S^r)$, then $h_* : \pi_{r+1}(E^r_+, S^{r-1}) \to \pi_{r+1}(S^r)$ is isomorphic onto for $r > 3$. As $\partial : \pi_{r+1}(E^r_+, S^{r-1}) \to \pi_r(S^{r-1})$ is isomorphic onto, there is a map $g : (E^{r+1}_+, S^r) \to (E^r_+, S^{r-1})$ such that $g \mid S^r : S^r \to S^{r-1}$ represents $\eta_{r-1} \in \pi_r(S^{r-1})$, and $g$ represents a generator of $\pi_{r+1}(E^r_+, S^{r-1})$, and therefore $hg$ represents the generator $\eta_r$ of $\pi_{r+1}(S^r)$.

We consider the diagram

$$\begin{array}{ccccccc} & \overset{p_*}{\nearrow} & \pi_r(X) & \xrightarrow{Y_r} & \pi_{r+1}(X) & \overset{p_*}{\nwarrow} & \\ \pi_r(B, Y_0) & & \big\downarrow \Delta_{r-1} & & \big\downarrow \Delta_r & & \pi_{r+1}(B, Y_0) \\ & \underset{\partial}{\searrow} & \pi_{r-1}(Y_0) & \xrightarrow{Y_{r-1}} & \pi_r(Y_0) & \underset{\partial}{\swarrow} & \end{array}$$

Let $f : (S^r, a_0) \to (X, x_0)$ and $f' : (E^r_+, S^{r-1}) \to (B, Y_0)$ be a representative of $\alpha \in \pi_r(X)$ and $p_*^{-1}(\alpha) \in \pi_r(B, Y_0)$ respectively, then both $pf'$ and $fh$ represent $\alpha$ and so $pf'$ is homotopic to $fh$. Hence, $pf'g$ is homotopic to $fhg$, and, as the latter represents $\alpha \circ \eta_r = Y_r(\alpha)$, $f'g$ represents $p_*^{-1}Y_r(\alpha)$. Thus $f'g \mid S^r : S^r \to Y_0$ represents $\partial p_*^{-1} Y_r(\alpha) = \Delta_r Y_r(\alpha)$. On the other hand, as $f'g \mid S^r$ is the composition of $g \mid S^r$ and $f' \mid S^{r-1}$ and these maps represent $\eta_{r-1}$ and $\partial p_*^{-1}(\alpha) = \Delta_{r-1}\alpha$ respectively, $f'g \mid S^r$ represents $(\Delta_{r-1}\alpha)\circ\eta_{r-1} = Y_{r-1}\Delta_{r-1}\alpha$. Thus we have $\Delta_r Y_r \alpha = Y_{r-1}\Delta_{r-1}\alpha$.

2.5. **Lemma.** *The map $\Delta_r : \pi_{r+1}(S^4) \to \pi_r(R_4)$ is isomorphic onto for $r = 7, 8$.*

For the case $\mathfrak{B} = \{R_5, p, S^4, R\ , R\ \}$, $Y_7 : \pi_7(S^4) \to \pi_8(S^4)$, $Y_6 : \pi_6(R_4) \to \pi_7(R_4)$ and $\Delta_6 : \pi_7(S^4) \to \pi_6(R_4)$ are onto by 1.2, 1.4 and 2.3 respectively, and therefore, by the lemma of 2.4, $\Delta_7$ is onto. Similarly, $\Delta_8$ is also onto. Finally, as $\pi_{r+1}(S^4)$ and $\pi_r(R_4)$ are the same type $2+2$, for $r = 7$ and 8, isomorphic properties are followed from ontoness.

2.6. Now we can determine $\pi_r(R_5)$.

**Proposition.** $\pi_6(R_5) = 0$, $\pi_7(R_5) = \infty = \{\gamma_7\}$ *and* $\pi_8(R_5) = 0$, *where* $\gamma_7$ *satisfies* $p_* \gamma_7 = 12\nu_4 \in \pi_7(S^4)$.

Consider the exact homotopy sequence of the bundle $\{R_5, p, S^4, R_4, R_4\}$:

$$\pi_{r+1}(S^4) \xrightarrow{\Delta_r} \pi_r(R_4) \to \pi_r(R_5) \xrightarrow{p_*} \pi_r(S^4) \xrightarrow{\Delta_{r-1}} \pi_{r-1}(R_4) \xrightarrow{i_*} \pi_{r-1}(R_5).$$

For $r = 6, 7$ and 8, $\Delta_r$ is onto by 2.3 and 2.5, and hence $p_*$ is isomorphic into by exactness. For the case $r = 6$, kernel $i_* = 2$ by 1.3, and $\pi_6(S^4) = 2$, and so $\Delta_5$ is isomorphic onto. This shows that $\pi_6(R_5) = 0$. For $r = 7$, kernel $\Delta_6 = \{12\nu_4\}$ by 2.3, and hence $\pi_7(R_5) = \infty$. Finally, for $r = 8$, as $\Delta_7$ is isomorphic onto, $\pi_8(R_5) = 0$.

## 3. The groups $\pi_r(R_n)$ for $n \geqslant 6$

**3.1. Proposition.** $\pi_6(R_n) = 0$ *for* $n \geqslant 6$.

In the homotopy sequence $\pi_6(R_5) \xrightarrow{i_*} \pi_6(R_6) \xrightarrow{p_*} \pi_6(S^5) \xrightarrow{\Delta} \pi_5(R_5) \to \pi_5(R_6)$, image $\Delta = 2$ and $\pi_6(S^5) = 2$ imply the ontoness of $i_*$, and hence $\pi_6(R_6) = 0$, because $\pi_6(R_5) = 0$. By 1.1, $\pi_6(R_n) \to \pi_6(R_{n+1})$ is onto for $n \geqslant 6$, and therefore we have 3.1.

**3.2. Proposition.** $\pi_7(R_6)$ *is equal to* i) $\infty + 2 = \{\gamma_7\} + \{\delta_7\}$ *or* ii) $\infty = \{\delta_7\}$, *where* $p_* \delta_7 = \eta_5 \circ \eta_6 \in \pi_7(S^5)$ *and, in the case* ii), $2\delta_7 = \gamma_7$. $\pi_8(R_6) = 24 = \{\delta_8\}$, *where* $p_* \delta_8 = \nu_5 \in \pi_8(S^5)$.

$\pi_8(S^5) = 24$ and $\pi_7(R_5) = \infty$ imply the image of the homomorphism $\Delta : \pi_8(S^5) \to \pi_7(R_5)$ is zero only, and therefore the above proposition follows immediately by making use of the homotopy sequence of the fibre bundle $\{R_6, p, S^5, R_5, R_5\}$.

3.3. Now, we consider some maps. Representing $S^7$ by Cayley numbers of absolute value 1, and taking a map $\bar{\rho} : S^7 \to R_7$ defined by $\bar{\rho}(c) \cdot c' = cc'c^{-1}$, where $c \in S^7$ and $c' \in S^6 = \{c \mid c \in S^7$ and the real part of $c$ is zero$\}$. Then $\bar{\rho}$ is a continuous map and it is known that $p\bar{\rho} : S^7 \to S^6$ represents a nonzero element of $\pi_7(S^6)$[1].

It is known that the bundle $R_8$ is equivalent to the product bundle $S^7 \times R_7$ and the map $\bar{\sigma} : S^7 \to R_8$, defined by $\bar{\sigma}(c) \cdot c' = cc'$, where $c, c' \in S^7$, is clearly a cross-section of this product bundle, and so, in the direct sum decomposition $\pi_r(R^8) \approx \pi_r(S^7) + \pi_r(R_7)$, the isomorphism of $\pi_r(S^7)$ into $\pi_r(R_8)$ is given by $\bar{\sigma}_*$[2]. Let $\varepsilon_7$ and $\zeta_7$ be elements represented by $\bar{\rho}$ and $\bar{\sigma}$ respectively, then

**3.4. Proposition.** i) $\pi_7(R_7) = \infty + 4 = \{\gamma_7\} + \{\varepsilon_7\}$ *and* $\pi_7(R_8) =$

1) $\bar{\rho}$ is equivalent to $\bar{f}$ of [12], (8.12), which has the property that $p\bar{f}$ represents $\eta_6$.
2) Cf. [7], (8.5), (8.6) and (17.8).

$\infty + 4 + \infty = \{\gamma_7\} + \{\varepsilon_7\} + \{\zeta_7\}$, *if* $\pi_7(R_6)$ *is the case* i) *of* 3.2; *or* ii) $\pi_7(R_7) = \infty = \{\varepsilon_7\}$ *and* $\pi_7(R_8) = \infty + \infty = \{\varepsilon_7\} + \{\zeta_7\}$, *if* $\pi_7(R_6)$ *is the case* ii) *of* 3.2; *and the relation* $2\varepsilon_7 = \delta_7$ *holds in* $\pi_7(R_7)$.

In the sequence $\pi_7(R_6) \xrightarrow{i_*} \pi_7(R_7) \xrightarrow{p_*} \pi_7(S^6) \xrightarrow{\Delta} \pi_6(R_6)$, because $\delta_5 \circ \eta_5 \in \pi_6(R_6) = 0$, kernel $i_* = T_{7*}\pi_7(S^5) = \{\delta_5 \circ \eta_5 \circ \eta_6\}$[1] $= 0$, and hence $i_*$ is isomorphic into. As $\pi_6(R_6) = 0$, $\pi_7(R_7)\,/\,\text{kernel}\ p_* \approx \pi_7(S^6) = 2$. On the other hand, in the homotopy sequence of $(R_7, R_5)$: $\pi_7(R_5) \xrightarrow{i_*} \pi_7(R_7) \xrightarrow{j_*} \pi_7(R_7, R_5) \rightarrow \pi_6(R_5)$, $i_*$ is isomorphic and $\pi_6(R_5) = 0$, and therefore $\pi_7(R_7)\,/\,\text{image}\ i_* = \pi_7(R_7)\,/\,\{\gamma_7\} \approx \pi_7(R_7, R_5) = 4$[2]. These relations imply the above proposition for $\pi_7(R_7)$. $\pi_7(R_8)$ follows from 3.3.

**3.5. Proposition.** $\pi_8(R_7) = 2 + 2 = \{\bar{\delta}_8\} + \{\varepsilon_8\}$ *and* $\pi_8(R_8) = 2 + 2 + 2 = \{\bar{\delta}_8\} + \{\varepsilon_8\} + \{\zeta_8\}$, *where* $\bar{\delta}_8 = i_* \delta_8$, $\varepsilon_8 = \varepsilon_7 \circ \eta_7$ *and* $\zeta_8 = \zeta_7 \circ \eta_7$.

In the sequence: $\pi_8(R_6) \xrightarrow{i^8_*} \pi_8(R_7) \xrightarrow{p_*} \pi_8(S^6) \rightarrow \pi_7(R_6) \xrightarrow{i^7_*} \pi_7(R_7)$, $i^7_*$ is isomorphic, and hence $p_*$ is onto. The kernel of $i^8_*$ is equal to $T_{7*}\pi_8(S^5)$ and its generator is $T_{7*}\nu_5 = \delta_5 \circ \nu_5 = 2\delta_8$, as $p_*(\delta_5 \circ \nu_5) = p_*(\delta_5) \circ \nu_5 = 2\iota_5 \circ \nu_5 = 2\nu_5$. Thus image $i^8_* \approx \{\delta_8\}\,/\,\{2\delta_8\} = 2$. On the other hand, as $p_*(\varepsilon_7 \circ \eta_7) = p_*(\varepsilon_7) \circ \eta_7 = \eta_6 \circ \eta_7 \neq 0$ in $\pi_8(S^6)$, the element $\varepsilon_8 = \varepsilon_4 \circ \eta_7$ of $\pi_8(R_7)$ does not belong to image $i^8_*$ and clearly has oder 2. Thus we have $\pi_8(R_7) = 2 + 2$ and the above proposition.

**3.6. Proposition.** *For* $n \geqslant 9$, *corresponding to the case* i) *or* ii) *of* 3.2, i) $\pi_7(R_n) = \infty + 8 = \{\gamma_7\} + \{\bar{\zeta}_7\}$, *or* ii) $\pi_7(R_n) = \infty = \{\bar{\zeta}_7\}$, *where* $\bar{\zeta}_7 = i_* \zeta_7$ *and the relation* $2\bar{\zeta}_7 = \varepsilon_7$ *holds.* $\pi_8(R_9) = 2 + 2 = \{\bar{\delta}_8\} + \{\zeta_8\}$, *and* $\pi_8(R_n) = 2 = \{\bar{\delta}_8\}$ *for* $n \geqslant 10$.

The groups $\pi_7(R_9)$ and $\pi_8(R_9)$ are the immediate consequence of the property that $T_9 : S^7 \rightarrow R_8$, the characteristic map of the principal bundle $\{R_9, p, S^8, R_8, R_8\}$, represents the element $-\varepsilon_7 + 2\zeta_7$ of $\pi_7(R_8)$, which can be proved by the same proofs of the fact that $T_5$ represents $-\alpha_3 + 2\beta_3$[3] by using Cayley numbers instead of quaternions.

The characteristic map $T_{10} : S^8 \rightarrow R_9$ is homotopic to the characteristic map $T'_5 : S^8 \rightarrow R_8$ of the unitary bundle[3]. Because $pT'_5 : S^8 \rightarrow S^7$ is essential[3], $T'_5$ represents $a\bar{\delta}_8 + b\varepsilon_8 + \zeta_8$ of $\pi_8(R_8)$, where $a, b = 0$ or 1. These properties show that $T_{10}$ represents the image of $a\bar{\delta}_8 + b\varepsilon_8 + \zeta_8$ under the map $i_* : \pi_8(R_8) \rightarrow \pi_8(R_9)$ and the latter is $a\bar{\delta}_8 + \zeta_8$, where $a = 0$ or 1. Thus we have 3.6.

1) $T_7$ represents $\delta_5 \in \pi_5(R_6)$, cf. proofs of the proposition of 1.3.
2) Cf. [1], Theorem 1.2.
3) Cf. [7], 23.6. Theorem, and 24.2 - 24.5.

By the results of §§2-3, we obtain Theorem 1 completely.

## 4. Some remarks on $\pi_r(S^n)$

**4.1.** It was proved by G. W. Whitehead that, if $\alpha \in \pi_r(R_n)$ and $p_*(\alpha) \in \pi_r(S^{n-1})$ is not zero, then $J(\alpha) \in \pi_{r+n}(S^n)$ is a non-zero element for $r < 2n-3$[1]; and A. L. Blakers and W. S. Massey generalized it for $n \leqslant 2n-3$ that, if $\alpha \in \pi_r(R_n)$ and the suspension $Ep_*(\alpha)$ is not zero, then $J(\alpha) \neq 0$[2]. By the analogous process and making use of the Hopf homomorphism $H_0: \pi_r(S^n) \to \pi_{r+1}(S^{2n})$ of [9], we can prove more generally

**4.2. Theorem.** *If $\alpha \in \pi_r(R_n)$ and the m-hold suspension $E^m p_*(\alpha)$ of $p_*(\alpha) \in \pi_r(S^{n-1})$ is not zero, then $J(\alpha)$ is a non-zero element of $\pi_{r+n}(S^n)$, where m is the minimum value of $n+1$ and $r-2n+4$.*

If $\alpha \in \pi_r(R_n)$, then $J(\alpha)$ is represented by the Hopf construction of the mapping $S^r \times S^{n-1} \to S^{n-1}$ of type $(p_*(\alpha), \iota_{n-1})$[1], and therefore $H_0(J(\alpha)) = (-1)^{r(n-1)} E(p_*(\alpha) * \iota_{n-1})$[3] $= (-1)^{r(n-1)} E(E^n p_*(\alpha)) = (-1)^{r(n-1)} E^{n+1} p_*(\alpha)$. As the suspension homomorphism $E: \pi_{r+p}(S^{n-1+p}) \to \pi_{r+p+1}(S^{n+p})$ is isomorphic for $p \geqslant r-2n+4$, 4.2 is established.

**4.3.** From 4.2, we can prove Theorem 2.

For the case $\pi_{12}(S^5)$, we consider the element $\gamma_7$ of $\pi_7(R_5)$. By 2.6, $p_* \gamma_7 = 12\nu_4$, and, because $E^m(12\nu_4) = 12\nu_{m+4} \in \pi_{m+7}(S^{m+4})$ is not zero, 5.2 implies $J(\gamma_7)$ is a non-zero element of $\pi_{12}(S^5)$.

For the case $\pi_{14}(S^6)$, $E^m p_* \delta_8 = E^m \nu_5 = \nu_{m+5}$ and hence $aJ(\delta_8) = J(a\delta_8)$ is not zero for $a = 1, 2, \cdots\cdots, 23$. Hence

**4.4. Proposition.** *$\pi_{14}(S^6)$ contains a cyclic subgroup whose order is a multiple of 24.*

$\pi_{15}(S^7)$ contains a non-zero element $J(\varepsilon_8)$, and hence, by [4, Theorem 4], $\nu_4 \circ J(\varepsilon_8)$ is a non-zero element of $\pi_{15}(S^4)$.

Now, we consider the homotopy sequence of the bundle $\{R_7, p, S^6, R_6, R_6\}$ :

$$\pi_9(R_7) \xrightarrow{p_*} \pi_9(S^6) \xrightarrow{\Delta} \pi_8(R_6) \xrightarrow{i_*} \pi_8(R_7).$$

As kernel $i_* = 12$ by 3.5 and $\pi_9(S^6) = 24$, image $p_* =$ kernel $\Delta = \{12\nu_6\} = 2$. Hence $\pi_9(R_7)$ contains a element $\varepsilon_9$ such that $p_* \varepsilon_9 = 12\nu_6$.

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1) Cf. [12], Corollary 5.14, for the definition of $J: \pi_r(R_n) \to \pi_{r+n}(S^n)$ and these results.
2) Cf. [2], (5.5).
3) Cf. [9], Corollary (3.6).

Therefore $\pi_{16}(S^7)$ contains a non-zero element $J(\varepsilon_9)$, and consequently $\pi_{16}(S^4)$ contains a non-zero element $\nu_4 \circ J(\varepsilon_9)$.

By the same manner as above, considering the sequence of the bundle $\{R_n, p, S^{n-1}, R_{n-1}, R_{n-1}\}$ and using 4.2, it follows immediately that

4.5. **Corollary.** *If the homomorphism* $i_* : \pi_{r-1}(R_{n-1}) \to \pi_{r-1}(R_n)$ *is isomorphic and the image of the suspension* $E^m : \pi_r(S^{n-1}) \to \pi_{r+m}(S^{n-1+m})$ *contains a cyclic subgroup of order* $p$, *where* $m = \min(n+1, r-2n+4)$, *then* $\pi_{r+n}(S^n)$ *contains a cyclic subgroup whose order is a multiple of* $p$.

As $i_* : \pi_{r-1}(R_7) \to \pi_{r-1}(R_8)$ is isomorphic, it follows immediately from this property that $\pi_{17}(S^8)$ and $\pi_{21}(S^8)$[1] is not zero.

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1) $E : \pi_{13}(S^7) \to \pi_{14}(S^8)$ is isomorphic, cf. [9], Appendix 2, viii).