INDECOMPOSABLE SEMIGROUPS

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In this note we prove, among other things, that if a topological semigroup with unit is a metric indecomposable continuum then it is a (topological) group.

A *clan* is a compact connected Hausdorff space together with a continuous associative multiplication with unit.

A subset C of a space is a *C-set* provided that if Q is a continuum and $Q \cap C \neq \square$ then $Q \subset C$ or $C \subset Q$. If X is a metric indecomposable continuum then any of its composants is a C-set, see [1] Chapter I. In this note a continuum is a compact connected Hausdorff space. The terminology of semigroups is that given in Clifford [2].

Theorem 1. *Let S be a clan, let C be a C-set and let I be a closed ideal of S. If $I \subset C$ and I meets $\overline{S \setminus C}$ then $I = C$.*

Proof. Suppose that there is a point p in $C \setminus I$. Then $IS \subset I \subset S \setminus p$, so there is an open set V including I and with $VS \subset S \setminus p$ because I and S are compact and $S \setminus p$ is open, see [3] and [4]. Since I meets $\overline{S \setminus C}$ and since V is open there is some $x \in V \setminus C$. Thus $x \in xS \subset VS \subset S \setminus p$. Clearly xS is a continuum and a closed right ideal. Now any right ideal meets every ideal so xS meets I. The continuum xS intersects C and we have neither $C \subset xS$ nor $xS \subset C$.

It is readily seen that if a C-set is closed and proper then it has no inner points. In this case we may omit the stipulation that I meets $\overline{S \setminus C}$.

Corollary. *If S is a metric indecomposable continuum, it is a topological group.*

Proof. Let C be a composant of S meeting the minimal closed ideal K (see [3] and [4]). Since S is indecomposable its composants are C-sets. If $C \subset K$ then $K = S$ and so the unit u of S is included in K and so S is a group (see [3] and [5]). Thus $K \subset C$. Since S is indecomposable, C has no inner point and $K = C$ by Theorem 1.

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But K is closed and C is dense so that $K = S$ and K is a group, as above.

As is well known, all solenoids are indecomposable continua. The corollary may be compared with the result (unpublished) that if a (classical) manifold is a clan it is a group. Certainly manifolds and indecomposable continua are antipodal points in the sphere of topology.

It is known (see [3], [4], and [5]) that if $H(u)$ denotes the maximal sub*group* of the clan S including its unit u, then $H(u)$ is a compact topological group.

Theorem 2. *Let S be a clan, and let C be a C-set without inner points. If the unit u of S is included in C then $C \subset H(u)$.*

Proof. Let K be the minimal closed ideal of S. If K meets C then either (i) $u \in C \subset K$ and S is a group and $S = H(u)$ or (ii) $K \subset C$ so that by Theorem 1 $K = C$ and again $u \in K$ so that $S = H(u)$. Thus we may assume that $K \cap C = \square$. Now if $x \in C$ then xS meets K (as in the proof of Theorem 1) and $x \in xS$ so that xS meets C. Since xS is a continuum we have $u \in C \subset xS$ and so $S = uS \subset xS^2 \subset xS$. It follows that x has an inverse and is thus included in $H(u)$, since $H(u)$ may be described as all elements of S with inverses, see [3] and [4].

Example 1. Coordinatize 3-space as the set of all (z, t) with z complex and t real. Let

$$C = \{(z, t) \mid \ |z| = 1 \ \text{ and } \ t = 0\},$$
$$W = \{(z, t) \mid \ z = e^{2\pi i s} \ \text{ and } \ t = e^{-s} \ \text{ and } \ s \geq 0\}.$$

If $S = C \cup W$ then, with coordinate-wise multiplication, S is a clan. It may be shown that, whatever multiplication be used, if S is a clan then C is its minimal closed ideal and its unit is the endpoint of S.

Example 2. Let

$$C = \{(x, y) \mid \ x = 0 \ \text{ and } \ |y| \geq 1\},$$
$$W = \{(x, y) \mid \ x = e^{-s} \ \text{ and } \ y = \cos s \ \text{ and } \ s \geq 0\}.$$

Let $S = C \cup W$. With the aid of Theorems 1 and 2 it may be shown that S *cannot* be a clan for any multiplication.

Example 3. Let S be the semigroup of matrices

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \ , \quad |x| + |y| \geq 1, \quad 0 \leq x, y.$$

We may regard S as a subset of the plane. Then S is a clan and its minimal closed ideal K is that part of the Y-axis included in S. Thus K is not a C-set.

Let us say that a *mob* is a Hausdorff semigroup. The following questions are, so far as I know, unsettled. If a compact connected mob has a *unique* left unit, is this also a right unit? If a clan is locally connected at no point, is it a group? An easy example shows that a clan can fail to be locally connected at every point save one without being a group. If a clan is a homogeneous space, is it a group?

Added in proof. It can be shown that no proper C-set of a continuum can contain an inner point. The hypotheses of the theorems can be relaxed in accordance with this remark.

References

[1] Moore, *Foundations of Point Set Theory*, New York, 1932.

[2] Clifford, Semigroups containing minimal ideals, *American Journal of Mathematics*, 70, (1948), pp. 521 - 526.

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