

NOTES ON BASIC RINGS

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Every ring R with minimum condition on left ideals can be completely described by means of a certain semi-primitive subring S and a number of positive integers $f_1, f_2, \dots, f_n$. By Brauer (see [6]), S is called the basic ring of R . In this paper, we shall define the basic rings for rings with milder condition and prove that the main properties of basic rings are also valid in our general case (Theorems 1 and 3 below).

1. Let R be an arbitrary ring. An element e of R is called an idempotent if $e^2 = e$. Two idempotents e and f are called *isomorphic* in R if there exist two elements a and b such that $ab = e$ and $ba = f$. We write then $e \cong f$. This is evidently a reflexive, symmetric, and transitive relation, by means of which the idempotents are classified into disjoint classes of isomorphic elements. A class containing an idempotent e is denoted by $C(e)$. We may assume in the above definition that $a \in eRf$ and $b \in fRe$ (see [1]).

Two idempotents e and f are isomorphic if and only if the left ideals Re and Rf are R -isomorphic. Further if $e \cong f$, then two subrings eRe and fRf are isomorphic under the mapping $x \rightarrow bxa$ ($x \in eRe$), where $e = ab$, $f = ba$ and $a \in eRf$, $b \in fRe$.

Lemma 1. *Let $e = e_1 + e_2 + \dots + e_t$ be a decomposition of an idempotent e into a sum of t orthogonal idempotents e_i . Similarly let $f = f_1 + f_2 + \dots + f_t$. If $e_i \cong f_i$ for every i , then $e \cong f$.*

Proof. Since $e_i = p_i q_i$ and $f_i = q_i p_i$ ($p_i \in e_i R f_i$, $q_i \in f_i R e_i$), we have $e = pq$ and $f = qp$, where $p = \sum p_i$ and $q = \sum q_i$.

In what follows we assume that R contains an identity 1. Let

$$(1) \quad 1 = \sum_{i=1}^n \sum_{\alpha=1}^{f(i)} e_{i,\alpha}$$

be a decomposition of the identity into a sum of orthogonal idempotents $e_{i,\alpha}$ such that all $e_{i,\alpha}$ with the same first subscript are isomorphic and no two $e_{i,\alpha}$'s with different subscripts are isomorphic. Since $e_{i,1} \cong e_{i,\alpha}$, there exist two elements $c_{i,1\alpha} \in e_{i,1} R e_{i,\alpha}$ and $c_{i,\alpha 1} \in e_{i,\alpha} R e_{i,1}$ such that

$$e_{i,1} = c_{i,1\alpha} c_{i,\alpha 1}, \quad e_{i,\alpha} = c_{i,\alpha 1} c_{i,1\alpha}.$$

We may set $c_{i,11} = e_{i,1}$. Further, if we put

$$(2) \quad c_{i,\alpha\beta} = c_{i,\alpha 1} c_{i,1\beta},$$

then $c_{i,\alpha\beta} \in e_{i,\alpha} R e_{i,\beta}$ and

$$e_{i,\alpha} = c_{i,\alpha\beta} c_{i,\beta\alpha}, \quad e_{i,\beta} = c_{i,\beta\alpha} c_{i,\alpha\beta}.$$

From (2) we have $c_{i,\alpha\alpha} = e_{i,\alpha}$. We see easily that

$$(3) \quad c_{i,\alpha\beta} c_{j,\kappa\lambda} = \delta_{ij} \delta_{\beta\kappa} c_{i,\alpha\lambda}.$$

Hence the $c_{i,\alpha\beta}$ are matrix units for a fixed i , and so $\{c_{i,\alpha\beta}\}$ is called a set of matrix units corresponding to the decomposition (1). It follows from $e_{i,\alpha} R e_{j,\beta} = c_{i,\alpha 1} R c_{j,1\beta}$ that

$$(4) \quad R = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} R c_{j,1\beta}.$$

If we set $e_{i,1} = e_i$ and $\sum e_i = e$, then eRe is a subring of R with an identity e . We have from (4)

$$(5) \quad R = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} (eRe) c_{j,1\beta},$$

so that every element a in R is expressed uniquely as

$$(6) \quad a = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} (c_{i,1\alpha} a c_{j,\beta 1}) c_{j,1\beta}.$$

Let R^* be a second ring with an identity 1^* and let

$$(7) \quad 1^* = \sum_{i=1}^n \sum_{\alpha=1}^{f(i)} e_{i,\alpha}^*$$

be a decomposition of the identity 1^* into a sum of orthogonal idempotents $e_{i,\alpha}^*$ such that $e_{i,\alpha}^* \cong e_{i,1}^*$ and $e_{i,\alpha}^* \not\cong e_{j,\beta}^*$ if $i \neq j$. We set $e_{i,1}^* = e_i^*$ and $\sum e_i^* = e^*$. Suppose that two rings eRe and $e^* R^* e^*$ are isomorphic under the mapping $x \rightarrow x^p$ ($x \in eRe$) such that $e_i^p = e_i^*$. Let $\{c_{i,\alpha\beta}^*\}$ be a set of matrix units corresponding to (7). It is easy to see that

$$(8) \quad \begin{aligned} a &= \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} (c_{i,1\alpha} a c_{j,\beta 1}) c_{j,1\beta} \\ &\rightarrow a^p = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1}^* (c_{i,1\alpha} a c_{j,\beta 1})^p c_{j,1\beta}^* \end{aligned}$$

gives the isomorphism between R and R^* . Moreover we have $c'_{i,\alpha\beta} = c_{i,\alpha\beta}^*$ and $a^\varphi = a^\theta$ for every a in eRe .

Now let

$$(9) \quad 1 = \sum_{i=1}^n \sum_{\alpha=1}^{f(i)} e'_{i,\alpha}$$

be a second decomposition of the identity of R into orthogonal idempotents $e'_{i,\alpha}$ such that $e'_{i,\alpha} \cong e_{i,\alpha}$ and let $\{c'_{i,\alpha\beta}\}$ be a set of matrix units corresponding to (9). We set $e'_{i,1} = e'_i$ and $\sum e'_i = e'$.

Lemma 2. *R contains a regular element s which satisfies*

$$e'Re' = s^{-1}(eRe)s.$$

Proof. By Lemma 1, $e_i = p_i q_i$, $e'_i = q_i p_i$ and $e = pq$, $e' = qp$, where $p = \sum p_i$ and $q = \sum q_i$. Since $e \cong e'$, eRe and $e'Re'$ are isomorphic under the mapping $x \mapsto qxp$. Hence we see that the mapping

$$\begin{aligned} a &= \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} (c_{i,1\alpha} a c_{j,\beta 1}) c_{j,1\beta} \\ &\rightarrow a^\varphi = \sum_{i,\alpha} \sum_{j,\beta} c'_{i,\alpha 1} (q c_{i,1\alpha} a c_{j,\beta 1} p) c'_{i,1\beta} \end{aligned}$$

gives an automorphism of R such that $c'_{i,\alpha\beta} = c_{i,\alpha\beta}^\varphi$ and $a^\varphi = qap$ for every a in eRe . We see that

$$a^\varphi = (\sum_{i,\alpha} c'_{i,\alpha 1} q c_{i,1\alpha}) a (\sum_{j,\beta} c_{j,\beta 1} p c'_{j,1\beta}).$$

If we put $s = \sum_{j,\beta} c_{j,\beta 1} p c'_{j,1\beta}$ and $t = \sum_{i,\alpha} c'_{i,\alpha 1} q c_{i,1\alpha}$, then

$$\begin{aligned} st &= \sum_{j,\beta} c_{j,\beta 1} p e'_j q c_{j,1\beta} \\ &= \sum_{j,\beta} c_{j,\beta 1} c_{j,1\beta} = \sum_{j,\beta} e_{j,\beta} = 1. \end{aligned}$$

Similarly $ts = 1$ and so $t = s^{-1}$. Hence $a^\varphi = s^{-1}as$. In particular, $qap = s^{-1}as$ for a in eRe and $c'_{i,\alpha\beta} = s^{-1}c_{i,\alpha\beta}s$.

Evidently two idempotents e and $t^{-1}et$ are isomorphic. The converse does not hold generally. But we have from Lemma 2

Lemma 3. *Let $1 = \sum_{i=1}^m f_i$ be a decomposition of the identity of R into a sum of orthogonal idempotents f_i . If $f_j \cong f_k$, then there exists a regular element t such that $f_k = t^{-1}f_j t$.*

2. In this section we assume that a ring R with an identity satisfies the following condition:

- (*) *R is decomposed into a direct sum of a finite number of indecomposable left ideals and this decomposition is unique up to R-isomorphism.*

For instance, if R is a ring with minimum condition on left ideals and with an identity, then R satisfies the condition (*).

If R satisfies the condition (*), the identity is decomposed into a sum of a finite number of orthogonal *primitive* idempotents:

$$(10) \quad 1 = \sum_{i=1}^n \sum_{\alpha=1}^{f(i)} e_{i,\alpha},$$

where $e_{i,\alpha} \cong e_{i,1}$ and $e_{i,\alpha} \not\cong e_{j,\beta}$ if $i \neq j$, and this decomposition is unique up to isomorphism. We put $e_{i,1} = e_i$.

Lemma 4. *The number of classes $C(e)$ of isomorphic idempotents is finite.*

Proof. Let g be an arbitrary idempotent. Then $1 = g + (1 - g)$, where g and $1 - g$ are orthogonal. Hence g is decomposed into a sum of orthogonal primitive idempotents: $g = \sum g_j$, and this decomposition is unique up to isomorphism. Moreover every g_j is isomorphic to one of e_i . We denote by s_i the number of g_j which are isomorphic to e_i . Then $0 \leq s_i \leq f(i)$. We say that an idempotent g is of type $(s_1, s_2, \dots, s_n)$, or simply of type (s_i) . Then we see that two isomorphic idempotents are of same type. The converse is also true. Hence a (1-1) correspondence between the classes $C(g)$ and the systems of positive integers $(s_1, s_2, \dots, s_n)$, $(0 \leq s_i \leq f(i))$ is established. Thus the number of classes $C(g)$ is equal to $\prod_i (1 + f(i))$.

Let $C(e)$ and $C(f)$ be of type (s_i) and (t_i) , and let $C(g)$ and $C(h)$ be of type $(\max(s_i, t_i))$ and $(\min(s_i, t_i))$. If we define the join and the meet of $C(e)$ and $C(f)$ by

$$C(e) \cup C(f) = C(g), \quad C(e) \cap C(f) = C(h),$$

then the classes of isomorphic idempotents form a lattice L and the structure of L is determined by a system of positive integers $(f(1), f(2), \dots, f(n))$.

Let e be an idempotent of type $(1, 1, \dots, 1)$. $R^0 = eRe$ is a subring of R with an identity e . R^0 is called the *basic ring* of R . The concept of basic ring first appeared in Nakayama [4] p. 617 (see [5] p. 335 too). The basic ring R^0 is uniquely determined up to isomorphism.

From (5) we have

$$(11) \quad R = \sum_{i, \alpha} \sum_{j, \beta} c_{i, \alpha 1} R^n c_{j, 1 \beta}.$$

By the results obtained in section 1, we have immediately

Theorem 1. *The structure of every ring R which satisfies the condition (*) is completely determined by its basic ring R^n and a system of positive integers $(f(1), f(2), \dots, f(n))$.*

Theorem 2. *Let R be a ring which satisfies the condition (*). Suppose that*

$$1 = \sum_{i=1}^n \sum_{\alpha=1}^{f(i)} e_{i, \alpha} = \sum_{i=1}^n \sum_{\alpha=1}^{f'(i)} e'_{i, \alpha}$$

are two decompositions of the identity into a sum of orthogonal primitive idempotents such that $e_{i, \alpha} \cong e_{i, 1}$, $e_{i, \alpha} \cong e_{j, \beta}$ and $e_{i, \alpha} \cong e'_{i, \alpha}$. If $\{c_{i, \alpha \beta}\}$ and $\{c'_{i, \alpha \beta}\}$ are two systems of matrix units corresponding to above decompositions of the identity, then there exists a regular element s such that $c'_{i, \alpha \beta} = s^{-1} c_{i, \alpha \beta} s$.

Corollary 1. *Let $\alpha \rightarrow \alpha^\pi$ be an automorphism of R and let $e_i^\pi \cong e_{\pi(i)}$, where $\pi(1), \pi(2), \dots, \pi(n)$ is a permutation of $1, 2, \dots, n$. There exists a regular element s such that $c_{i, \alpha \beta}^\pi = s^{-1} c_{\pi(i), \alpha \beta} s$.*

Corollary 2. *If two idempotents f and g are isomorphic, then there exists a regular element s such that $g = s^{-1}fs$.*

Theorem 3. *The basic ring R^n of R is uniquely determined up to inner automorphism.*

3. We consider a ring with minimum condition on left ideals and with an identity.

Lemma 5. *Let $\mathfrak{z}$ be a two-sided ideal of R . If $e \cong f$ and $e \in \mathfrak{z}$, then $f \in \mathfrak{z}$.*

Proof. Since $e = ab \in \mathfrak{z}$, $f = bea \in \mathfrak{z}$.

If $\mathfrak{z}$ is a two-sided ideal of R , then $\mathfrak{z}'' = R^n \cap \mathfrak{z} = e \mathfrak{z} e$ is a two-sided ideal of R^n and $R \mathfrak{z}'' R = \mathfrak{z}$, while if $\mathfrak{z}''$ is a two-sided ideal of R^n , then $\mathfrak{z} = R \mathfrak{z}'' R$ is a two-sided ideal of R and $R^n \cap \mathfrak{z} = \mathfrak{z}''$, as can easily be seen from (6). Hence $\mathfrak{z} \leftrightarrow \mathfrak{z}'' = R^n \cap \mathfrak{z}$ gives a (1-1) correspondence between two-sided ideals of R and R^n .

Lemma 6. *Let $\mathfrak{z}$ be a two-sided ideal of R . Then $(R/\mathfrak{z})'' \cong R^n/\mathfrak{z}''$, where $\mathfrak{z}'' = R^n \cap \mathfrak{z}$.*

Let N be the radical of R . We denote by $\bar{R}$ the residue class ring of R modulo N . Then

$$(12) \quad \bar{R} = R/N = \sum_i \sum_{\alpha} \bar{R}\bar{e}_{i,\alpha},$$

where every $\bar{R}\bar{e}_{i,\alpha}$ is a simple left ideal of $\bar{R}$. The radical of R is $N^0 = R^0 \cap N = eNe$ and

$$(13) \quad (R/N)^0 \cong R^0/N^0.$$

The basic ring R^0 is a semi-primitive ring, since the residue class ring R^0/N^0 is a direct sum of division rings.

It is well known that two idempotents f and g are isomorphic in R if and only if $\bar{f}$ and $\bar{g}$ are isomorphic in $\bar{R}$.

We say that two indecomposable left ideals Rf and Rg belong to the same block if there is a sequence of indecomposable left ideals $Rf = Rf_1, Rf_2, \dots, Rf_i = Rg$ such that $f_i^2 = f_i$ and each Rf_i has a composition factor R -isomorphic to one of the composition factors of Rf_{i+1} . When Rf and Rg belong to the same block, we say also that f and g belong to the same block and write $f \sim g$. This is a reflexive, symmetric, and transitive relation, by means of which the primitive idempotents are classified into disjoint classes. Evidently if two primitive idempotents are isomorphic, then they belong to the same block.

Let $R = R_1 + R_2 + \dots + R_s$ be the direct decomposition of R into indecomposable two-sided ideals. It was shown that two primitive idempotents f and g belong to the same block if and only if they belong to the same two-sided component ([3], see [2] p. 74 too).

Theorem 4. *Let R^* be the residue class ring of R modulo N^2 . Two primitive idempotents e and f belong to the same block if and only if e^* and f^* in R^* belong to the same block.*

Proof. Since the "if" part is obvious, we shall prove the "only if" part. Let E_i be the identity of R_i . The E_i are the primitive idempotents of the center Z of R . We have

$$R^* = R/N^2 = R^*E_1^* + R^*E_2^* + \dots + R^*E_s^*,$$

where $R^*E_i^*$ are the two-sided ideals of R^* . If we can show that every $R^*E_i^*$ is indecomposable, then our proof is complete. Suppose that one of E_i , say, E_1 is decomposed into a sum of orthogonal idem-

potents f_1 and f_2 such that f_1^* and f_2^* lie in the center of R^* . Then we have $f_1^* R^* f_2^* = f_2^* R^* f_1^* = 0$, so that $f_1 R f_2, f_2 R f_1 \subset N^2$. Hence

$$f_1 R f_2 = f_1 N^2 f_2, \quad f_2 R f_1 = f_2 N^2 f_1.$$

Now we have for $n_1, n_2 \in N$

$$\begin{aligned} f_1 n_1 n_2 f_2 &= \sum_i f_1 n_1 E_i n_2 f_2 \\ &= f_1 n_1 f_1 n_2 f_2 + f_1 n_1 f_2 n_2 f_2 \in f_1 N^3 f_2. \end{aligned}$$

This implies

$$f_1 R f_2 = f_1 N f_2 = \dots = f_1 N^p f_2 = 0,$$

since $N^p = 0$. Similarly $f_2 R f_1 = 0$. Consequently $R f_1 = f_1 R$ and $R f_2 = f_2 R$ are the two-sided ideals and $R E_1 = R f_1 + R f_2$. This contradicts to our assumption.

Theorem 5. *Let $R = R_1 + R_2 + \dots + R_s$ be the direct decomposition of R into indecomposable two-sided ideals. Then*

$$R/N^2 = R_1/(R_1 \cap N^2) + \dots + R_s/(R_s \cap N^2)$$

is the direct decomposition of R/N^2 into indecomposable two-sided ideals.

Let V be an R -space. We assume that V satisfies 1. $v = v$ for $v \in V$. Then we have $V = RV = \sum_{i, \alpha} c_{i, \alpha} V$. If V is an R -space, then eV is an R^0 -space and $ReV = V$, while if V_0 is an R^0 -space, then $V = \sum_{i, \alpha} c_{i, \alpha} V_0$ is an R -space under the assumption $a(bv) = (ab)v$ for $a, b \in R$ and $v \in V$. Further we see that $eV = V_0$. Hence $V \rightleftharpoons eV$ gives a (1-1) correspondence between the R -spaces and the R^0 -spaces.

Lemma 7. *Let U be an R -subspace of an R -space V , and let V/U be simple and $\cong \bar{R}\bar{e}_i$. Then $eV/eU \cong \bar{R}^0\bar{e}_i$.*

Proof. It follows from $V = ReV$ and $U = ReU$ that eU is a proper subspace of eV . Since $e_j V = e_j U$ for every $j \neq i$, we have $eV/eU \cong \bar{R}^0\bar{e}_i$.

Theorem 6. *Let V be an R -space satisfying the minimum condition and let*

$$V = V_1 \supset V_2 \supset \dots \supset V_m \supset (0)$$

be a composition series for V . Suppose that $V_k/V_{k+1} \cong \bar{R}\bar{e}_i$. Then

$$eV = eV_1 \supset eV_2 \supset \cdots \supset eV_m \supset (0)$$

is a composition series of the R^0 -space eV , and $eV_k/eV_{k+1} \cong \bar{R}^0 \bar{e}_i$.

We shall study in a forthcoming paper the connection between a ring and its basic ring. This problem was studied in [6] in the case when R is an algebra over an algebraically closed field.

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