

# COMPACTIFICATION OF TOPOLOGICAL SPACES

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One of the writers<sup>1)</sup> has given the compactification of a  $T$ -space<sup>2)</sup>  $R$  as follows:

Let us denote by  $R^*$  the totality of all ultrafilters in  $R$ . Then it is proved that the family

$$\{U^{*3)} \mid U \text{ is an open set in } R\}^{4)}$$

can be taken as a basis of open sets in  $R^*$  and  $R^*$  becomes a compact<sup>5)</sup>  $T$ -space containing the set:

$$\tilde{R} = \{\mathfrak{F}_x^{6)} \mid x \in R; \mathfrak{F}_x \text{ is the ultrafilter containing } x\}$$

as a dense subset, moreover,  $\tilde{R}$  is homeomorphic with  $R$  by the mapping  $\varphi$  defined by

$$\varphi(x) = \mathfrak{F}_x.$$

It is the purpose of this note to make clear some relations among our compactification and Wallman's of a  $T_0$ -space and Čech's of a completely regular space.

**§1. An extension theorem of continuous functions.** First of all we shall prove the

**Theorem 1.** *Let  $f$  be a real valued bounded continuous function defined on  $R$ . Then there exists a real valued bounded continuous function  $f^*$  defined on  $R^*$  such that*

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x) \text{ and } f^*(\mathfrak{F}_x) = f(x).$$

1) T. Inagaki: Contribution à la topologie I, Math. Journ. of Okayama Univ., Vol. I (1952), pp. 158 - 166.

2) A  $T$ -space is a topological space which satisfies the conditions:

1)  $\bar{\phi} = \phi$ ; 2)  $\bar{M} \supset M$ ; 3)  $\overline{M \cup N} = \bar{M} \cup \bar{N}$ ; 4)  $\bar{\bar{M}} = \bar{M}$ ,

where  $M$  and  $N$  are subsets of  $R$  and  $\phi$  is the null set as usual.

3)  $M^*$  is the totality of all ultrafilters in  $R$ , which contains  $M$ .

4) The notation  $\{A \mid B\}$  means the totality of all sets  $A$  satisfying the condition  $B$ .

5) A  $T$ -space  $R$  is called compact if each filter in  $R$  has at least one cluster point.

6) In future, we shall denote by  $\mathfrak{F}$  a ultrafilter in  $R$  and by  $\mathfrak{F}_x$  the ultrafilter containing  $x$ .

Since  $\varphi$  is the homeomorphism of  $R$  on  $\tilde{R}$ , if we regard  $\tilde{R}$  as the space  $R$ , then the theorem says that  $f$  can be extended on  $R^*$ .

*Proof.* Let  $\mathfrak{F} \in R^*$  and let

$$t = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) \quad \text{and} \quad t' = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x).$$

Then it is clear that  $t \geq t'$ , but we can show that  $t = t'$ . In fact, by definition of  $t$ , for every positive number  $\varepsilon$  there exists a set  $A_1 \in \mathfrak{F}$  such that  $t \leq \sup_{x \in A_1} f(x) < t + \varepsilon$ , and hence for some point  $x_1 \in A_1$ ,  $t - \varepsilon < f(x_1) < t + \varepsilon$ . Therefore, if we denote by  $B$  the set  $\{x \mid t - \varepsilon < f(x) < t + \varepsilon\}$ , then  $B \neq \emptyset$ . We now show that  $B \in \mathfrak{F}$ . Since  $\mathfrak{F}$  is an ultrafilter, if  $B \notin \mathfrak{F}$ , then  $CB^1 \in \mathfrak{F}$  and  $A_1 \cap CB \in \mathfrak{F}$ . Let  $x$  be a point of  $A_1 \cap CB$ , then  $f(x) < t + \varepsilon$  by definition of  $A_1$  and  $f(x) \leq t - \varepsilon$  or  $t + \varepsilon \leq f(x)$  by definition of  $B$ . Therefore we can say that if  $x \in A_1 \cap CB$  then  $f(x) \leq t - \varepsilon$ , and hence  $\sup_{x \in A_1 \cap CB} f(x) \leq t - \varepsilon$ . This contradicts with the definition of  $t$ . Thus we have  $B \in \mathfrak{F}$  and  $\inf_{x \in B} f(x) \geq t - \varepsilon$ . Therefore, by definition of  $t'$ , we have  $t - \varepsilon \leq t'$ , and this shows that  $t \leq t'$ ; hence  $t = t'$ .

From what we have just proved above, we can define a function  $f^*$  defined on  $R^*$  by the equality

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x).$$

It follows evidently that  $f^*$  is bounded and  $f^*(\mathfrak{F}_x) = f(x)$ .

To prove that  $f^*$  is continuous, let  $t = f^*(\mathfrak{F})$ . In proving that  $t \leq t'$  above, we have shown that for every positive number  $\varepsilon$ , the set  $G = \{x \mid t - \frac{\varepsilon}{2} < f(x) < t + \frac{\varepsilon}{2}\}$  belongs to  $\mathfrak{F}$ . Since  $f$  is continuous,  $G$  is open in  $R$  and so  $G^*$  is open in  $R^*$ . Now, it follows from the definition of  $f^*$  that  $f^*(G) \subset [t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}] \subset (t - \varepsilon, t + \varepsilon)$ . This shows that  $f^*$  is continuous, Q.E.D.

*Remark.* As it is easily seen, if  $f$  is continuous, then for any set  $A$ ,  $\sup_{x \in A} f(x) = \sup_{x \in \bar{A}} f(x)$  and  $\inf_{x \in A} f(x) = \inf_{x \in \bar{A}} f(x)$ ; therefore the function  $f^*$  can be defined by the equality

$$(1) \quad f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in \bar{A}} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in \bar{A}} f(x).$$

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1)  $CB$  denote the complement of  $B$ , that is,  $CB = R - B$ .

**§2. Compactification of a  $T_0$ -space<sup>1)</sup>.** In this place, we suppose that  $R$  is a  $T_0$ -space. For a point  $\mathfrak{F} \in R^*$  we define  $\overline{\mathfrak{F}}$  by

$$\overline{\mathfrak{F}} = \{F \mid F \in \mathfrak{F} \text{ and } F \text{ is closed in } R\}.$$

For two points  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  of  $R^*$ , we write

$$\mathfrak{F}_1 \sim \mathfrak{F}_2, \text{ if } \overline{\mathfrak{F}_1} = \overline{\mathfrak{F}_2}.$$

Obviously the relation  $\sim$  is an equivalence relation, hence the relation  $\sim$  divides  $R^*$  into disjoint classes of equivalent points.

We introduce the notations:

$R^\circ$  = the totality of all classes of equivalent points;

$[\mathfrak{F}]$  = the class of equivalent points, which contains  $\mathfrak{F}$ .

It is important to remark that if  $x \neq y$  then  $[\mathfrak{F}_x] \neq [\mathfrak{F}_y]$ . For, since  $x \neq y$  and  $R$  is a  $T_0$ -space, at least one of  $x \in \overline{y}$  and  $y \in \overline{x}$  holds, from which  $\overline{x} \neq \overline{y}$ ; hence it is not hard to see that  $\overline{\mathfrak{F}_x} \neq \overline{\mathfrak{F}_y}$ .

We define the mapping  $\phi_1$  of the set  $R^*$  on the set  $R^\circ$  such that

$$[\mathfrak{F}] = \phi_1(\mathfrak{F}).$$

Then it is evident that  $\phi_1$  is one-to-one mapping between  $\widetilde{R}$  and  $\widetilde{R}^\circ$ , by setting

$$\widetilde{R}^\circ = \phi_1(\widetilde{R}).$$

Now it is not difficult to see that the family

$$\Gamma = \{F^\circ \mid \phi_1^{-1}(F^\circ) = F^*, \text{ where } F \text{ is closed in } R\}$$

can be taken as a basis of closed sets in  $R^\circ$ , and, moreover,  $R^\circ$  becomes a  $T$ -space and  $\phi_1$  is continuous.

From this definition, we can prove the

**Theorem 2.**  $R^\circ$  is a compact  $T_0$ -space and contains a dense subset  $\widetilde{R}^\circ$  which is homeomorphic with  $R$ .

*Proof.*  $R^\circ$  is compact, because  $R^*$  is compact, and  $\phi_1$  is continuous in the topology introduced in  $R^\circ$ .

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1) A  $T_0$ -space is a  $T$ -space such that, for any two different points  $x$  and  $y$ , at least one of  $x \in \overline{y}$  or  $y \in \overline{x}$  holds.

To show that  $R^\circ$  is a  $T_0$ -space, we shall prove first that

$$\phi_1^{-1}(\phi_1(U^*)) = U^* \quad \text{for any open set } U \text{ in } R.$$

In fact, if  $\mathfrak{F} \in \phi_1^{-1}(\phi_1(U^*))$ , there exists a point  $\mathfrak{F}_1 \in U^*$  such that  $\mathfrak{F} = \mathfrak{F}_1$ . Hence  $U \in \mathfrak{F}_1$  and  $CU \in \mathfrak{F}_1$ . Since  $CU$  is closed, we have  $CU \in \mathfrak{F}$ ; and so  $U \in \mathfrak{F}$ . This shows that  $\phi_1^{-1}(\phi_1(U^*)) \subset U^*$ . Since it is evident that  $\phi_1^{-1}(\phi_1(U^*)) \supset U^*$ , we have  $\phi_1^{-1}(\phi_1(U^*)) = U^*$ .

Under this remark, we shall show that  $R^\circ$  is a  $T_0$ -space. Let  $[\mathfrak{F}_1]$  and  $[\mathfrak{F}_2]$  be two different points of  $R^\circ$ , then  $\mathfrak{F}_1 \neq \mathfrak{F}_2$ . Hence, at least one of  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$ , say  $\mathfrak{F}_1$ , contains a closed set  $F$  such that  $F \in \mathfrak{F}_2$ , and so  $CF \in \mathfrak{F}_1$  and  $CF \in \mathfrak{F}_2$ . Since  $CF$  is open, we get  $\phi_1^{-1}(\phi_1((CF)^*)) = (CF)^*$ , and clearly  $[\mathfrak{F}_1] \in \phi_1((CF)^*)$  and  $[\mathfrak{F}_2] \in \phi_1((CF)^*)$ . Hence it follows that  $R^\circ$  is a  $T_0$ -space.

In order to show that  $R$  and  $\tilde{R}^\circ$  are homeomorphic, as it is readily seen, it is sufficient to prove that the mapping  $\phi_1\varphi$  sends an open set in  $R$  to an open set in  $\tilde{R}^\circ$ . Now let  $G$  be an open set in  $R$ , then

$$\phi_1\varphi(G) = \phi_1(G^* \cap \tilde{R}) = \phi_1(\bigcup_{x \in G} \mathfrak{F}_x) = \bigcup_{x \in G} \phi_1(\mathfrak{F}_x) = \tilde{R}^\circ \cap \phi_1(G^*),$$

from which  $\phi_1\varphi(G)$  is open in  $\tilde{R}^\circ$ , because  $\phi_1(G^*)$  is open in  $R^\circ$ , Q.E.D.

**Theorem 3.** *Let  $f$  be a real valued bounded continuous function defined on  $R$ . Then there exists a real valued bounded continuous function  $f^\circ$  defined on  $R^\circ$  such that*

$$f^\circ([\mathfrak{F}]) = f^*(\mathfrak{F}).$$

*Proof.* Since  $f$  is continuous, the function  $f^*$  defined by the equation (1) in §1 takes the same value at each point which belongs to an equivalence class in  $R^*$ . Thus we can define a function  $f^\circ$  such that

$$f^\circ([\mathfrak{F}]) = f^*(\mathfrak{F}).$$

To prove that  $f^\circ$  is continuous, let  $[\mathfrak{F}]$  be a point of  $R^\circ$  and  $t = f^\circ([\mathfrak{F}]) = f^*(\mathfrak{F})$ . Since  $f^*$  is continuous, for every neighborhood  $U(t)$  of  $t$ , there exists an open set  $G$  in  $R$  such that  $G \in \mathfrak{F}$  and  $U(t) \supset f^*(G^*)$ . On the other hand, as we proved in the proof of Theorem 2,  $G \in \mathfrak{F}$  implies  $[\mathfrak{F}] \subset G^*$  and  $\phi_1(G^*)$  is open in  $R^\circ$ . Hence

$U(t) \supset f^*(G^*) = f^\circ(\phi_1(G^*))$ , this shows that  $f^\circ$  is continuous, Q.E.D.

Let us denote by  $\alpha(R)$  the totality of all dual prime ideals in the lattice  $\mathfrak{L}$  composed of all closed sets in  $R$ , then we have the

**Lemma 1.** *There is an one-to-one mapping of the set  $R^\circ$  on  $\alpha(R)$ .*

*Proof.* To a point  $[\mathfrak{F}] \in R^\circ$  we correspond the set  $\overline{\mathfrak{F}}$ , and we write

$$\overline{\mathfrak{F}} = \psi([\mathfrak{F}]).$$

First of all we prove that  $\overline{\mathfrak{F}}$  is a dual prime ideal. It is evident that  $\overline{\mathfrak{F}}$  is a dual ideal. In order to show that  $\overline{\mathfrak{F}}$  is prime, let  $F_1$  and  $F_2$  be closed sets in  $R$  such that  $F_1 \cup F_2 \in \overline{\mathfrak{F}}$ . If we suppose that  $F_1 \notin \overline{\mathfrak{F}}$ , then, since  $F_1$  is closed and  $\mathfrak{F}$  is an ultrafilter in  $R$ , we have  $F_1 \in \mathfrak{F}$  and so  $CF_1 \in \mathfrak{F}$ . Hence it follows that  $CF_1 \cap (F_1 \cup F_2) \in \mathfrak{F}$ , from which  $F_2 \in \mathfrak{F}$  and  $F_2 \in \overline{\mathfrak{F}}$ , since  $CF_1 \cap (F_1 \cup F_2) \subset F_2$  and  $F_2$  is closed. This shows that  $\overline{\mathfrak{F}}$  is a dual prime ideal, and hence  $\overline{\mathfrak{F}} \in \alpha(R)$ . Moreover, it is evident that if  $[\mathfrak{F}_1] \neq [\mathfrak{F}_2]$ , then  $\psi([\mathfrak{F}_1]) \neq \psi([\mathfrak{F}_2])$ .

We shall now prove that  $\psi(R^\circ) = \alpha(R)$ . In fact, let  $\mathfrak{M}$  be a dual prime ideal in  $\mathfrak{L}$  and let

$$\mathfrak{N} = \{G \mid G \text{ is open in } R \text{ and } CG \in \mathfrak{M}\}.$$

Since  $R \in \mathfrak{M}$ , we have  $\phi \notin \mathfrak{N}$ . Next, in order to show that  $\mathfrak{N}$  has the finite intersection property, take two sets  $G_1$  and  $G_2$  of  $\mathfrak{N}$ . Then  $CG_1 \in \mathfrak{M}$ ,  $CG_2 \in \mathfrak{M}$  and  $\mathfrak{M} \ni CG_1 \cup CG_2 = C(G_1 \cap G_2)$ , since  $\mathfrak{M}$  is prime. Therefore,  $G_1 \cap G_2 \in \mathfrak{N}$  and  $G_1 \cap G_2 \neq \phi$ , from which we say that  $\mathfrak{N}$  has the finite intersection property. Now let  $F \in \mathfrak{M}$  and  $G \in \mathfrak{N}$ . If we suppose that  $F \cap G = \phi$ , then  $F \subset CG$  and so  $CG \in \mathfrak{M}$ . This contradicts with  $CG \notin \mathfrak{M}$ , and hence  $F \cap G \neq \phi$ . From what we have proved above, we can say that the totality of all sets  $F \cap G$ , where  $F \in \mathfrak{M}$  and  $G \in \mathfrak{N}$ , forms a basis of a filter. Hence there exists an ultrafilter  $\mathfrak{F}$  which contains the above basis:  $\mathfrak{M} \cup \mathfrak{N} \subset \mathfrak{F}$ . For this ultrafilter  $\mathfrak{F}$ , we can show that  $\overline{\mathfrak{F}} = \mathfrak{M}$ . In fact, if  $F \in \overline{\mathfrak{F}}$ , then  $F \in \mathfrak{F}$  and  $CF \in \mathfrak{F}$ , from which  $CF \in \mathfrak{M}$ . Hence  $F \in \mathfrak{M}$  by definition of  $\mathfrak{M}$ , and so  $\overline{\mathfrak{F}} \subset \mathfrak{M}$ .

From what we have just proved, it follows that  $\psi(R^\circ) = \alpha(R)$  and  $\psi$  is an one-to-one mapping between  $R^\circ$  and  $\alpha(R)$ , Q.E.D.

We note here that a set  $F^\circ \subset R^\circ$  belongs to the closed basis  $\Gamma$  of  $R^\circ$ , if and only if there exists a closed set  $F$  in  $R$  such that  $\psi(F^\circ) = \alpha(F)$  by setting

$$\alpha(F) = \{\mathfrak{M} \mid \mathfrak{M} \text{ is a dual prime ideal in } \mathfrak{L} \text{ and } F \in \mathfrak{M}\}.$$

In fact, if  $F^\circ$  belongs to  $\Gamma$ , there exists a closed set  $F$  in  $R$  such that  $\phi_1^{-1}(F^\circ) = F^*$ . Hence  $\psi(F^\circ) = \psi(\phi_1(F^*)) = \psi(\{[\mathfrak{F}] \mid F \in \mathfrak{F}\}) = \alpha(F)$ .

Conversely, let  $F$  be a closed set in  $R$  and  $\psi(F^\circ) = \alpha(F)$ . Then, since  $\psi$  is one-to-one, we have  $\phi_1^{-1}(F^\circ) = \phi_1^{-1}(\psi^{-1}(\psi(F^\circ))) = \phi_1^{-1}(\psi^{-1}(\alpha(F))) = \phi_1^{-1}(\{[\mathfrak{F}] \mid F \in \mathfrak{F}\}) = F^*$ , and hence  $F^\circ$  is contained in the closed basis  $\Gamma$  of  $R^\circ$ .

Thus we have the

**Lemma 2.** *If we introduce the topology in  $\alpha(R)$  such that the family  $\{\alpha(F) \mid F \text{ is closed in } R\}$  is a closed basis of  $\alpha(R)$ , then the mapping  $\psi([\mathfrak{F}]) = \mathfrak{F}$  is a homeomorphism of the space  $R^\circ$  on the space  $\alpha(R)$ .*

**§3. Compactification of a  $T_1$ -space.** In this section, we suppose that  $R$  is a  $T_1$ -space. Now let

$$\begin{aligned}\beta^\circ(R) &= \{[\mathfrak{F}] \mid \mathfrak{F} \text{ is a maximal dual ideal in } \mathfrak{L}\}, \\ \beta(R) &= \psi(\beta^\circ(R)).\end{aligned}$$

Then, it is evident that  $\beta(R)$  is a subset of  $\alpha(R)$  and consists of all maximal dual prime ideals in  $\mathfrak{L}$ . Moreover, since  $\beta^\circ(R)$  and  $\beta(R)$  are regarded as the subspaces of  $R^\circ$  and  $\alpha(R)$  respectively,  $\beta^\circ(R)$  and  $\beta(R)$  are homeomorphic with each other.

Since  $R$  is a  $T_1$ -space, it is important to remark that  $[\mathfrak{F}_*]$  is  $\mathfrak{F}_*$  itself and hence  $\tilde{R} = \tilde{R}^\circ \subset \beta^\circ(R)$ .

Under these remarks, we have the well known

**Wallman's Theorem.** *The space  $\beta(R)$  is a compact  $T_1$ -space and contains a dense subset  $\tilde{R}^\circ$  which is homeomorphic with  $R$ .*

But we give a proof of this theorem for the purpose to make clear the relation among the spaces considered in this note.

*Proof.* Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be any two distinct points of  $\beta(R)$ . Then, since they are maximal ideals in  $\mathfrak{L}$ , any one of them, say  $\mathfrak{M}_1$ , contains a closed set  $F$  in  $R$  such that  $F \notin \mathfrak{M}_2$ . However,  $\beta(R) - \alpha(F)$  is open in  $\beta(R)$  and contains  $\mathfrak{M}_2$  and not  $\mathfrak{M}_1$ , hence  $\beta(R)$  is a  $T_1$ -space.

To show that  $\beta(R)$  is compact, we take an ultrafilter  $F$  in  $\beta(R)$ . Obviously, since  $F$  is a filter in  $\alpha(R)$  which is compact, there is a cluster point  $\mathfrak{F}$  of  $F$ . As we know, there is an ultrafilter  $\mathfrak{F}_1$  in  $R$  such that  $\mathfrak{F}_1 \in \beta^\circ(R)$  and  $\mathfrak{F} \subset \mathfrak{F}_1$ . Since  $\mathfrak{F} \subset \mathfrak{F}_1$ , for any closed set  $F$  in  $R$ ,  $\alpha(R) - \alpha(F) \ni \mathfrak{F}_1$  implies  $\alpha(R) - \alpha(F) \ni \mathfrak{F}$ , and hence we can say

that in the space  $\alpha(R)$  each neighborhood of  $\mathfrak{F}_1$  is also a neighborhood of  $\mathfrak{F}$ . Hence  $\mathfrak{F}_1$  is a cluster point of  $F$ , and therefore  $\beta(R)$  is compact.

It is almost evident that the subset  $\tilde{R}^c = \tilde{R}$  of  $\beta(R)$  is dense and homeomorphic with  $R$ , Q.E.D.

By using Theorem 3, we can prove the

**Theorem 4.** *A real valued bounded continuous function  $f$  defined on  $R$  is extendable to a real valued bounded continuous function  $f_\beta$  defined on  $\beta(R)$  such that*

$$f_\beta(\tilde{\mathfrak{F}}) = f^*(\tilde{\mathfrak{F}}).$$

Finally we give the

**Theorem 5.** *In order that  $\beta(R)$  be normal, it is necessary and sufficient that  $R$  be normal.*

*Proof.* Suppose that  $\beta(R)$  is normal, and let  $F_1$  and  $F_2$  be two disjoint closed sets in  $R$ . Then, the sets  $F_\beta^1 = \beta(R) \cap \alpha(F_1)$  and  $F_\beta^2 = \beta(R) \cap \alpha(F_2)$  are disjoint closed sets in  $\beta(R)$ . Hence there exists a continuous function  $f_\beta$  defined on  $\beta(R)$  such that  $f_\beta = 0$  on  $F_\beta^1$ ,  $f_\beta = 1$  on  $F_\beta^2$  and  $0 \leq f_\beta \leq 1$  on  $\beta(R)$ . If we define a function  $f$  by the equality  $f(x) = f_\beta(\tilde{\mathfrak{F}}_x)$ , it is clear that  $f$  is continuous and  $0 \leq f \leq 1$  on  $R$ . Moreover, if  $x \in F_1$ , then  $\tilde{\mathfrak{F}}_x \in \beta(R) \cap \alpha(F_1)$  and hence  $f(x) = f_\beta(\tilde{\mathfrak{F}}_x) = 0$ . Similarly, if  $x \in F_2$ , then  $f(x) = 1$ . This shows that  $R$  is normal.

Conversely, let  $R$  be normal and let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be two distinct points in  $\beta(R)$ . Then, as  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are maximal dual ideals in  $\mathfrak{L}$ , there exist disjoint closed sets  $F_1$  and  $F_2$  such that  $F_1 \in \mathfrak{M}_1$  and  $F_2 \in \mathfrak{M}_2$ . Since  $R$  is normal, there exists a continuous function  $f$  such that  $f(x) = 0$  on  $F_1$ ,  $f(x) = 1$  on  $F_2$  and  $0 \leq f \leq 1$  on  $R$ . Let  $f_\beta$  be the function extended from  $f$  by Theorems 1 and 4, then it is clear that  $f_\beta(\mathfrak{M}_1) = 0$  since  $F_1 \in \mathfrak{M}_1$ , and similarly  $f_\beta(\mathfrak{M}_2) = 1$ . This shows that  $\beta(R)$  is a Hausdorff space, and hence, as  $\beta(R)$  is compact,  $\beta(R)$  is normal, Q.E.D.

**§ 4. Compactification of a completely regular space.** In this section, we suppose first that  $R$  is a complete Hausdorff space.<sup>1)</sup> By considering the remark in §1, it is easily seen that, for two points

1) We mean by a complete Hausdorff space the Hausdorff space such that, for any two distinct points  $x$  and  $y$ , there exists a real valued bounded continuous function taking different values at  $x$  and  $y$ .

$\mathfrak{F}_1$  and  $\mathfrak{F}_2$ , the following propositions are equivalent:

- ( $\alpha$ ). *There is no real valued continuous function  $f^*$  defined on  $R^*$  such that  $f^*(\mathfrak{F}_1) = 0$ ,  $f^*(\mathfrak{F}_2) = 1$  and  $0 \leq f^* \leq 1$  on  $R^*$ .*  
 ( $\beta$ ). *Any real valued bounded continuous function  $f^*$  defined on  $R^*$  takes the same value at  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$ .*

If two points  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  satisfies the proposition, we write  $\mathfrak{F}_1 \approx \mathfrak{F}_2$ . Evidently the relation  $\approx$  is an equivalence relation, hence the relation divides  $R^*$  into disjoint classes of equivalent points.

We introduce the notations:

- $r(R)$  = the totality of all classes of equivalent points;  
 $\{\mathfrak{F}\}$  = the class which contains  $\mathfrak{F}$ .

Moreover, we define the mapping  $\phi_2$  of  $R^*$  on  $r(R)$  such that

$$\phi_2(\mathfrak{F}) = \{\mathfrak{F}\}.$$

We shall give the

**Lemma 3.** *If  $\mathfrak{F}_1 \subset \mathfrak{F}_2$ , then  $f^*(\mathfrak{F}_1) = f^*(\mathfrak{F}_2)$ , for every real valued bounded continuous function  $f^*$  on  $R^*$ .*

*Proof.* Let  $f(x) = f^*(\mathfrak{F}_x)$ , then  $f$  is a real valued bounded continuous function and equality (1) in §1 holds. On the other hand, since  $\mathfrak{F}_1 \subset \mathfrak{F}_2$ , it follows that  $\inf_{A \in \mathfrak{F}_1} \sup_{x \in A} f(x) \geq \inf_{A \in \mathfrak{F}_2} \sup_{x \in A} f(x)$  and  $\sup_{A \in \mathfrak{F}_1} \inf_{x \in A} f(x) \leq \sup_{A \in \mathfrak{F}_2} \inf_{x \in A} f(x)$ . Therefore, it is clear that  $f^*(\mathfrak{F}_1) = f^*(\mathfrak{F}_2)$ , Q.E.D.

From the lemma, it is not difficult to see that:

- (a).  $[\mathfrak{F}] \subset \{\mathfrak{F}\}$ ;  
 (b).  $\{\mathfrak{F}\}$  contains a point  $\mathfrak{F}_1$  such that  $[\mathfrak{F}_1] \in \beta^\circ(R)$ .

Since  $R$  is a complete Hausdorff space, for two distinct points  $x$  and  $y$ , there exists a continuous function  $f$  such that  $f(x) = 0$ ,  $f(y) = 1$  and  $0 \leq f \leq 1$  on  $R$ . Then, by the equality (1) in §1,  $f^*(\mathfrak{F}_x) = 0$ ,  $f^*(\mathfrak{F}_y) = 1$  and  $0 \leq f^* \leq 1$ . Thus we have

- (c). If  $x \neq y$ , then  $\phi_2(\mathfrak{F}_x) \neq \phi_2(\mathfrak{F}_y)$ .

Hence, if we put

$$\tilde{r}(R) = \phi_2(\tilde{R}),$$

then  $\phi_2$  gives an one-to-one correspondence between  $\tilde{R}$  and  $\tilde{r}(R)$ .

Moreover, if we take the set:

$$\{F_\gamma \mid \phi_2^{-1}(F_\gamma) \text{ is closed in } R^*\},$$

as the totality of all closed sets in  $\tau(R)$ , then  $\tau(R)$  is a  $T$ -space and  $\phi_2$  is a continuous mapping.

Thus we can prove more precisely the

**Theorem 6.** *The space  $\tau(R)$  is a compact Hausdorff space and contains a dense subset  $\tilde{\tau}(R)$ . Moreover, a real valued bounded continuous function  $f$  defined on  $R$  can be extended to the function  $f_\gamma$  defined on  $\tau(R)$  such that*

$$f_\gamma(\{\mathfrak{F}\}) = f^*(\mathfrak{F}).$$

*Proof.* Since  $\phi_2$  is continuous, it follows that  $\tau(R)$  is compact.

Let  $\{\mathfrak{F}_1\}$  and  $\{\mathfrak{F}_2\}$  be two distinct points in  $\tau(R)$ . Then  $\mathfrak{F}_1 \neq \mathfrak{F}_2$  and, therefore, there exists a continuous function  $f^*$  defined on  $R^*$  such that  $f^*(\mathfrak{F}_1) = 0$ ,  $f^*(\mathfrak{F}_2) = 1$  and  $0 \leq f^* \leq 1$  on  $R^*$ . Hence, if we put  $U_1^* = f^{*-1}([0, \frac{1}{2}))$  and  $U_2^* = f^{*-1}([\frac{1}{2}, 1])$ , then  $U_1^*$  and  $U_2^*$  are disjoint open sets in  $R^*$ . Moreover, it is evident, from the definition of the relation  $\approx$ , that if  $\mathfrak{F} \in U_i^*$  then  $\{\mathfrak{F}\} \in U_i^*$ , from which it is easily seen that  $U_i^* = \phi_2^{-1}(\phi_2(U_i^*))$ , ( $i = 1, 2$ ). Hence, it follows that  $\phi_2(U_1^*)$  and  $\phi_2(U_2^*)$  are disjoint open sets in  $\tau(R)$  such that  $\{\mathfrak{F}_1\} \in \phi_2(U_1^*)$  and  $\{\mathfrak{F}_2\} \in \phi_2(U_2^*)$ . Thus  $\tau(R)$  is a Hausdorff space.

To prove that  $f_\gamma$  is the function extended from  $f$ , it is sufficient to verify that  $f_\gamma$  is continuous. Now, let  $\{\mathfrak{F}\}$  be a point of  $\tau(R)$ ,  $t = f_\gamma(\{\mathfrak{F}\})$ , and let  $U(t)$  be an open set containing  $t$ . Then, since  $f_\gamma(\{\mathfrak{F}\}) = f^*(\mathfrak{F})$ , the set  $U^* = f^{*-1}(U(t))$  is open in  $R^*$  which contains  $\mathfrak{F}$ . As it is easily seen, from  $U^* = f^{*-1}(U(t))$  and the definition of  $f_\gamma$ , that  $f_\gamma^{-1}(U(t)) = \phi_2(f^{*-1}(U(t))) = \phi_2(U^*)$ . As we proved above,  $\phi_2(U^*)$  is open in  $\tau(R)$ , which contains  $\{\mathfrak{F}\}$ , thus  $f_\gamma$  is continuous, Q.E.D.

*Remark.* In the same manner as we used above, we can define two spaces  $\tau_1(R)$  and  $\tau^\circ(R)$  from  $R^\circ$  and  $\beta^\circ(R)$  respectively. That is, if  $\phi_3$  and  $\phi_4$  are the mapping of  $R^\circ$  on  $\tau_1(R)$  and that of  $\beta^\circ(R)$  on  $\tau^\circ(R)$  respectively, then

$$\begin{aligned} \phi_3([\mathfrak{F}]) &= \phi_1(\phi_2(\phi_1^{-1}([\mathfrak{F}]))), & [\mathfrak{F}] \in R^\circ; \\ \phi_4([\mathfrak{F}]) &= \phi_1(\phi_2(\phi_1^{-1}([\mathfrak{F}]))), & [\mathfrak{F}] \in \beta^\circ(\mathfrak{F}). \end{aligned}$$

Therefore, if we denote by  $\psi_3$  and  $\psi_4$  the mapping of  $\tau(R)$  on  $\tau_1(R)$  and that of  $\tau(R)$  on  $\tau^\circ(R)$  respectively, such that  $\psi_3(\{\mathfrak{F}\}) = \phi_1(\{\mathfrak{F}\})$ ,

$\psi_4(\{\mathfrak{F}\}) = \phi_1(\{\mathfrak{F}\})$ , then by considering the properties (a) and (b), we can prove that  $r(R)$ ,  $r_1(R)$  and  $r^\circ(R)$  are homeomorphic with each other.

The space  $\tilde{r}(R)$  in the Theorem 6 is a continuous image of  $\tilde{R}$ , but not necessarily homeomorphic with  $\tilde{R}$ . As the condition of that  $\tilde{r}(R)$  and  $\tilde{R}$  be homeomorphic with each other, we have the

**Lemma 4.** *In order that  $\tilde{r}(R)$  and  $\tilde{R}$  be homeomorphic, it is necessary and sufficient that  $R$  be a completely regular space.*

*Proof.* The necessity is evident.

Conversely, suppose that  $R$  be completely regular and we shall show that the mapping  $\phi_2^{-1}$  of  $\tilde{r}(R)$  on  $\tilde{R}$  is continuous.

Let  $F$  be a closed set of  $R$  and  $\{\mathfrak{F}_x\}$  be a point of  $\tilde{r}(R) - \phi_2(F^* \cap \tilde{R})$ . Then  $x$  does not belong to  $F$ , and, since  $R$  is completely regular, there exists a real valued continuous function defined on  $R$  such that  $f(x) = 0$ ,  $f(y) = 1$  for every point  $y \in F$  and  $0 \leq f \leq 1$ . Let  $f_\gamma$  be the function extended from  $f$  by the Theorems 1 and 6, then it is clear that  $f_\gamma(\{\mathfrak{F}_x\}) = 0$  and  $f_\gamma(\{\mathfrak{F}\}) = 1$  for every point  $\{\mathfrak{F}\} \in \phi_2(F^*)$ . This implies that the open set  $f_\gamma^{-1}([0, \frac{1}{2}))$  of  $r(R)$  contains  $\{\mathfrak{F}_x\}$  and does not intersect with  $\phi_2(F^*)$ , and hence the open set  $f_\gamma^{-1}([0, \frac{1}{2})) \cap \tilde{r}(R)$  of  $\tilde{r}(R)$  contains  $\{\mathfrak{F}_x\}$  and is contained in  $\tilde{r}(R) - \phi_2(F^* \cap \tilde{R})$ . This shows that  $\tilde{r}(R) - \phi_2(F^* \cap \tilde{R})$  is open in  $\tilde{r}(R)$  and so  $\phi_2(F^* \cap \tilde{R})$  is closed in  $\tilde{r}(R)$ , Q.E.D.

Thus, as we know, there is the well known

**Čech's Theorem.** *For a completely regular space  $R$ , there is a space  $W$  satisfying the following conditions:*

- (1)  $W$  is a compact Hausdorff space;
- (2)  $R \subset W$  and  $\bar{R} = W$ ;
- (3) Any real valued bounded continuous function defined on  $R$  can be extended on  $W$ .

Moreover, the spaces which satisfies the three conditions given above are homeomorphic with each other.

In this place, we will give a proof of this theorem for the purpose to make clear the structure of the space  $W$ .

*Proof.* The space  $r(R)$  is certainly a space satisfying the conditions (1), (2) and (3), thinking  $\tilde{r}(R) = \tilde{R}$  be  $R$ . Hence the existence is true.

Let  $W$  be a space satisfying the conditions (1), (2) and (3), and we will prove that  $W$  and  $r(R)$  are homeomorphic, by dividing the proof into seven parts.

(a). Let  $M$  be a subset of  $W$ . Then, by using the condition (1), it is not difficult to see that an ultrafilter  $\mathfrak{F}(M)$  in  $M$  converges to only one point  $w$  which belongs to  $\overline{M}$ , and conversely, if  $w \in \overline{M}$ , then there exists an ultrafilter  $\mathfrak{F}(M)$  in  $M$  such that  $\mathfrak{F}(M)$  converges to  $w$ .

(b). Let  $g(w)$  be a real valued bounded continuous function defined on  $W$ . Let  $\mathfrak{F}$  be an ultrafilter in  $R$  such that  $\mathfrak{F}$  converges to a point  $w$  and let  $g(w) = t$ . Since  $g(w)$  is continuous, for every neighborhood  $U(t)$  of  $t$  there exists a neighborhood  $V(w)$  such that  $U(t) \supset g(V(w))$ , and from which we have:

$$g(w) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} g(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} g(x).$$

If we define a function  $f$  such that

$$f(x) = g(x), \quad x \in R \subset W,$$

then  $f(x)$  is continuous and the function  $f^*$  defined by

$$f^*(\mathfrak{F}) = \inf_{A \in \mathfrak{F}} \sup_{x \in A} f(x) = \sup_{A \in \mathfrak{F}} \inf_{x \in A} f(x)$$

is the function obtained in the Theorem 1. Hence it is evident that if  $\mathfrak{F}$  converges to  $w$ , then

$$g(w) = f^*(\mathfrak{F}).$$

Conversely, let  $f^*$  be a continuous function defined on  $R^*$ . Then the function  $f$  defined by  $f(x) = f^*(\mathfrak{F}_x)$  is continuous on  $R$ . On the other hand, by the condition (3), there exists a continuous function  $g(w)$  defined on  $W$  such that  $f(x) = g(x)$  for  $x \in R$ . Since  $g(w)$  is continuous, from what we have proved above, we have

$$g(w) = f^*(\mathfrak{F}),$$

for any ultrafilter  $\mathfrak{F}$  in  $R$ , which converges to  $w$ .

(c). For a point  $w \in W$ , if we denote by  $\{\mathfrak{F}\}_w$  the family of all ultrafilters  $\mathfrak{F}$  in  $R$  which converges to  $w$  in  $W$ , then  $R^*$  is divided into disjoint classes  $\{\mathfrak{F}\}_w$ . From (b), it follows that  $\{\mathfrak{F}\}_w \subset \{\mathfrak{F}\}$ .

(d). Let  $w_1$  and  $w_2$  be two distinct points of  $W$ , then there exists a real valued bounded continuous function  $g(w)$  defined on  $W$  such that  $g(w_1) \neq g(w_2)$ . On the other hand, by (a) and the condition (2), there exist two ultrafilters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  in  $R$  such that  $\mathfrak{F}_1$  converges to  $w_1$  and  $\mathfrak{F}_2$  converges to  $w_2$ . Therefore, by (b), it follows that  $f^*(\mathfrak{F}_1) \neq f^*(\mathfrak{F}_2)$ , from which  $\mathfrak{F}_1 \neq \mathfrak{F}_2$ . This shows that  $\{\mathfrak{F}\} \subset \{\mathfrak{F}\}_w$  for every point  $w \in W$ . Then we have, for  $\{\mathfrak{F}\}$  there exists a point  $w \in W$  such that  $\{\mathfrak{F}\} = \{\mathfrak{F}\}_w$ .

(e). By (d), we can define the function  $\phi_s$  of  $W$  on  $\gamma(R)$  such that

$$\phi_s(w) = \{\mathfrak{F}\}_w.$$

It is evident that  $\phi_s$  is one-to-one.

(f). We shall prove that  $\phi_s$  is continuous. Let  $\phi_s(w_0) = \{\mathfrak{F}_0\}$  and  $U_\gamma$  be an open set of  $\gamma(R)$  containing  $\{\mathfrak{F}_0\}$ . By the normality of  $\gamma(R)$ , there exists a real valued continuous function  $f_\gamma$  defined on  $\gamma(R)$  such that  $f_\gamma(\{\mathfrak{F}_0\}) = 0$ ,  $f_\gamma(\{\mathfrak{F}\}) = 1$  on  $\gamma(R) - U_\gamma$  and  $0 \leq f_\gamma \leq 1$  on  $\gamma(R)$ . If we define the function  $f^*$  on  $R^*$  such that  $f^*(\mathfrak{F}) = f_\gamma(\{\mathfrak{F}\})$ , then  $f^*$  is a real valued bounded continuous function on  $R^*$ , and, by (b), we get a continuous function  $g$  defined on  $W$ .

We shall show that the open set  $V = g^{-1}([0, \frac{1}{2}))$  of  $W$  contains  $w_0$  and  $\phi_s(V) \subset U_\gamma$ . By (b),  $g(w_0) = f^*(\mathfrak{F}_0) = f_\gamma(\{\mathfrak{F}_0\}) = 0$  and so  $V$  contains  $w_0$ . Let  $w$  be a point of  $V$ , then  $f_\gamma(\phi_s(w)) = f^*(\phi_s^{-1}(\phi_s(w))) = g(w) \in [0, \frac{1}{2})$  and hence  $\phi_s(w)$  does not belong to  $\gamma(R) - U_\gamma$  and this shows that  $\phi_s(w) \in U_\gamma$ . Thus the proof of the continuity of  $\phi_s$  is established.

(g). Since  $W$  is compact and  $\gamma(R)$  is a Hausdorff space, from (e) and (f), the mapping  $\phi_s$  is a homeomorphism. Thus the theorem is completely proved, Q.E.D.

*Remark.* Finally, if  $\gamma(R)$  is homeomorphic with  $\beta(R)$ , then  $\beta(R)$  is normal. Then, by Theorem 5, the space  $R$  is normal.

Conversely, let  $R$  be normal and let  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  be two distinct points of  $\beta(R)$ . Since  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are maximal dual ideals in  $\mathfrak{L}$ , there exists two distinct closed sets  $F_1$  and  $F_2$  in  $R$  such that  $F_1 \in \mathfrak{F}_1$ ,  $F_1 \notin \mathfrak{F}_2$ ,  $F_2 \in \mathfrak{F}_2$  and  $F_2 \notin \mathfrak{F}_1$ . Hence it is evident that  $\beta(R) - \alpha(F_1)$  and  $\beta(R) - \alpha(F_2)$  are disjoint open sets in  $\beta(R)$  and the former contains  $\mathfrak{F}_2$ , the latter  $\mathfrak{F}_1$ . Then  $\beta(R)$  is a Hausdorff space. Hence  $\beta(R)$

satisfies the three conditions (1), (2) and (3) given in the Čech's Theorem, and, therefore,  $\gamma(R)$  is homeomorphic with  $\beta(R)$ .

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