

ON FAMILIES OF CONTINUOUS VECTOR FIELDS OVER SPHERES

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It is well known that¹⁾, for $n \equiv 3 \pmod{4}$, the sphere S^n admits a set of three continuous vector fields independent at each point, i.e., a 3-field. G. W. Whitehead [3] has proved that, for $n \equiv 3 \pmod{8}$, S^n does not admit a 4-field, but his proof assumes the assertion of Pontrjagin [6] that $\pi_5(S^3) = 0$. But it was proved independently by Pontrjagin [7] and G. W. Whitehead [4] that $\pi_5(S^3)$ is cyclic of order two.

In this note we will prove that the result of G. W. Whitehead is true.

1. Let R_{n+1} be the group of rotations of $(n+1)$ -dimensional Euclidean space E^{n+1} and S^n the unit sphere of E^{n+1} . Then R_{n+1} is the bundle space over S^n with group and fibre R_n . Let $T_{n+1}: S^{n-1} \rightarrow R_n$ be the characteristic map²⁾ of its normal form²⁾. The next lemma is known.

Lemma 1³⁾. *The following two properties of S^n are equivalent: (i) T_{n+1} is homotopic in R_n to a map of S^{n-1} into R_k , and (ii) S^n admits a continuous $(n-k)$ -field.*

Let Sp_{m+1} be the symplectic group operating on the space of $m+1$ quaternion variables $(q_0, \dots, q_m)$. Then Sp_{m+1} is the bundle space over S^{4m+3} with group Sp_m , and let $T''_{m+1}: S^{4m+2} \rightarrow Sp_m$ be the characteristic map of its normal form. It is known that⁴⁾ T''_{m+1} is represented by the equation

$$(1) \quad T''_{m+1}(x) = \|\delta_j^i - 2q_i(1 + q_m)^{-2}\bar{q}_j\|, \quad i, j = 0, 1, \dots, m-1;$$

where $x = (q_0, \dots, q_m)$, $\sum_{i=0}^m |q_i|^2 = 1$ and the real part of q_m is 0 and δ_j^i is the Kronecker δ . It holds the following lemma concerning T and T'' .

1) Cf. Reference [1], p.142, 27.10. Theorem.

2) Cf. [1], pp.96-97, 18.1.

3) Cf. [1], p.141, 27.6. Theorem.

4) Cf. [1], p.130, 24.11.

Lemma 2¹⁾. $T_{4m+1}: S^{4m+2} \rightarrow R_{4m+3}$ is homotopic in R_{4m+3} to $T''_{m+1}: S^{4m+2} \rightarrow Sp_m$.

Let $x_{4m-1} = (q_0, \dots, q_m) \in S^{4m+2}$ be the point such that $q_i = 0$ for $i < m-1$, $q_{m-1} = 1$ and $q_m = 0$, and $p'': Sp_m \rightarrow S^{4m-1}$ the bundle projection defined by $p''(s) = s(x_{4m-1})$, $s \in Sp_m$. Then we have the theorem:

Theorem 1. If m is even, $p'' T''_{m+1}: S^{4m+2} \rightarrow S^{4m-1}$ is essential and homotopic to $(4m-5)$ -hold suspension²⁾ of a map of S^7 on S^4 with odd Hopf invariant³⁾.

Proof. Let $(r_0, \dots, r_{m-1}) \in S^{4m-1}$ be the coordinates of $p'' T''_{m+1}(x)$, then, from (1), $p'' T''_{m+1}$ is given by the equations

$$(2) \quad \begin{aligned} r_i &= -2q_i(1 + q_m)^{-2} \bar{q}_{m-1}, & i = 0, 1, \dots, m-2, \\ r_{m-1} &= 1 - 2q_{m-1}(1 + q_m)^{-2} \bar{q}_{m-1}; \end{aligned}$$

where $\sum_{i=1}^m q_i \bar{q}_i = 1$ and the real part of q_m is 0. G. W. Whitehead⁴⁾ proved that the map given by (2) has the property that it is homotopic to the $(4m-5)$ -hold suspension of a map of S^7 on S^4 with odd Hopf invariant. Hence the theorem holds.

2. We obtain the theorem concerning to the property of the boundary operation ∂ of homotopy sequence.

Theorem 2. If $n \equiv 0 \pmod{8}$ and $\neq 0$, and consider the composition of homomorphism and isomorphism

$$\pi_{n+3}(R_{n+3}, R_n) \xrightarrow{\partial} \pi_{n+2}(R_n, R_{n-1}) \xrightarrow{p_*} \pi_{n+2}(S^{n-1}),$$

where p_* is the induced isomorphism of the projection $p: (R_n, R_{n-1}) \rightarrow S^{n-1}$. Then the element of $p_* \partial(\pi_{n+3}(R_{n+3}, R_n)) \subset \pi_{n+2}(S^{n-1})$ is represented by a map of S^{n+2} into S^{n-1} which is the $(n-5)$ -hold suspension of a map of S^7 into S^4 with even Hopf invariant.

To prove this theorem, we use three lemmas. Let ε_0 be the identity of R_n/R_p and $\pi_{p,q}^n$ be the natural projection of R_n/R_{n-q} into R_n/R_{n-p} . In addition, let I^n be the n -cube and $\dot{I}^n$ its boundary, then $\dot{I}^n$ is homeomorphic with S^{n-1} .

Lemma 3⁵⁾. If $n \equiv 0 \pmod{8}$ and $\neq 0$, there is a map $\phi_0^*: (I^{n+3},$

1) Cf. [1], p. 128, 24.5. Corollary.

2) Cf. [1], pp. 111 - 112, 21.3.

3) Cf. [1], p. 123, 21.6.

4) See the proof of Lemma 1 of [3].

5) Lemmas 3 and 5 are proved by G. W. Whitehead, see [3], Lemmas 2 and 3.

$\dot{I}^{n+3} \rightarrow (R_{n+3}/R_{n-1}, \xi_0)$ such that $\pi_{1,1}^{n+3} \phi_0^* : (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+3}/R_{n+2}, \xi_0) \cong (S^{n+2}, x_0)$ is essential.

Lemma 4. Under the same assumption for n , there is a map $\phi_1^* : (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+2}/R_{n-1}, \xi_0)$ such that $\pi_{3,1}^{n+2} \phi_1^* : (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+2}/R_{n+1}, \xi_0) \cong (S^{n+1}, x_0)$ is essential.

Lemma 5¹⁾ Under the same assumption for n , if ϕ_2 maps $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+1}/R_{n-1}, R_n/R_{n-1})$ and we consider the map $\phi_2 | \dot{I}^{n+3}$ is defined on S^{n+2} , then the last map is homotopic to the $(n-5)$ -hold suspension of a map of S^7 on S^1 with even Hopf invariant.

Proof of Lemma 4. It is easy to see that R_n/R_{n-1} can be considered as the set of all $n \times p$ matrices A such that $A'A = I$, the $p \times p$ identity matrix. Let S^{n+3} be represented by coordinates $(x_0, x_1, \dots, x_m)$, where $m = n/8$, x_0 is a quaternion and $x_1, \dots, x_m$ are Cayley numbers such that $\sum_{i=1}^m |x_i|^2 = 1$. The matrices of the linear transformations $y \rightarrow xy$ and $y \rightarrow yx$ are denoted by $L(x)$ and $R(x)$ respectively for a quaternion, and $L_i(x)$ and $R_i(x)$ for a Cayley number. If $x_0 \neq 0$, let $f_0(x)$ be the 3×3 matrix obtained from $L(x_0)R(x_0) | x_0 |^{-1}$ by deleting the first row and the first column; while $x_0 = 0$, $f_0(x)$ be the 3×3 matrix of zeros. For $i = 1, \dots, m-1$, let $f_i(x)$ be the 8×3 matrix formed from $L_i(x_i)$ by deleting the last five columns. If $x_m = 0$, let $f_m(x)$ be the 7×3 matrix of zeros; while if $x_m \neq 0$, $f_m(x)$ be the 7×3 matrix obtained from $L_1(x_m)R_1(x_m) | x_m |^{-1}$ by deleting the first row and the first four and the last columns. Let f be the map defined by $f'(x) = (f'_0(x), f'_1(x), \dots, f'_m(x))$, then $f(x)$ is a $(n+2) \times 3$ matrix and it is easy to see that $f'(x)f(x) = I$, the 3×3 identity matrix. Hence f maps S^{n+3} into R_{n+2}/R_{n-1} and $f'(1, 0, \dots, 0) = (I, 0, \dots, 0)$. Let g maps I^{n+3} on S^{n+3} with degree 1 so that $g(\dot{I}^{n+3}) = (1, 0, \dots, 0)$, and let $\phi_1^* = fg$. Clearly $\phi_1^*(\dot{I}^{n+3}) = \xi_0$.

To prove the last assertion, we shall show that $h = \pi_{3,1}^{n+2} f : S^{n+3} \rightarrow S^{n+1}$ is essential. The map h is given in real coordinates by

$$\left. \begin{aligned} h_1(x) &= (y_1^2 + y_2^2 - y_3^2 - y_4^2) / |y|, \\ h_2(x) &= 2(y_2 y_3 + y_1 y_4) / |y|, \\ h_3(x) &= 2(y_2 y_4 - y_1 y_3) / |y|, \\ h_i(x) &= 0, \quad i = 1, 2, 3, \end{aligned} \right\} \quad \begin{aligned} &\text{where } |y| \neq 0, \\ &\text{where } |y| = 0. \end{aligned}$$

1) See the footnote 5) of p. 50.

$$\begin{aligned}
 (3) \quad & h_i(x) = y_{i+1}, \quad i = 4, 5, \dots, 8m-5, \\
 & \left. \begin{aligned} h_{8m-5+j}(x) &= 2(z_{j+1}z_5 - z_1z_{j+5}) / |z|, \quad j = 1, 2, 3, \\ h_{8m-1}(x) &= [z_1^2 + z_5^2 - \sum_{j=2}^4 (z_j^2 + z_{j+4}^2)] / |z|, \\ h_{8m-1+j}(x) &= 2(z_{j+5}z_5 + z_1z_{j+1}) / |z|, \quad j = 1, 2, 3, \\ h_i(x) &= 0, \quad i = 8m-4, \dots, 8m+2, \end{aligned} \right\} \text{where } |z| \neq 0, \\
 & \text{where } |z| = 0;
 \end{aligned}$$

where $x_0 = y_1 + y_2i + y_3j + y_4k$, $|y| = (\sum_{i=1}^4 y_i^2)^{\frac{1}{2}}$, $x_i = \sum_{\alpha=1}^8 y_{8i-1+\alpha} e_\alpha$ ($i = 2, \dots, m-1$), $x_m = \sum_{\alpha=1}^8 z_\alpha e_\alpha$ and $|z| = (\sum_{i=1}^8 z_i^2)^{\frac{1}{2}}$. This map is a composition of two maps $h^{(1)}: S^{8m+3} \rightarrow S^{8m+2}$ defined by

$$\begin{aligned}
 (4) \quad & \left. \begin{aligned} h_1^{(1)}(y) &= (y_1^2 + y_2^2 - y_3^2 - y_4^2) / |y|, \\ h_2^{(1)}(y) &= 2(y_2y_3 + y_1y_4) / |y|, \\ h_3^{(1)}(y) &= 2(y_2y_1 - y_1y_3) / |y|, \\ h_i^{(1)}(y) &= 0, \quad i = 1, 2, 3, \\ h_i^{(1)}(y) &= y_{i+1}, \end{aligned} \right\} \text{where } |y| \neq 0, \\
 & \text{where } |y| = 0, \quad i = 4, 5, \dots, 8m+3;
 \end{aligned}$$

and $h^{(2)}: S^{8m+2} \rightarrow S^{8m+1}$ defined by

$$\begin{aligned}
 (5) \quad & \left. \begin{aligned} h_i^{(2)}(y) &= y_i, \quad i = 1, 2, \dots, 8m-5, \\ h_{8m-5+j}^{(2)}(y) &= 2(z_{j+1}z_5 - z_1z_{j+5}) / |z|, \quad j = 1, 2, 3, \\ h_{8m-1}^{(2)}(y) &= [z_1^2 + z_5^2 - \sum_{j=2}^4 (z_j^2 + z_{j+4}^2)] / |z|, \\ h_{8m-1+j}^{(2)}(y) &= 2(z_{j+5}z_5 + z_1z_{j+1}) / |z|, \quad j = 1, 2, 3, \\ h_i^{(2)}(y) &= 0, \quad i = 8m-4, \dots, 8m+2, \end{aligned} \right\} \text{where } |z| \neq 0, \\
 & \text{where } |z| = 0.
 \end{aligned}$$

The map $h^{(1)}$ is the $8m$ -hold suspension of the map of S^3 on S^2 obtained by setting $y_i = 0$ for $i = 4, 5, \dots, 8m+3$ in (4), and the last map is the Hopf map¹⁾ $H: S^3 \rightarrow S^2$. Hence $h^{(1)} \sim E^{8m}H$. The map $h^{(2)}$ is the $(8m-5)$ -hold suspension of the map of S^7 on S^6 obtained by setting $y_i = 0$ for $i = 1, 2, \dots, 8m-5$ in (5), and this map is essential²⁾, and so the 4-hold suspension of the Hopf map H . Hence $h^{(2)} \sim E^{8m-1}H$. Thus $h \sim (E^{8m-1}H) \cdot (E^{8m}H) \sim E^{8m-1}(H \cdot EH)$. As $H \cdot EH$ represents a non zero element of $\pi_4(S^2)^{(3)}$ and $E: \pi_{k+2}(S^k) \rightarrow \pi_{k+3}(S^{k+1})$ is the isomorphism onto³⁾, h is essential. Thus the proof of Lemma 4 is complete.

1) Cf. [1], p. 126, 24.3, equation (9).

2) See [2], p. 140, equation (8).

3) See [1], p. 113, 21.7.

4) Cf. [7] or [4] for $k = 2$, [5] for $k = 3$ and [8] for $k > 3$.

Proof of Theorem 2. Since $\pi_{n,q}^n: \pi_i(R_n/R_{n-q}, R_{n-q}/R_{n-1}) \rightarrow \pi_i(R_n/R_{n-1})$ is an isomorphism onto, we shall show that *an element of the image of the map $\partial: \pi_{n+3}(R_{n+3}/R_{n+1}, R_n/R_{n-1}) \rightarrow \pi_{n+2}(R_n/R_{n-1}) \cong \pi_{n+2}(S^{n-1})$ is represented by a map of S^{n+2} into S^{n-1} being the $(n-5)$ -fold suspension of a map of S^1 into S^1 with even Hopf invariant.*

Let ϕ_0 be a map of $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+3}/R_{n-1}, R_n/R_{n-1})$. We assert that *there exists a map $\phi_1: (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+2}/R_{n-1}, R_n/R_{n-1})$ such that $\phi_0|_{\dot{I}^{n+3}} = \phi_1|_{\dot{I}^{n+3}}$.* If $\pi_{4,1}^{n+3}\phi_0$ is inessential, let $h: (I^{n+3} \times I, \dot{I}^{n+3} \times I) \rightarrow (S^{n+2}, x_0)$ be a homotopy of $\pi_{4,1}^{n+3}\phi_0$ into a point. Since $\pi_{4,1}^{n+3}: R_{n+3}/R_{n-1} \rightarrow R_{n+3}/R_{n+2}$ is a bundle projection, there exists a covering homotopy¹⁾ $h^*: (I^{n+3} \times I, \dot{I}^{n+3} \times I) \rightarrow (R_{n+3}/R_{n-1}, R_n/R_{n-1})$ of ϕ_0 such that $\pi_{4,1}^{n+3}h^* = h$ and $h^*(y, t) = \phi_0(y)$ for $(y, t) \in \dot{I}^{n+3} \times I$. Let $\phi_1 = h^*|_{(I^{n+3} \times 1, \dot{I}^{n+3} \times 1)}$, then ϕ_1 maps $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+2}/R_{n-1}, R_n/R_{n-1})$ and $\phi_1|_{\dot{I}^{n+3}} = \phi_0|_{\dot{I}^{n+3}}$. If $\pi_{4,1}^{n+3}\phi_0$ is essential, let ϕ_0^* be the map of Lemma 3. Since both $\pi_{4,1}^{n+3}\phi_0$ and $\pi_{4,1}^{n+3}\phi_0^*$ are essential, they represent the same element of $\pi_{n+3}(S^{n+2})$. As $\pi_{4,1}^{n+3}: \pi_{n+3}(R_{n+3}/R_{n-1}, R_{n+2}/R_{n-1}) \rightarrow \pi_{n+3}(S^{n+2})$ is an isomorphism, ϕ_0 and ϕ_0^* represent the same element of $\pi_{n+3}(R_{n+3}/R_{n-1}, R_{n+2}/R_{n-1})$, hence $\phi_0|_{\dot{I}^{n+3}}$ is homotopic in R_{n+2}/R_{n-1} to $\phi_0^*|_{\dot{I}^{n+3}} = \hat{\phi}_0$. By this homotopy, we obtain the map $\phi_1: (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+2}/R_{n-1}, R_n/R_{n-1})$ such that $\phi_0|_{\dot{I}^{n+3}} = \phi_1|_{\dot{I}^{n+3}}$.

For this ϕ_1 , *there is a map $\phi_2: (I^{n+3}, \dot{I}^{n+3}) \rightarrow (R_{n+1}/R_{n-1}, R_n/R_{n-1})$ such that $\phi_1|_{\dot{I}^{n+3}} = \phi_2|_{\dot{I}^{n+3}}$.* If $\pi_{3,1}^{n+3}\phi_1$ is inessential, we can construct ϕ_2 by the analogous process of the first case of the above. If $\pi_{3,1}^{n+3}\phi_1$ is essential, we can also construct ϕ_2 by Lemma 4 and the analogous process of the second case of the above.

The last map ϕ_2 maps $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+1}/R_{n-1}, R_n/R_{n-1})$, and so, by Lemma 5, $\phi_2|_{\dot{I}^{n+3}}$, considered as defined on S^{n+2} , is homotopic to the $(n-5)$ -fold suspension of a map of S^1 on S^1 with even Hopf invariant. By the construction, $\phi_2|_{\dot{I}^{n+3}} = \phi_0|_{\dot{I}^{n+3}}$, and hence $\phi_0|_{\dot{I}^{n+3}}$ has the same property. This completes the proof of Theorem 2.

3. Here we have the principal result.

Theorem 3. *If $n \equiv 3 \pmod{8}$, S^n does not admit a continuous 4-field.*

By Lemma 1, to prove this theorem, it is sufficient to prove

Theorem 4. *If $n \equiv 0 \pmod{8}$ and $\neq 0$, then the characteristic map $T_{n+4}: S^{n+2} \rightarrow R_{n+3}$ is not homotopic to a map of S^{n+2} into R_{n-1} .*

1) Cf. [1], p. 50, 11.3. Theorem.

Proof. Consider the diagram

$$\begin{array}{ccccc}
 \pi_{n+2}(R_n) & \xrightarrow{i_*} & \pi_{n+2}(R_{n+3}) & & \\
 \downarrow j_* & & \downarrow k_* & & \\
 \pi_{n+3}(R_{n+3}, R_n) & \xrightarrow{\partial} \pi_{n+2}(R_n, R_{n-1}) & \xrightarrow{m_*} & \pi_{n+2}(R_{n+3}, R_{n-1}), &
 \end{array}$$

where i_* , j_* , k_* and m_* are the induced homomorphisms of the inclusion maps i , j , k and m respectively, and the lower line is from the homotopy sequence of the triple (R_{n+3}, R_n, R_{n-1}) .

Let $q = n/4$ so that q is even. By Lemma 2, T_{q+1} is homotopic to $T''_{q+1}: S^{n+2} \rightarrow Sp_q$. Since $Sp_q \subset R_n$, T''_{q+1} represents an element α of $\pi_{n+2}(R_n)$ such that $i_*\alpha$ is represented by T_{n+1} . As the composition.

$$S^{n+2} \xrightarrow{T''_{q+1}} Sp_q \xrightarrow{l} R_n \xrightarrow{j} (R_n, R_{n-1}) \xrightarrow{p} S^{n-1},$$

where l is the inclusion map and p is the projection, is just the map $p''T''_{q+1}$, $p''T''_{q+1}$ represents the element $p_*j_*\alpha$. As q is even, Theorem 1 implies that $p''T''_{q+1}$ is the $(n-5)$ -fold suspension of the map of S^1 on S^1 with odd Hopf invariant. Hence, by Theorem 2, $p_*j_*\alpha$ is not contained in $p_*\partial(\pi_{n+3}(R_{n+3}, R_n))$, and so $j_*\alpha$ is not contained in $\partial(\pi_{n+3}(R_{n+3}, R_n))$. Exactness of homotopy sequence implies that the kernel of m_* does not contain $j_*\alpha$, and so $m_*j_*\alpha \neq 0$. From $mj = ki$, it follows that $k_*i_*\alpha \neq 0$. Therefore kT_{n+1} represents a non-zero element of $\pi_{n+2}(R_{n+3}, R_{n-1})$. This is equivalent to the desired conclusion. The proof of Theorem 4 is completed.

4. For a field of tangent hyperplanes of S^n , it is known that¹⁾, for $2k \leq n$, S^n admits a continuous field of tangent k -planes if and only if it admits a continuous k -field. Theorem 3 implies immediately

Theorem 5. *If $n \equiv 3 \pmod{8}$, the n -sphere does not admit a continuous field of k -planes for $4 \leq k \leq n-4$.*

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1) Cf. [1], p. 144, §7.16, Theorem.

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