ON THE SPACES WITH NORMAL PROJECTIVE CONNEXIONS AND SOME IMBEDDING PROBLEM OF RIEMANNIAN SPACES II[1]

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In the present paper, we shall investigate the conditions under which a given Riemannian space V_n can be imbedded, as a hypersurface, into a Riemannian space V_{n+1} which has the following properties I) and II).

I) *The group of holonomy of the space with a normal projective connexion corresponding to* V_{n+1} *fixes a hyperquadric and* V_n *is its image in* V_{n+1}, *that is, the locus of points lying on the parallel displaced hyperquadrics, regarded as points in the tangent projective spaces.*

If V_{n+1} has the property above, there exist a scalar y such that the hypersurface is given by the relation $y = 0$.

II) *The orthogonal trajectories of the family of the hypersurfaces on which* y *is constant are geodesics in* V_{n+1}.

If the group of holonomy of the space with a normal projective connexion corresponding to a V_{n+1} fixes a hyperquadric, it is projectively equivalent to an Einstein space[2]. In the previous paper, the author have studied the problem of the same kind as this under the conditions I) and

II′) V_{n+1} *is an Einstein space.*

The imbedding problem of V_n into V_{n+1} under the only condition I) is very complicated in structure. The purpose of the present paper is also to search for the methods dealing with the problem, as the previous one.

§ 1

Let V_n be a given n-dimensional Riemannian space with positive definite line element

1) T. Ōtsuki, On the spaces with normal projective connexions and some imbedding problem of Riemannian spaces, Math. Jour. of Okayama University, Vol. 1, 1952, pp. 69-98.
2) T. Ōtsuki, On projectively connected spaces whose groups of holonomy fix a hyperquadric, Jour. of the Math. Soc. of Japan, Vol. 1, No. 4, 1950, pp. 251-263.

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu$$ [3)]

in each of its coordinate neighborhoods. By virtue of Theorem 3 in the previous paper, in oder that we can imbed V_n in a Riemannian space V_{n+1} with the property I), it is necessary and sufficient that the following system of equations with respect to symmetric tensors $g_{ab}(x, y)$, $h_{ab}(x, y)$, $T_{ab}(x, y)$, a vector $L_a(x, y)$ and scalars $\psi(x, y)$, $U(x, y)$

$$(1) \qquad \frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},$$

$$(2) \qquad \frac{\partial h_a^b}{\partial y} = \frac{n}{2y}(h_a^b + \psi T_a^b) + \psi(hh_a^b - R_a^b) + \psi_a^b,$$ [4)]

$$(3) \qquad \frac{\partial \psi}{\partial y} = \psi^2 h + \psi^3(2T + U) + \frac{2y\psi^3}{n}(h^2 - h_\lambda^\mu h_\mu^\lambda - R),$$

$$(4) \qquad \frac{\partial T_a^b}{\partial y} = \psi(h_\lambda^b T_a^\lambda - h_a^\lambda T_\lambda^b) + \psi_{,a}L^b + g^{b\lambda}\psi_{,\lambda}L_a,$$

$$(5) \qquad \frac{\partial L_a}{\partial y} = -\psi h_a^\lambda L_\lambda - T_a^\lambda \psi_{,\lambda} - \psi_{,a}(T + U) - \frac{1}{n}(h_{,a} - h_{a,\lambda}^\lambda),$$

$$(6) \qquad \frac{\partial U}{\partial y} = \frac{1}{y\psi}(h + \psi T) + \frac{2}{n}(h^2 - h_\lambda^\mu h_\mu^\lambda - R)$$

is integrable under the conditions

$$(7) \qquad L_a + \frac{1}{\psi^2}\psi_{,a} - \frac{2y}{n}(h_{,a} - h_{a,\lambda}^\lambda) = 0,$$

$$(8) \qquad L_{a,b} + T_a^\lambda h_{b\lambda} + h_{ab}(T + U) + \frac{1}{2y\psi^2}(h_{ab} + \psi T_{ab}) = 0,$$

$$(9) \qquad T_{ab,c} - L_a h_{bc} - L_b h_{ac} = 0,$$

$$(10) \qquad U_{,a} - \frac{2}{n\psi}(h_{,a} - h_{a,\lambda}^\lambda) = 0$$

and

$$(g_{ab}(x, y))_{y=0} = g_{ab}(x),$$

where a comma denotes the covariant derivative of $V_n(y)$ with line element

3) Indices take the following values:

$$a, b, c, \cdots\cdots : \lambda, \mu, \nu, \cdots\cdots = 1, 2, \cdots\cdots, n.$$

4) $R_{ab} = R_a{}^\lambda{}_{b\lambda}$, $R_a{}^c{}_{bd} = \frac{\partial \Gamma_{ab}^c}{\partial x^d} - \frac{\partial \Gamma_{ad}^c}{\partial x^b} + \Gamma_a{}^\lambda{}_b \Gamma_\lambda{}^c{}_d - \Gamma_a{}^\lambda{}_d \Gamma_\lambda{}^c{}_b$, where $\Gamma_b{}^a{}_c$ denotes the Christoffel symbol made by g_{ab}.

$$ds^2 = g_{\lambda\mu}(x, y)dx^\lambda dx^\mu.$$

Then, in the coordinate neighborhood $x^1, \cdots\cdots, x^n, y$, the line element of V_{n+1} is given by

$$ds^2 = g_{\lambda\mu}(x, y)dx^\lambda dx^\mu + (\psi(x, y))^2.$$

On the other hand, the orthogonal trajectories of the family of the hypersurfaces in V_{n+1} on which y is constant is given by $x^a =$ constant. In order that these trajectories are geodesics, it is necessary and sufficient that

$$\psi_{,a} = \frac{\partial \psi}{\partial x^a} = 0. \tag{11}$$

Then, the above relations become respectively

$$\frac{\partial h_a^b}{\partial y} = \frac{n}{2y}(h_a^b + \psi T_a^b) + \psi(h h_a^b - R_a^b), \tag{2'}$$

$$\frac{\partial T_a^b}{\partial y} = \psi(h_\lambda^b T_a^\lambda - h_a^\lambda T_\lambda^b), \tag{4'}$$

$$\frac{\partial L_a}{\partial y} = -\psi h_a^\lambda L_\lambda - \frac{1}{n}(h_{,a} - h_{a,\lambda}^\lambda), \tag{5'}$$

$$L_a - \frac{2y}{n}(h_{,a} - h_{a,\lambda}^\lambda) = 0. \tag{7'}$$

Let us put

$$Q_{ab} = h h_{ab} - h_a^\lambda h_{\lambda b} - R_{ab}, \qquad Q = g^{\lambda\mu}Q_{\lambda\mu}.$$

By virtue of (3), (11) and (9), we get the relation

$$h_{,a} + \psi(U_{,a} + 4h_a^\lambda L_\lambda) + \frac{2y\psi}{n}Q_{,a} = 0. \tag{12}$$

Furtheremore, from (10) and (7′) we get

$$L_a = y\psi U_{,a}. \tag{13}$$

Putting this into (5′), by (6), (9), (10) and (11) we get

$$h_{,a} + 3\psi h_a^\lambda L_\lambda + \frac{2y\psi}{n}Q_{,a} + \frac{3}{2}\psi U_{,a} + yU_{,a}\left\{\psi^2 h + \psi^3(2T + U) + \frac{2y}{n}\psi^3 Q\right\} = 0.$$

Hence, putting (12) and (13) into the relation, we obtain

$$(14)\qquad \left\{\frac{1}{2} + y\psi h + y\psi^2(2T + U) + \frac{2y^2}{n}\psi^2 Q\right\}U_{,a} - y\psi h_a^\lambda U_{,\lambda} = 0.$$

On the other hand, we get from (4′)

$$\frac{\partial T}{\partial y} = 0,$$

that is, T depends only on x^a. Accordingly, from the relation

$$T_{,a} = 2L_\lambda h_a^\lambda = 2y\psi U_{,\lambda} h_a^\lambda,$$

we get

$$U_{,\lambda} h_a^\lambda = 0,$$

hence T is a constant. Accordingly, by virtue of the relation, we get from (14), for sufficient small values of y,

$$(15)\qquad U_{,a} = 0.$$

Now, by means of (3), (6), (11) and (15), we see that $\frac{1}{2\psi^2} + y(U + T)$ is a constant. Since ψ depends only on y, we may put $[\psi]_{y=0} = 1$, then we have the relation

$$(16)\qquad \frac{1}{2\psi^2} + y(T + U) - \frac{1}{2} = 0.$$

Then, by (13) and (15), (9) and (10) become

$$(9')\qquad T_{ab,c} = 0,$$

$$(10')\qquad h_{,a} - h_{a,\lambda}^\lambda = 0.$$

We get from (1) and (11)

$$(17)\qquad \frac{\partial \Gamma_{bc}^a}{\partial y} = \psi(h_{bc,\lambda} g^{a\lambda} - h_{b,c}^a - h_{c,b}^a).$$

Accordingly, by (2), (9′), (11) and (17) we get

$$\begin{aligned}\frac{\partial}{\partial y}(h_{,a} - h_{a,\lambda}^\lambda) &= \left(\frac{\partial h}{\partial y}\right)_{,a} - \left(\frac{\partial h_a^\lambda}{\partial y}\right)_{,\lambda} - h_a^\mu \frac{\partial}{\partial y}\Gamma_{\mu\lambda}^\lambda + h_\mu^\lambda \frac{\partial}{\partial y}\Gamma_{a\lambda}^\mu \\ &= \frac{1}{2}Q_{,a} + \psi\left(R_{a,\lambda}^\lambda - \frac{1}{2}R_{,a}\right) = \frac{1}{2}Q_{,a} = 0.\end{aligned}$$

Hence, we obtain by (12), (15) and (10′) the relations

$$(18)\qquad h_{,a} = 0, \qquad h_{a,\lambda}^\lambda = 0,$$

and

$$h^{\mu}_{\lambda} h^{\lambda}_{\mu,a} + \frac{1}{2} R_{,a} = 0.$$

Thus, we see that the system of relations (1) – (11) is equivalent to the system (α) as follows:

(1) $$\frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},$$

(2′) $$\frac{\partial h^b_a}{\partial y} = \frac{n}{2y}(h^b_a + \psi T^b_a) + \psi(h h^b_a - R^b_a),$$

(3) $$\frac{\partial \psi}{\partial y} = \psi^2 h + \psi^3(T + W) + \frac{2y\psi^3}{n} Q,$$

(4′) $$\frac{\partial T^b_a}{\partial y} = \psi(h^b_\lambda T^\lambda_a - h^\lambda_a T^b_\lambda),$$

(6′) $$\frac{\partial W}{\partial y} = \frac{1}{y\psi}(h + \psi T) + \frac{2}{n} Q,$$

(11) $$\psi_{,a} = 0$$

(15′) $$W_{,a} = 0,$$

(18) $$h_{,a} = 0, \qquad h^\lambda_{a,\lambda} = 0,$$
$$h^\mu_\lambda h^\lambda_{\mu,\lambda} + \frac{1}{2} R_{,a} = 0,$$

(8′) $$T^\lambda_a h_{\lambda b} + \frac{1}{2y}\left(h_{ab} + \frac{1}{\psi} T_{ab}\right) = 0,$$

(9′) $$T_{ab,c} = 0,$$

(16′) $$\frac{1}{2\psi^3} + yW - \frac{1}{2} = 0,$$

where $$W = T + U.$$

§ 2

It follows from (18) that

$$\frac{\partial}{\partial y} h_{,a} = -\psi R_{,a} = 0,$$

hence

$$R_{,a} = 0, \qquad h^\mu_\lambda h^\lambda_{\mu,a} = 0.$$

By virtue of (8′) and the symmetricity of h_{ab} and T_{ab}, we get

$$T^{\lambda}_{b} h_{\lambda b} - T^{\lambda}_{b} h_{\lambda a} = 0,$$

hence

$$\frac{\partial T^{b}_{a}}{\partial y} = 0. \tag{4''}$$

That is, T^{b}_{a} is dependent only on x^{a}.

Using matrices, let be

$$\mathfrak{H} = (h^{b}_{a}), \qquad \mathfrak{T} = (T^{b}_{a}),$$

where a and b denote rows and columns respectively. Then, we can rewrite (8′) as

$$(1 + 2y\mathfrak{T})\mathfrak{H} = -\frac{1}{\psi}\mathfrak{T},$$

Hence, for sufficiently small values of y, we get the relation

$$\mathfrak{H} = -\frac{1}{\psi}\frac{\mathfrak{T}}{1 + 2y\mathfrak{T}} = -\frac{1}{\psi}\sum_{m=0}^{\infty}(-2y)^{m}\mathfrak{T}^{m+1}. \tag{19}$$ [5]

If we put

$$\mathfrak{F} = (f^{b}_{a}) = -\psi\mathfrak{H} = \frac{\mathfrak{T}}{1 + 2y\mathfrak{T}}, \tag{20}$$

it follows by (4″) that

$$\frac{\partial}{\partial y}\mathfrak{F} \equiv \left(\frac{\partial}{\partial y}f^{b}_{a}\right) = -2\mathfrak{F}^{2}. \tag{21}$$ [6]

Substituting (21) into (2′), we get easily the relation

$$\mathfrak{R} + \frac{2}{\psi^{2}}\mathfrak{F}^{2} - \frac{n}{2y}\left(\mathfrak{T} - \frac{1}{\psi^{2}}\mathfrak{F}\right) + \mathfrak{F}\left\{-\frac{2f}{\psi^{2}} + T + W + \frac{2y}{n\psi^{2}}(f^{2} - f^{\mu}_{\lambda}f^{\lambda}_{\mu}) - \frac{2y}{n}R\right\} = 0,$$

where $\mathfrak{R} = (R^{b}_{a})$. By virtue of (16′), we get

$$\mathfrak{T} - \frac{1}{\psi^{2}}\mathfrak{F} = 2y(\mathfrak{T}\mathfrak{F} + W\mathfrak{F}),$$

5) The right hand side is uniformly convergent for y such that $2\,|y|\cdot\|\mathfrak{T}\| < 1$, where $\|\mathfrak{T}\|$ denotes the norm of $\mathfrak{T}$. 1 denotes the unit matrix.

6) In the following, for any matrix $\mathfrak{F} = (f^{b}_{a})$, $\frac{\partial}{\partial y}\mathfrak{F}$ and $\mathfrak{F}_{,c}$ stand for the matrices $\left(\frac{\partial}{\partial y}f^{b}_{a}\right)$ and $(f^{b}_{a,c})$ respectively.

therefore we can rewrite the above relation as

$$\mathfrak{R} = n\mathfrak{T}\mathfrak{F} - 2(1-2yW)\mathfrak{F}^2$$
$$+ \left\{2\left(f - \frac{y}{n}(f^2 - f^\mu_\lambda f^\lambda_\mu)\right)(1-2yW) - T + (n-1)W + \frac{2y}{n}R\right\}\mathfrak{F}.$$

This relation shows *that* R^b_a *is a function of* T^μ_λ, W *and* y *as*

$$R^b_a = \varphi^b_a(T^\mu_\lambda, W, y).$$

On the other hand, from (9′), (19) and (20) we obtain

$$h^b_{a,c} = -\frac{1}{\psi}f^b_{a,c} = -\frac{1}{\psi}\sum\frac{\partial f^b_a}{\partial T^\mu_\lambda}T^\mu_{\lambda,c} = 0.$$

Collecting these results, we obtain the following theorem.

Theorem 1. *In order that we can imbed a given* V_n *with line element*

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu$$

into a V_{n+1} *which has the properties I) and II), it is necessary and sufficient that the system of relations*

$$(\beta)\begin{cases} (1') \quad \dfrac{\partial g_{ab}}{\partial y} = 2f^\lambda_a g_{\lambda b}, \\ (4'') \quad \dfrac{\partial T^b_a}{\partial y} = 0, \\ (6') \quad \dfrac{\partial W}{\partial y} = 2(Wf + T^\mu_\lambda f^\lambda_\mu) + \dfrac{2}{n}(f^2 - f^\mu_\lambda f^\lambda_\mu)(1-2yW) - \dfrac{2}{n}\varphi^\lambda_\lambda \\ \qquad\qquad \equiv p(T^\mu_\lambda, W, y) \end{cases}$$

where

$$(19') \quad (f^b_a) \equiv \mathfrak{F} = \frac{\mathfrak{T}}{1+2y\mathfrak{T}},$$

$$(22) \quad (\varphi^b_a) \equiv \mathfrak{R} = n\mathfrak{T}\mathfrak{F} - 2(1-2yW)\mathfrak{F}^2$$
$$+ \left\{2\left(f - \frac{y}{n}(f^2 - f^\mu_\lambda f^\lambda_\mu)\right)(1-2yW) - T + (n-1)W + \frac{2y}{n}\varphi\right\}\mathfrak{F},$$

is integrable under the conditions

$$(\gamma)\qquad \begin{cases} (15) \quad \xi_a \equiv W_{,a} = 0, \\ (9') \quad \zeta_a{}^b{}_c \equiv T^b_{a,c} = 0, \\ \qquad\;\; \eta^b_a \equiv R^b_a - \varphi^b_a = 0 \end{cases}$$

and

$$[g_{ab}(x, y)]_{y=0} = g_{ab}(x).$$

For it is easily verified that (β) *and* (γ) *imply* (α).

§ 3

Let $g_{ab}(x, y)$, $T_a^b(x)$ and $W(x, y)$ be any solutions of the system of equations (β) and ξ_a, $\zeta_a{}^b{}_c$ and η_a^b be quantities made by them according to the left hand sides of (γ).

Now, we get easily

$$\frac{\partial \xi_a}{\partial y} = \sum (\partial p / \partial T_\lambda^\mu)\zeta_\lambda{}^\mu{}_a + (\partial p / \partial W)\xi_a. \tag{23}$$

Since we get from (1′) the relation analogous to (17) such that

$$\frac{\partial}{\partial y}\Gamma_a{}^b{}_c = -f_{ac,\lambda}g^{b\lambda} + f_a{}^b{}_{,c} + f_c{}^b{}_{,a},$$

we get, by means of (β), the relation

$$\begin{aligned}\frac{\partial}{\partial y}\zeta_a{}^b{}_c = &-g^{b\mu}T_{a\lambda}(\partial f_c^\lambda / \partial T_\nu^\rho)\zeta^\rho_{\nu\mu} + T_a^\lambda(\partial f_\lambda^b / \partial T_\mu^\nu)\zeta^\nu_{\mu c} \\ &+ T_a^\lambda(\partial f_c^b / \partial T_\mu^\nu)\zeta^\nu_{\mu\lambda} + g_{a\lambda}T^{b\mu}(\partial f_c^\lambda / \partial T_\nu^\rho)\zeta^\rho_{\nu\mu} \\ &- T_\lambda^b(\partial f_a^\lambda / \partial T_\mu^\nu)\zeta^\nu_{\mu c} - T_\lambda^b(\partial f_c^\lambda / \partial T_\mu^\nu)\zeta^\nu_{\mu a}.\end{aligned} \tag{24}$$

Accordingly, if we take solutions such that $W_{,a} = 0$ and $T^b_{a,c} = 0$ for $y = 0$, then by means of (23) and (24) the relations $\xi_a = 0$ and $\zeta_a{}^b{}_c = 0$ hold good for each value of y. Then W is depend only on y and T is a constant. In the following, we shall consider only such solutions. Then, we get easily the relation

$$\frac{\partial R_a^b}{\partial y} = -2R_a^\lambda f_\lambda^b. \tag{25}$$

We get from (22)

$$\begin{aligned}&\Big(1 - \frac{2yf}{n}\Big)\{\varphi - (f^2 - f_\lambda^\mu f_\mu^\lambda)(1 - 2yW)\} \\ &\quad = nT_\lambda^\mu f_\mu^\lambda + (f^2 - f_\lambda^\mu f_\mu^\lambda)(1 - 2yW) + \{(n-1)W - T\}f,\end{aligned}$$

accordingly we have the relation

$$\begin{aligned}&2\Big\{f - \frac{y}{n}(f^2 - f_\lambda^\mu f_\mu^\lambda)\Big\}(1 - 2yW) + (n-1)W - T + \frac{2y}{n}\varphi \\ &\quad = \frac{1}{1 - \frac{2yf}{n}}\Big[2yT_\lambda^\mu f_\mu^\lambda + (n-1)W - T\end{aligned}$$

$$+ 2\left\{f - \frac{y}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\}(1 - 2yW)\Big].$$

Making use of (19), the right hand side becomes

$$\frac{1}{1 - \frac{2yf}{n}}\Big[(n-1)W - f + 2\left\{f - \frac{y}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\}(1 - 2yW)\Big].$$

Therefore φ^b_a determined indirectly in (22) can be represented as follows:

$$\text{(22}'\text{)}\quad \mathfrak{K} = n\mathfrak{T}\mathfrak{F} - 2(1 - 2yW)\mathfrak{F}^2$$
$$+ \frac{1}{1 - \frac{2yf}{n}}\Big[(n-1)W - f + 2\left\{f - \frac{y}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\}(1 - 2yW)\Big]\mathfrak{F}.$$

Now, by virtue of (22′), we obtain likewise the relation

$$p = 2(Wf + T^\mu_\lambda f^\lambda_\mu) - \frac{2}{n}\{\varphi - (f^2 - f^\mu_\lambda f^\lambda_\mu)(1 - 2yW)\}$$
$$= \frac{2}{n - 2yf}\{W(f - 2yf^\mu_\lambda f^\lambda_\mu) + f^\mu_\lambda f^\lambda_\mu\}.$$

Hence, by (21) and this relation, (6′) becomes

$$\frac{\partial W}{\partial y} = -W\frac{\partial}{\partial y}\log\left(1 - \frac{2yf}{n}\right) + \frac{2f^\mu_\lambda f^\lambda_\mu}{n - 2yf},$$

that is

$$\frac{\partial}{\partial y}\left\{W\left(1 - \frac{2yf}{n}\right)\right\} = -\frac{1}{n}\frac{\partial f}{\partial y}.$$

Accordingly, $W\left(1 - \frac{2yf}{n}\right) + \frac{f}{n} = \left(W + \frac{f}{n}\right)_{y=0} = \left(W + \frac{T}{n}\right)_{y=0}$, that is

$$W = \frac{c - f}{n - 2yf}$$

where c is a constant. Then, since

$$1 - 2yW = \frac{n - 2yc}{n - 2yf}$$

and

$$(n-1)W - f + 2\left\{f - \frac{y}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\}(1 - 2yW)$$
$$= \frac{n}{n - 2yf}\left\{\frac{(n-1)c}{n} + \frac{f}{n} - \frac{2y}{n}(2cf + f^\mu_\lambda f^\lambda_\mu) + \frac{4y^2c}{n^2}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\},$$

(22′) becomes

(26) $$\mathfrak{K} \equiv n\mathfrak{T}\mathfrak{F} - 2\frac{n-2yc}{n-2yf}\mathfrak{F}^2 + \frac{n}{(n-2yf)^2}\left\{(n-1)c + f - 2y(2cf + f^\mu_\lambda f^\lambda_\mu) + \frac{4y^2c}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)\right\}\mathfrak{F}.$$

From the above relation, we get

(27) $$\frac{\partial}{\partial y}\mathfrak{K} = \left(\frac{\partial \varphi^h_\alpha}{\partial y}\right) = -2\mathfrak{K}\mathfrak{F} + 4\frac{n-2yc}{n-4yf}\mathfrak{F}^3 - 2\frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right)\mathfrak{F}^2 + \sigma\mathfrak{F},$$

where

$$\sigma = \sigma(T^\mu_\lambda, y, c) \equiv \frac{4nf(f-c)}{(n-2yf)^3} + \frac{8n(n-2yc)}{(n-2yf)^3}\left[-f^\mu_\lambda f^\lambda_\mu + yf^\mu_\lambda f^\nu_\mu f^\lambda_\nu + \frac{2y^2}{n}\{(f^\mu_\lambda f^\lambda_\mu)^2 - ff^\mu_\nu f^\nu_\mu f^\lambda_\nu\}\right] + \frac{4n}{(n-2yf)^2}(f^\mu_\lambda f^\lambda_\mu - yf^\mu_\lambda f^\nu_\mu f^\lambda_\nu),$$

which we obtain by a long computation. Hence we get by (25) and (27) the relation

$$\frac{\partial}{\partial y}(\mathfrak{N} - \mathfrak{K}) = -2(\mathfrak{N} - \mathfrak{K})\mathfrak{F} - 4\frac{n-2yc}{n-2yf}\mathfrak{F}^3 + 2\frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right)\mathfrak{F}^2 - \sigma\mathfrak{F}.$$

By the last relation, it must hold good that

$$4\frac{n-2yc}{n-2yf}\mathfrak{F}^3 - 2\frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right)\mathfrak{F}^2 + \sigma\mathfrak{F} = 0.$$

Making use of (19′), this is equivalent to

$$\sigma\mathfrak{T} + 2\left\{2y\sigma - \frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right)\right\}\mathfrak{T}^2 + 4\left\{y^2\sigma - y\frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right) + \frac{n-2yc}{n-2yf}\right\}\mathfrak{T}^3 = 0.$$

On the other hand, by some comutations we get the relations

(28) $$\left(1 - \frac{2yf}{n}\right)^3\left\{2y\sigma - \frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right)\right\} = -\frac{2}{n}(f-c) + \frac{4y}{n}\{3f(f-c) - nf^\mu_\lambda f^\lambda_\mu\}$$

$$- \frac{8y^2}{n}\left\{\frac{3}{n}(f-c)f^{\mu}_{\lambda}f^{\lambda}_{\mu} - f^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu}\right\}$$
$$- \frac{16y^3}{n^2}\left\{(f+2c)f^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu} - \frac{c}{n}ff^{\mu}_{\lambda}f^{\lambda}_{\mu} - 2(f^{\mu}_{\lambda}f^{\lambda}_{\mu})^2\right\}$$
$$- \frac{64y^4c}{n^3}\{(f^{\mu}_{\lambda}f^{\lambda}_{\mu})^2 - ff^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu}\} \equiv 2\beta(T^{\mu}_{\lambda}, c, y)$$

and

$$\left(1-\frac{2yf}{n}\right)^3\left\{y^2\sigma - y\frac{\partial}{\partial y}\left(\frac{n-2yc}{n-2yf}\right) + \frac{n-2yc}{n-2yf}\right\}$$
$$= 1 - \frac{6y}{n}f + \frac{12y^2}{n^2}f^2 - \frac{4y^3}{n^2}\{2(2f-c)f^{\mu}_{\lambda}f^{\lambda}_{\mu} + n(2cf^2 - f^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu})\}$$
$$(29) \qquad - \frac{8y^4}{n^2}\{ff^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu} - 2(f^{\mu}_{\lambda}f^{\lambda}_{\mu})^2\} - \frac{32y^5c}{n^3}\{(f^{\mu}_{\lambda}f^{\lambda}_{\mu})^2 - ff^{\mu}_{\lambda}f^{\nu}_{\mu}f^{\lambda}_{\nu}\}$$
$$\equiv \gamma(T^{\mu}_{\lambda}, c, y).$$

Therefore, we have the relation

$$(30) \qquad \mathfrak{T}\{\alpha(T^{\mu}_{\lambda}, c, y) + \beta(T^{\mu}_{\lambda}, c, y)\mathfrak{T} + \gamma(T^{\mu}_{\lambda}, c, y)\mathfrak{T}^2\} = 0,$$

where

$$\alpha(T^{\mu}_{\lambda}, c, y) = \frac{1}{4}\left(1-\frac{2yf}{n}\right)^3\sigma(T^{\mu}_{\lambda}, c, y).$$

§ 4

Since (30) holds good for any value of y, putting $y=0$, we get by virtue of (27), (28) and (29) the relation

$$(31) \qquad \mathfrak{T}^3 = A\mathfrak{T} + B\mathfrak{T}^2.$$

where

$$A = -\frac{1}{n^2}a_1(a_1-c) + \frac{a_2}{n} = -\alpha(T^{\mu}_{\lambda}, c, o)$$
$$B = \frac{1}{n}(a_1-c) = -\beta(T^{\mu}_{\lambda}, c, o),$$
$$a_m = trace\ \mathfrak{T}^m, \qquad m = 1, 2, 3, \cdots\cdots.$$

Hence, we obtain inductively from (31) the relation

$$(32) \qquad \mathfrak{T}^m = AM_{m-3}\mathfrak{T} + M_{m-2}\mathfrak{T}^2, \qquad m = 3, 4, \cdots\cdots,$$

where

$$M_m = B^m + \binom{m-1}{1}B^{m-2}A + \binom{m-2}{2}B^{m-4}A + \cdots\cdots + \binom{m-r}{r}B^{m-2r}A^r + \cdots\cdots,$$
$$m = 0, 1, 2, \cdots\cdots.$$

For

$$\mathfrak{T}(AM_{m-3}\mathfrak{T} + M_{m-2}\mathfrak{T}^2) = AM_{m-2}\mathfrak{T} + (AM_{m-3} + BM_{m-2})\mathfrak{T}^2,$$
$$AM_{m-3} + BM_{m-2} = B^{m-1} + \sum_{r=1}^{[(m-1)/2]} \left\{\binom{m-2-r}{r} + \binom{m-2-r}{r-1}\right\}B^{m-1-2r}A^r$$
$$= B^{m-1} + \sum \binom{m-1-r}{r}B^{m-1-2r}A^r = M_{m-1}.$$

Then, by virtue of these relations, we get

$$a_m = AM_{m-3}a_1 + M_{m-2}a_2 = nM_m + cM_{m-1}, \quad m = 1, 2, 3, \cdots\cdots. \tag{33}$$

Now, by (19) and (33) we have

$$f = \sum_{m=0}^{\infty}(-2y)^m a_{m+1} = c + (n - 2cy)\sum_{m=0}^{\infty}(-2y)^m M_{m+1}.$$

Since

$$\sum(-2y)^m M_{m+1} = B - 2y(B^2 + A) + \sum_{m=2}^{\infty}(-2y)^m M_{m+1}$$
$$= \frac{B - 2Ay}{1 + 2By - 4Ay^2},$$

it follows that

$$f = \frac{c + nB - 2nAy}{1 + 2By - 4Ay^2}, \tag{34}$$

whence

$$\frac{n - 2yc}{n - 2yf} = 1 + 2By - 4Ay^2.$$

Then, by an analogous computation we get

$$f^\mu_\lambda f^\lambda_\mu = -\frac{1}{2}\frac{\partial f}{\partial y} = \frac{nA + Bc + nB^2 - 4A(c + nB)y + 4nA^2y^2}{(1 + 2By - 4Ay^2)^2}.$$

Now, for the coefficient of $\mathfrak{F}$ in (26), we obtain by virtue of the above relations

$$(n-1)c + f - 2y(2cf + f^\mu_\lambda f^\lambda_\mu) + \frac{4y^2c}{n}(f^2 + f^\mu_\lambda f^\lambda_\mu)$$
$$= nc\left(1 - \frac{2yf}{n}\right)^2 + \frac{(n - 2yc)^2(B - 4Ay)}{n(1 + 2By - 4Ay^2)^2},$$

whence

$$\frac{1}{n-2yf}\left\{(n-1)c+f-2y(2cf+f^{\mu}_{\lambda}f^{\lambda}_{\mu})+\frac{4y^2c}{n}(f^2+f^{\mu}_{\lambda}f^{\lambda}_{\mu})\right\}$$
$$=c+B-4Ay.$$

Thus, (26) may become the simple relation such that

$$\mathfrak{K}\equiv n\mathfrak{T}\mathfrak{F}-2(1+2By-4Ay^2)\mathfrak{F}^2+(c+B-4Ay)\mathfrak{F}.$$

Hence we get from this

$$\frac{\partial}{\partial y}\mathfrak{K}=-2\mathfrak{K}\mathfrak{F}+4(1+2By-4Ay^2)\mathfrak{F}^3-4(B-4Ay)\mathfrak{F}^2-4A\mathfrak{F}.$$

Therefore, (30) may become the relation such that

$$(1+2By-4Ay^2)\mathfrak{F}^3-(B-4Ay)\mathfrak{F}^2-A\mathfrak{F}=0,$$

which is equivalent to

$$(1+2By-4Ay^2)\mathfrak{T}^3-(B-4Ay)\mathfrak{T}^2(1+2y\mathfrak{T})-A\mathfrak{T}(1+2y\mathfrak{T})^2$$
$$=-A\mathfrak{T}-B\mathfrak{T}^2+\mathfrak{T}^3=0$$

by virtue of (19′). Thus we get a conclusion as follows.

Theorem 1′. *For a given Riemannian space V_n, in order that we can imbed it into a V_{n+1} with the properties I) and II), it is necessary and sufficient that the system of equations with respect to T^b_a and a constant c:*

(35) $$\mathfrak{R}=(n-2)\mathfrak{T}^2+(c+B)\mathfrak{T},$$

(31) $$\mathfrak{T}^3=A\mathfrak{T}+B\mathfrak{T}^2,$$

(9′) $$T^b_{a,c}=0,$$

(36) $$A=\frac{1}{n}T^{\mu}_{\lambda}T^{\lambda}_{\mu}-\frac{1}{n^2}T(T-c),\qquad B=\frac{1}{n}(T-c)$$

is solvable.

In the following, let be $n>2$. In place of c, take k such that

(37) $$c+B=\frac{T+(n-1)c}{n}=2(n-2)k,$$

then (35) is rewritten as

(35′) $$\mathfrak{R}=(n-2)(\mathfrak{T}^2+2k\mathfrak{T}).$$

By virtue of (37), we get from (36)

$$B = \frac{1}{n-1}\{T - 2(n-2)k\},$$
$$A = \frac{R}{n(n-2)} - \frac{1}{n(n-1)}(T^2 + 2kT).$$

Now, putting (35′) into (31), we get

$$\frac{1}{n-1}(T + 2k)\mathfrak{T}^2 + A\mathfrak{T} - \frac{1}{n-2}\mathfrak{R}\mathfrak{T} = 0,$$

which we can rewrite by the same substitution as

$$(38) \quad \frac{T+2k}{(n-1)(n-2)}\mathfrak{R} - \frac{1}{n-2}\mathfrak{R}\mathfrak{T} + \frac{1}{n}\left\{\frac{R}{n-2} - \frac{(T+2k)(T+2nk)}{n-1}\right\}\mathfrak{T} = 0.$$

Accordingly, the system of equations in Theorem 1′ may be replaced by the one of (35′), (38) and (9′).

Furthermore, let us put

$$\mathfrak{V} = \mathfrak{T} + k\mathbf{1},$$

then (35′) and (38) become

$$(39) \quad \mathfrak{R} + (n-2)k^2\mathbf{1} = (n-2)\mathfrak{V}^2,$$

$$(40) \quad \frac{V+k}{(n-1)(n-2)}\mathfrak{R} - \frac{1}{n-2}\mathfrak{R}\mathfrak{V} + \frac{1}{n}\left(\frac{R}{n-2} - \frac{(V+nk)(V-(n-2)k)}{n-1}\right)(\mathfrak{V} - k\mathbf{1}) = 0.$$

On the other hand, from (35′) and (9′) the relation $R^b_{a,c} = 0$ must be satisfied in the space. Thus we can describe Theorem 1′ as follows: *For a given V_n, in order that we can solve our problem in the small, it is necessary and sufficient that $R^b_{a,c} = 0$ and there exist $\mathfrak{V}$ and k such that they satisfy* (39), (40) *and* $\mathfrak{V}_{,a} = 0$.

Now, we see that for the given space V_n, if the relation

$$\frac{1}{n-2} \min \{\text{eigen values of } \mathfrak{R}\} + k^2 \geqslant 0$$

holds good at each point, we may have in general 2^n solutions such that

$$\mathfrak{V} = \left(\frac{1}{n-2}\mathfrak{R} + k^2\mathbf{1}\right)^{\frac{1}{2}} = \mathfrak{F}(R^\mu_\lambda, k).$$

Therefore, if $k \neq 0$ and

$$\frac{1}{(n-2)k^2} \text{ norm } (\mathfrak{R}) < 1,$$

we have a function

$$(41) \qquad \mathfrak{F}_0 = k\Big\{1 + \frac{1}{2(n-2)k^2}\mathfrak{R} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot 3\cdots\cdots(2m-3)}{2\cdot 4\cdots\cdots(2m)}\Big(\frac{\mathfrak{R}}{(n-2)k^2}\Big)^m\Big\}$$

which is uniformly convergent and satisfy (39). Since, in V_n, there exist in general 2^n tensors $A = (\lambda^{\nu}_{\mu}(x))$ such that

$$(42) \qquad A\mathfrak{R} = \mathfrak{R}A, \qquad A^2 = 1,$$

we see that $\mathfrak{F} = A\mathfrak{F}_0$.

§ 5

In the paragraph, we shall investigate the conditions for a given V_n $(n > 2)$ under which we can solve the problem stated in Theorem 1 for each (sufficient large) value of k.

Then, (41) is the fundamental solution for (39). Since we may consider k as an arbitrary constant, let be

$$\frac{1}{k^2} = z,$$

and

$$(43) \qquad \Phi = 1 + \frac{z}{2(n-2)}\mathfrak{R} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot 3\cdots\cdots(2m-3)z^m}{2\cdot 4\cdots\cdots 2m(n-2)^m}\mathfrak{R}^m.$$

According to the previous consideration, in a neighborhood of $z = 0$, we must have at least a A such that

$$(44) \qquad \frac{(\text{trace } A\Phi + 1)z}{(n-1)(n-2)}\mathfrak{R} - \frac{z}{n-2}\mathfrak{R}A\Phi + \frac{1}{n}\Big\{\frac{Rz}{n-2} - \frac{(\text{trace } A\Phi + n)(\text{trace } A\Phi - (n-2))}{n-1}\Big\}(A\Phi - 1) = 0.$$

Hence, if we put $z \to 0$ in (44), we get by (43) the relation

$$(\text{trace } A + n)(\text{trace } A - (n-2))(A - 1) = 0.$$

This relation and the assumption for k imply that we may consider the cases $A = \pm 1$ or trace $A = n - 2$.

I. *Case* $\Lambda = 1$.

Then, (44) is written as

$$(44_{\mathrm{I}}) \qquad \frac{(\varphi+1)z}{(n-1)(n-2)}\mathfrak{R} - \frac{z}{n-2}\mathfrak{R}\Phi + \frac{1}{n}\left\{\frac{Rz}{n-2} - \frac{(\varphi+n)(\varphi-n+2)}{n-1}\right\}(\Phi-1) = 0,$$

where $\varphi = \mathrm{trace}\,\Phi$. In (44_{I}), the coefficient of z is identically zero. From the coefficient of z^2 we may have the relation

$$(45) \qquad \mathfrak{R}^2 = \frac{R}{n}\mathfrak{R},$$

hence by induction

$$(46) \qquad \mathfrak{R}^m = \left(\frac{R}{n}\right)^{m-1}\mathfrak{R}, \qquad m = 2, 3, 4, \cdots\cdots.$$

Accordingly we get from (43)

$$(47) \qquad \varphi = n\sqrt{1 + \frac{Rz}{n(n-2)}}$$

and putting this and (46) into (43)

$$\Phi = 1 + \frac{z}{2(n-2)}\mathfrak{R} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot 3\cdots\cdots(2m-3)}{2\cdot 4\cdots\cdots 2m}\frac{z^m}{(n-2)^m}\left(\frac{R}{n}\right)^{m-1}\mathfrak{R},$$

that is

$$(48) \qquad \Phi = 1 + \frac{1}{R}(\varphi - n)\mathfrak{R}.$$

If we put (47) and (48) into (44_{I}), we get

$$\frac{(\varphi+1)z}{(n-1)(n-2)}\mathfrak{R} - \frac{z}{n-2}\mathfrak{R}\Phi + \frac{1}{n}\left\{\frac{Rz}{n-2} - \frac{(\varphi+n)(\varphi-n+2)}{n-1}\right\}(\Phi-1)$$
$$= \frac{z(\varphi-n)}{R(n-2)}\left(\frac{R}{n}\mathfrak{R} - \mathfrak{R}^2\right) = 0.$$

Since

$$\frac{\varphi-n}{R} = \frac{z}{2(n-2)} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot 3\cdots\cdots(2m-3)z^m}{2\ 4\cdots\cdots 2m(n-2)^m}\left(\frac{R}{n}\right)^{m-1} \neq 0,$$

the above relation is equivalent to

$$(45) \qquad \mathfrak{R}^2 - \frac{R}{n}\mathfrak{R} = 0.$$

The above argument shows that (44_{I}) is equivalent to (45). (45) is clearly equivalent to the relation

$$\mathfrak{R} = \frac{R}{n}\mathbf{1}. \tag{45'}$$

II. *Case* $\Lambda = -1$.

Then, (44) is written as

$$\frac{(\varphi-1)z}{(n-1)(n-2)}\mathfrak{R} - \frac{z}{n-2}\mathfrak{R}\Phi + \frac{1}{n}\left\{\frac{Rz}{n-2} - \frac{(\varphi-n)(\varphi+n-2)}{n-1}\right\}(\Phi+1) = 0. \tag{44_{II}}$$

The coefficient of z in the left hand side is identically zero. From the one of z^2 it must follow that

$$\mathfrak{R}^2 = \frac{R}{n-1}\mathfrak{R} - \frac{1}{n}\left(\frac{R^2}{n-1} - r_2\right)\mathbf{1}, \tag{49}$$

where $r_m = \text{trace } \mathfrak{R}^m$, $m = 1, 2, 3, \cdots\cdots$. From the coefficient of z^3 in 44_{II}), we have

$$\mathfrak{R}^3 = \frac{2}{n}\left(r_3 - \frac{r_2 R}{n-1}\right)\mathbf{1} + \frac{1}{n(n-1)}(r_2 + R^2)\mathfrak{R}. \tag{50}$$

On the other hand, by virtue of (49) we get the relation

$$\mathfrak{R}^3 = \frac{1}{n}\left\{\left(\frac{R}{n-1}\right)^2 + r_2\right\}\mathfrak{R} - \frac{R}{n(n-1)}\left(\frac{R^2}{n-1} - r_2\right)\mathbf{1}. \tag{51}$$

From this we get

$$r_3 - \frac{r_2 R}{n-1} = -\frac{R^3}{n(n-1)} + \frac{r_2 R}{n},$$

hence putting the relation into (50), we can rewrite it as

$$\mathfrak{R}^3 = \frac{1}{n(n-1)}(R^2 + r_3)\mathfrak{R} - \frac{2R}{n^2}\left(\frac{R^2}{n-1} - r_2\right)\mathbf{1}. \tag{50'}$$

Hence, by (51) and (50′) we get

$$\left(\frac{R^2}{n-1} - r_2\right)\left(\mathfrak{R} - \frac{R}{n}\mathbf{1}\right) = 0.$$

In the following, we divide the case into two cases.

(i) *Case* $\dfrac{R^2}{n-1} - R_\lambda^\mu R_\mu^\lambda \neq 0.$

Then, we get the relation $\mathfrak{R} = \frac{R}{n}1$. Putting this into (43), we get easily

$$\varphi = \text{tracc}\, \Phi = n\sqrt{1 + \frac{zR}{n(n-2)}}, \qquad \Phi = \frac{\varphi}{n}1.$$

Then, (44_{II}) is clearly satisfied by them and

$$\frac{R^2}{n-1} - R_\lambda^\mu R_\mu^\lambda = \frac{R^2}{n(n-1)}.$$

Therefore, if $\mathfrak{R} = \frac{R}{n}1$, $R \neq 0$, all the conditions are satisfied by these quantities.

(ii) *Case* $\frac{R^2}{n-1} - R_\lambda^\mu R_\mu^\lambda = 0$.

Then, we get from (49) the relation $\mathfrak{R}^2 = \frac{R}{n-1}\mathfrak{R}$, in general

$$\mathfrak{R}^m = \left(\frac{R}{n-1}\right)^{m-1}\mathfrak{R}, \qquad m = 2, 3, \cdots\cdots.$$

Accordingly, we get from (43) the relation

$$\varphi = 1 + (n-1)\sqrt{1 + \frac{zR}{(n-1)(n-2)}},$$

and

$$\Phi = 1 + \frac{\varphi - n}{R}\mathfrak{R}.$$

Then, we see easily that (44_{II}) is also satisfied by them. Therefore, if $\mathfrak{R}^2 = \frac{R}{n-1}\mathfrak{R}$, all the conditions are satisfied by these quantities.

III. *Case trace* $A = n - 2$.

In this case, from the left hand side of (44) it must follow that

$$\text{the coefficient of } z = \frac{1}{n-2}\left(\mathfrak{R} - \frac{R}{n}1 + \frac{\text{trace}\,(A\mathfrak{R})}{n}1\right)(1 - A)$$
$$= 0,$$

hence by virtue of (42) trace $(\mathfrak{R}A) = \mathrm{R}$. Putting this into the above relation, we get $\mathfrak{R}A = A\mathfrak{R} = \mathfrak{R}$. Then we get from (43)

(52) $$A\Phi = \Phi + A - 1$$

and trace $(A\Phi) = \varphi - 2$. Putting these relations into (44), it is replaced by

$$(44_{\rm III})\qquad \frac{(\varphi-1)z}{(n-1)(n-2)}\mathfrak{R}-\frac{z}{n-2}\mathfrak{R}\Phi + \frac{1}{n}\left\{\frac{Rz}{n-2}-\frac{(\varphi+n-2)(\varphi-n)}{n-1}\right\}(\Phi+A-2\mathbf{1}) = 0.$$

From the coefficient of z^2 in this relation, we obtain by a simple computation

$$\mathfrak{R}^2 = \frac{R}{n-1}\mathfrak{R}+\frac{1}{2n}\left(\frac{R^2}{n-1}-r_2\right)(\mathbf{1}-A),$$

whence $r_2=\frac{R^2}{n-1}$. Hence, the above relation becomes $\mathfrak{R}^2=\frac{R}{n-1}\mathfrak{R}$. Therefore, as the case II, (ii), we get

$$\varphi = 1+(n-1)\sqrt{1+\frac{zR}{(n-1)(n-2)}},$$

$$\Phi = 1+\frac{\varphi-n}{R}$$

Conversely, we see easily that these relations satisfy $(44_{\rm III})$.

In the above consideration, we did not discuss the condition $(A\Phi)_{,a}=0$. Since $A_{,a}=0$ in the cases I, II, the above condition is clearly satisfied. In the third case, this may not be satisfied by the A such that trace $A=n-2$ but $\mathfrak{R}$ must satisfy the same conditions in Case II, (ii). Accordingly, we get the following theorem.

Theorem 2. *In order that we can imbed a given n-dimensional Riemannian space V_n $(n>2)$ into a V_{n+1} with the properties I) and II), irrespective of values of k, it is necessary and sufficient that it is an Einstein space or such a space as its Ricci tensor R_a^b satisfy the conditions*

$$R_{a,c}^b = 0, \qquad R_a^\lambda R_\lambda^b = \frac{R}{n-1}R_a^b.$$

§ 6

In the paragraph, we consider the case $n=2$. Let $K(x)$ be the Gaussian total curvature of V_2. Then $\mathfrak{R}=K\mathbf{1}$. Let us consider the conditions in Theorem 1′. From (35) and (36), we get

$$K\mathbf{1} = \frac{1}{2}(T+c)\mathfrak{T}.$$

By (9′), it follows that K is a constant. Furthermore, we get

$$2K = \frac{1}{2}(T + c)T,$$

that is $T = \dfrac{-c \pm \sqrt{c^2 + 16K}}{2}$. Hence we get

$$\mathfrak{T} = -\frac{-c \mp \sqrt{c^2 + 16K}}{4}1.$$

By (36) and the above relation, we get

$$A = -\frac{c(c \mp \sqrt{c^2 + 16K}}{8}.$$

Then, we can easily see that (31) is satisfied by these quantities. Thus we obtain the following theorem.

Theorem 3. *In order that we can imbed a given V_2 into a V_3 with the properties I) and II), it is necessary and sufficient that V_2 is a surface with constant curvature.*

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