ON THE CARTAN INVARIANTS OF ALGEBRAS

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1. Let $A$ be an algebra with unit element over an algebraically closed field $K$ and let

$$A = A^* + N$$

be a splitting of $A$ into a direct sum of a semisimple subalgebra $A^*$ and the radical $N$ of $A$. We shall denote by

$$A^* = A_1^* + A_2^* + \cdots + A_k^*$$

the unique splitting of $A^*$ into a direct sum of simple invariant subalgebras. Let $e_{i_1, \alpha\beta}(\alpha, \beta = 1, 2, \ldots, f(i))$ be a set of matrix units for the simple algebra $A_\alpha^*$. We denote by $F_1, F_2, \ldots, F_k$ the distinct irreducible representations of $A$ and we set for $a$ in $A$

$$F_i(a) = (f_{i\alpha\beta}^i(a)).$$

Let

$$e_{i_1, u_\alpha b_{\beta}} e_{j_1, 1\beta} \quad u = 1, 2, \ldots, t \quad \alpha = 1, 2, \ldots, f(i_u) \quad \beta = 1, 2, \ldots, f(j_u)$$

be the Cartan basis\(^1\) of $A$. An element $a$ of $A$, expressed in terms of the Cartan basis elements will have the form

$$a = \sum_{u, \alpha\beta} h_{u\alpha\beta}^i(a) e_{i_1, u_\alpha b_{\beta}} e_{j_1, 1\beta}.$$

For a fixed $u$, we arrange the coefficients $h_{u\alpha\beta}^i(a)$ in a matrix $H_u(a) = (h_{u\alpha\beta}^i(a))$. The additive group $H_u(a)$ is called an elementary module of $A$. In particular, for $b_i = e_{i_1, 11}$ we have $H_i(a) = F_i(a)$, that is,

$$h_{u\alpha\beta}^i(a) = f_{i\alpha\beta}^i(a) \quad (i = 1, 2 \ldots k).$$

Let $d_1, d_2, \ldots, d_n$ be a basis $(d_i)$ of $A$. Then

$$d_i = \sum_{u, \alpha\beta} h_{u\alpha\beta}^i(d_i) e_{i_1, u_\alpha b_{\beta}} e_{j_1, 1\beta}.$$

1) See Nesbitt [3], Scott [5].
or in matrix form

\[(d_i) = (e_{i_u}, a_i b_u e_{j_u}, \beta)(h_{\alpha\beta}^n(d_i))\]

\((u, \alpha, \beta\) row index: \(s\) column index). Since \((d_i)\) is a basis of \(A\), \((h_{\alpha\beta}^n(d_i))\) is a non-singular matrix. Hence we have

**Lemma 1.** If \((d_i)\) is a basis of \(A\), then \(h_{\alpha\beta}^n(d_i)\) \((u = 1, 2, \ldots, t; \alpha = 1, 2, \ldots, f(i_u); \beta = 1, 2, \ldots, f(j_u))\) are linearly independent.

In particular, we obtain from (6)

**Lemma 2.** If \((d_i)\) is a basis of \(A\), then \(f_{\alpha\beta}^i(d_i)\) \((i = 1, 2, \ldots, k; \alpha, \beta = 1, 2, \ldots, f(i))\) are linearly independent.

We denote by \(\chi_i\) the character of \(F_i\). Then \(\chi_i(a) = \sum_a f_{\alpha\beta}^i(a)\). By Lemma 2

**Theorem 1.** Let \((d_i)\) be a basis of \(A\). Then \(\chi_1(d_i), \chi_2(d_i), \ldots, \chi_k(d_i)\) are linearly independent.

Now we can prove the following theorem by a procedure similar to that of Brauer and Nesbitt\(^1\).

**Theorem 2.** Let \(M_1\) and \(M_2\) be two representations of \(A\). If both \(M_1(d_i)\) and \(M_2(d_i)\) have the same characteristic roots for every \(d_i\) of a basis \((d_i)\), then \(M_1\) and \(M_2\) have the same irreducible constituents: \(M_1 \sim M_2\).

2. In this section we assume that \(A\) is an algebra with unit element over an algebraic number field \(K\) and that the irreducible representations \(Z_1, Z_2, \ldots, Z_k\) of \(A\) in \(K\) are absolutely irreducible. Let \(J\) be a domain of integrity in the algebra \(A\) in the following sense\(^2\): (1) \(J\) is a subring of \(A\); (2) \(J\) contains \(n\) linearly independent elements of \(A\); (3) the elements of \(J\) when expressed by a basis \(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\) of \(A\) have the form \(\sum a_i \varepsilon_i\) with \(a_i = b_i / w\) where \(w\) is a fixed denominator in \(K\) and \(b_i\) are integers of \(K\); (4) \(J\) contains the ring of all integers of \(K\). Every ideal \(m\) of \(o\) generates the ideal of \(J\) which may be denoted by \(m\) again. Let \(p\) be a fixed prime ideal of \(o\). We denote by \(o^*\) the ring of all \(p\)-integers of \(K\). Then \(o^*\) and \(J\) generate a subring \(J^*\) of \(A\). \(J^*\) has a basis \(\eta_1, \eta_2, \ldots, \eta_n\) such that every \(\alpha\) of \(J^*\) can be written uniquely in the form

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1) Cf. Brauer and Nesbitt [2], p. 3.

2) See Brauer [1].
\[ \alpha = c_1 \eta_1 + c_2 \eta_2 + \cdots + c_n \eta_n, \quad \text{where } \eta_i \text{ in } \mathfrak{v}^*. \]

The \( \eta_i \) can be chosen in \( J \). The ideal \( \mathfrak{v} \) generates an ideal of \( \mathfrak{v}^* \) and an ideal of \( J^* \), both of which will be denoted by \( \mathfrak{v}^* \). We denote the residue class of an element \( \alpha \) of \( J^* \) (mod \( \mathfrak{v}^* \)) by \( \bar{\alpha} \). We have

\[ \bar{\mathfrak{v}} = \mathfrak{v}^*/\mathfrak{v}^* \cong \mathfrak{v}/\mathfrak{v} ; \quad \bar{J} = J^*/\mathfrak{v}^* \cong J/\mathfrak{v} \]

for the residue class field and residue class algebra. The elements \( \bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_n \) form a basis of \( \bar{A} \) with regard to \( \bar{\mathfrak{v}} \). Let \( S(\alpha) \) and \( \bar{R}(\alpha) \) be the left and the right regular representations of \( A \), formed by means of the basis \( (\eta_i) \). Every \( \alpha \) of \( J^* \) is then represented by matrices \( S(\alpha) \) and \( \bar{R}(\alpha) \) with coefficients in \( \mathfrak{v}^* \). Hence \( \bar{\alpha} \to S(\bar{\alpha}) \) and \( \bar{\alpha} \to \bar{R}(\bar{\alpha}) \) give the left and the right regular representations of \( \bar{A} \), formed by means of the basis \( (\bar{\eta}_i) \). We denote by \( F_1, F_2, \ldots, F_m \) the distinct absolutely irreducible representations of \( \bar{A} \). Let us assume here that all \( F_\lambda \) lie already in \( \bar{\mathfrak{v}} \). Then we have

\[ S(\alpha) \bar{R}'(\bar{\beta}) \leftrightarrow \sum_{i,j}^n c_{ij} Z_i(\alpha) \times Z_j(\beta), \]

\[ S(\bar{\alpha}) \bar{R}'(\bar{\beta}) \leftrightarrow \sum_{i,\lambda}^n \bar{c}_{i\lambda} F_i(\bar{\alpha}) \times F_\lambda(\bar{\beta}) \]

where \( c_{ij} \) and \( \bar{c}_{i\lambda} \) denote the Cartan invariants of \( A \) and \( \bar{A} \) respectively. We may assume that \( Z_i \) represents the elements of \( J^* \) by matrices with coefficients in \( \mathfrak{v}^* \). Then \( Z_i(\bar{\alpha}) \) gives a representation of \( \bar{A} \). Let \( d_{i\lambda} \) denote the multiplicity of \( F_\lambda(\bar{\alpha}) \) in \( Z_i(\bar{\alpha}) \):

\[ Z_i(\bar{\alpha}) \leftrightarrow \sum_{\lambda} d_{i\lambda} F_\lambda(\bar{\alpha}). \]

The \( d_{i\lambda} \) are called the decomposition numbers of \( A \).

**Theorem 3.** Let \( c_{ij}, \bar{c}_{i\lambda} \) be the Cartan invariants of \( A \) and \( \bar{A} \). Then

\[ c_{i\lambda} = \sum_{i,j} d_{i\lambda} c_{ij} d_{j\lambda}, \]

where \( d_{i\lambda} \) are the decomposition numbers of \( A \).

**Proof.** From (13) we have

\[ \sum_{i,j} c_{ij} Z_i(\bar{\alpha}) \times Z_j(\bar{\beta}) \leftrightarrow \sum_{i,j} c_{ij} \sum_{\lambda} d_{i\lambda} F_\lambda(\bar{\alpha}) \times (\sum_{\lambda} d_{j\lambda} F_\lambda(\bar{\beta})) = \sum_{\lambda, \lambda} \sum_{i,j} d_{i\lambda} c_{ij} d_{j\lambda} F_\lambda(\bar{\alpha}) \times F_\lambda(\bar{\beta}). \]

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1) See Osima [4].
By (11), \( S(\tilde{\alpha}) R'(\tilde{\beta}) \) and \( \sum_{i,j} c_{ij} Z_i(\tilde{\alpha}) \times Z_j(\tilde{\beta}) \) have the same characteristic roots for every \( \tilde{\alpha} \) and \( \tilde{\beta} \). Hence it follows from Theorem 2 that

\[
(14) \quad S(\tilde{\alpha}) R'(\tilde{\beta}) \leftrightarrow \sum_{i,j} c_{ij} Z_i(\tilde{\alpha}) \times Z_j(\tilde{\beta}).
\]

Consequently we have from (12)

\[
\sum_{\varepsilon, \lambda} \phi_{\varepsilon, \lambda} F_{\varepsilon}(\tilde{\alpha}) \times F_{\lambda}(\tilde{\beta}) \leftrightarrow \sum_{\varepsilon, \lambda} \left( \sum_{i,j} d_{\varepsilon, ij} c_{ij} d_{\lambda, ij} \right) F_{\varepsilon}(\tilde{\alpha}) \times F_{\lambda}(\tilde{\beta}),
\]

so that we obtain

\[
c_{\varepsilon, \lambda} = \sum_{i,j} d_{\varepsilon, ij} c_{ij} d_{\lambda, ij}.
\]

We set \( C = (c_{ij}) \), \( D = (d_{ij}) \) and \( C^* = (c_{ij}^*) \). Then

\[
(15) \quad C^* = D'CD.
\]

This shows that if \( C \) is a symmetric matrix, then \( C^* \) is also symmetric. If \( A \) is semisimple, then \( C \) is a unit matrix. Hence, from (15) we obtain

\[
(16) \quad C^* = D'D.
\]

**REFERENCES**


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*(Received January 20, 1952)*