# ON THE CARTAN INVARIANTS OF ALGEBRAS

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**1.** Let $A$ be an algebra with unit element over an algebraically closed field $K$ and let

$$A = A^* + N \tag{1}$$

be a splitting of $A$ into a direct sum of a semisimple subalgebra $A^*$ and the radical $N$ of $A$. We shall denote by

$$A^* = A_1^* + A_2^* + \cdots\cdots + A_k^* \tag{2}$$

the unique splitting of $A^*$ into a direct sum of simple invariant subalgebras. Let $e_{i,\alpha\beta}(\alpha, \beta = 1, 2, \cdots\cdots, f(i))$ be a set of matrix units for the simple algebra $A_i^*$. We denote by $F_1, F_2, \cdots\cdots, F_k$ the distinct irreducible representations of $A$ and we set for $a$ in $A$

$$F_i(a) = (f^i_{\alpha\beta}(a)). \tag{3}$$

Let

$$e_{i_u,\alpha 1}b_u e_{j_u,1\beta} \qquad \begin{array}{l} u = 1, 2, \cdots\cdots, t \\ \alpha = 1, 2, \cdots\cdots, f(i_u) \\ \beta = 1, 2, \cdots\cdots, f(j_u) \end{array} \tag{4}$$

be the Cartan basis[1] of $A$. An element $a$ of $A$, expressed in terms of the Cartan basis elements will have the form

$$a = \sum_{u,\alpha\beta} h^u_{\alpha\beta}(a) e_{i_u,\alpha 1} b_u e_{j_u,1\beta}. \tag{5}$$

For a fixed $u$, we arrange the coefficients $h^u_{\alpha\beta}(a)$ in a matrix $H_u(a) = (h^u_{\alpha\beta}(a))$. The additive group $H_u(a)$ is called an elementary module of $A$. In particular, for $b_i = e_{i,11}$ we have $H_i(a) = F_i(a)$, that is,

$$h^i_{\alpha\beta}(a) = f^i_{\alpha\beta}(a) \qquad (i = 1, 2 \cdots\cdots k). \tag{6}$$

Let $d_1, d_2, \cdots\cdots, d_n$ be a basis $(d_s)$ of $A$. Then

$$d_s = \sum_{u,\alpha\beta} h^u_{\alpha\beta}(d_s) e_{i_u,\alpha 1} b_u e_{j_u,1\beta} \tag{7}$$

1) See Nesbitt [3], Scott [5].

or in matrix form

$$(8) \qquad (d_s) = (e_{i_u,\alpha 1} b_u e_{j_u,1\beta})(h^u_{\alpha\beta}(d_s))$$

($u$, $\alpha$, $\beta$ row index: $s$ column index). Since $(d_s)$ is a basis of $A$, $(h^u_{\alpha\beta}(d_s))$ is a non-singular matrix. Hence we have

**Lemma 1.** *If $(d_s)$ is a basis of $A$, then $h^u_{\alpha\beta}(d_s)$ $(u = 1, 2, \cdots\cdots, t;$ $\alpha = 1, 2, \cdots\cdots, f(i_u);$ $\beta = 1, 2, \cdots\cdots, f(j_u))$ are linearly independent.*

In particular, we obtain from (6)

**Lemma 2.** *If $(d_s)$ is a basis of $A$, then $f^i_{\alpha\beta}(d_s)$ $(i = 1, 2, \cdots\cdots, k;$ $\alpha, \beta = 1, 2, \cdots\cdots, f(i))$ are linearly independent.*

We denote by $\chi_i$ the character of $F_i$. Then $\chi_i(a) = \sum_\alpha f^i_{\alpha\alpha}(a)$. By Lemma 2

**Theorem 1.** *Let $(d_s)$ be a basis of $A$. Then $\chi_1(d_s), \chi_2(d_s), \cdots\cdots, \chi_k(d_s)$ are linearly independent.*

Now we can prove the following theorem by a procedure similar to that of Brauer and Nesbitt[1].

**Theorem 2.** *Let $M_1$ and $M_2$ be two representations of $A$. If both $M_1(d_s)$ and $M_2(d_s)$ have the same characteristic roots for every $d_s$ of a basis $(d_s)$, then $M_1$ and $M_2$ have the same irreducible constituents: $M_1 \leftrightarrow M_2$.*

**2.** In this section we assume that $A$ is an algebra with unit element over an algebraic number field $K$ and that the irreducible representations $Z_1, Z_2, \cdots\cdots, Z_k$ of $A$ in $K$ are absolutely irreducible. Let $J$ be a domain of integrity in the algebra $A$ in the following sense[2]: (1) $J$ is a subring of $A$; (2) $J$ contains $n$ linearly independent elements of $A$; (3) the elements of $J$ when expressed by a basis $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_n$ of $A$ have the form $\sum a_i \varepsilon_i$ with $a_i = b_i / w$ where $w$ is a fixed denominator in $K$ and $b_i$ are integers of $K$; (4) $J$ contains the ring $\mathfrak{o}$ of all integers of $K$. Every ideal $\mathfrak{m}$ of $\mathfrak{o}$ generates the ideal of $J$ which may be denoted by $\mathfrak{m}$ again. Let $\mathfrak{p}$ be a fixed prime ideal of $\mathfrak{o}$. We denote by $\mathfrak{o}^*$ the ring of all $\mathfrak{p}$-integers of $K$. Then $\mathfrak{o}^*$ and $J$ generate a subring $J^*$ of $A$. $J^*$ has a basis $\eta_1, \eta_2, \cdots\cdots, \eta_n$ such that every $\alpha$ of $J^*$ can be written uniquely in the form

1) Cf. Brauer and Nesbitt [2], p. 3.
2) See Brauer [1].

$$\alpha = c_1\eta_1 + c_2\eta_2 + \cdots\cdots + c_n\eta_n, \quad c_i \text{ in } \mathfrak{o}^*. \tag{9}$$

The $\eta_i$ can be chosen in $J$. The ideal $\mathfrak{p}$ generates an ideal of $\mathfrak{o}^*$ and an ideal of $J^*$, both of which will be denoted by $\mathfrak{p}^*$. We denote the residue class of an element $\alpha$ of $J^*$ (mod $\mathfrak{p}^*$) by $\bar{\alpha}$. We have

$$\bar{\mathfrak{o}} = \mathfrak{o}^* / \mathfrak{p}^* \cong \mathfrak{o} / \mathfrak{p}; \qquad \bar{A} = J^* / \mathfrak{p}^* \cong J / \mathfrak{p} \tag{10}$$

for the residue class field and residue class algebra. The elements $\bar{\eta}_1, \bar{\eta}_2, \cdots\cdots, \bar{\eta}_n$ form a basis of $\bar{A}$ with regard to $\bar{\mathfrak{o}}$. Let $S(\alpha)$ and $R(\alpha)$ be the left and the right regular representations of $A$, formed by means of the basis $(\eta_i)$. Every $\alpha$ of $J^*$ is then represented by matrices $S(\alpha)$ and $R(\alpha)$ with coefficients in $\mathfrak{o}^*$. Hence $\bar{\alpha} \to S(\bar{\alpha})$ and $\bar{\alpha} \to R(\bar{\alpha})$ give the left and the right regular representations of $\bar{A}$, formed by means of the basis $(\bar{\eta}_i)$. We denote by $F_1, F_2, \cdots\cdots, F_m$ the distinct absolutely irreducible representations of $\bar{A}$. Let us assume here that all $F_\kappa$ lie already in $\bar{\mathfrak{o}}$. Then we have[1)]

$$S(\alpha)R'(\beta) \leftrightarrow \sum_{i,j} c_{ij}Z_i(\alpha) \times Z'_j(\beta), \tag{11}$$

$$S(\bar{\alpha})R'(\bar{\beta}) \leftrightarrow \sum_{\kappa,\lambda} c^*_{\kappa\lambda}F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}) \tag{12}$$

where $c_{ij}$ and $c^*_{\kappa\lambda}$ denote the Cartan invariants of $A$ and $\bar{A}$ respectively. We may assume that $Z_i$ represents the elements of $J^*$ by matrices with coefficients in $\mathfrak{o}^*$. Then $Z_i(\bar{\alpha})$ gives a representation of $\bar{A}$. Let $d_{i\kappa}$ denote the multiplicity of $F_\kappa(\bar{\alpha})$ in $Z_i(\bar{\alpha})$:

$$Z_i(\bar{\alpha}) \leftrightarrow \sum_\kappa d_{i\kappa}F_\kappa(\bar{\alpha}). \tag{13}$$

The $d_{i\kappa}$ are called the decomposition numbers of $A$.

**Theorem 3.** *Let $c_{ij}$, $c^*_{\kappa\lambda}$ be the Cartan invariants of $A$ and $\bar{A}$. Then*

$$c^*_{\kappa\lambda} = \sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda},$$

*where $d_{i\kappa}$ are the decomposition numbers of $A$.*

*Proof.* From (13) we have

$$\sum_{i,j} c_{ij}Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta}) \leftrightarrow \sum_{i,j} c_{ij}(\sum_\kappa d_{i\kappa}F_\kappa(\bar{\alpha})) \times (\sum_\lambda d_{j\lambda}F'_\lambda(\bar{\beta}))$$
$$= \sum_{\kappa,\lambda}(\sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda})F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}).$$

1) See Osima [4].

By (11), $S(\bar{\alpha})R'(\bar{\beta})$ and $\sum_{i,j} c_{ij}Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta})$ have the same characteristic roots for every $\bar{\alpha}$ and $\bar{\beta}$. Hence it follows from Theorem 2 that

$$S(\bar{\alpha})R'(\bar{\beta}) \leftrightarrow \sum_{i,j} c_{ij}Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta}). \tag{14}$$

Consequently we have from (12)

$$\sum_{\kappa,\lambda} c^*_{\kappa\lambda}F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}) \leftrightarrow \sum_{\kappa,\lambda}(\sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda})F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}),$$

so that we obtain

$$c^*_{\kappa\lambda} = \sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda}.$$

We set $C = (c_{ij})$, $D = (d_{i\kappa})$ and $C^* = (c^*_{\kappa\lambda})$. Then

$$C^* = D'CD. \tag{15}$$

This shows that if $C$ is a symmetric matrix, then $C^*$ is also symmetric. If $A$ is semisimple, then $C$ is a unit matrix. Hence, from (15) we obtain

$$C^* = D'D. \tag{16}$$

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