ON BICOMPACT SEMIGROUPS

KATSUMI NUMAKURA

We shall investigate in this note the structure of a minimal ideal of a bicompact semigroup, and to extend the theory of Suschkewitsch's kernel [1] (which he calls "Kerngruppe") of finite semigroups to bicompact semigroups.

If S is a bicompact semigroup, then S has a minimal ideal K, and K is completely simple in the sense of Rees [2] and is decomposed into groups which are isomorphic one another and have no element in common. Further, we shall show that minimal ideals (left, right and two-sided) of S are bicompact and closed in S. If S contains zero element, then K is zero alone, while if S has no zero, then K is a completely simple semigroup without zero.

1. A set S is called a *semigroup*, if in S a single-valued product ab is defined for every pair a, b of S such that for product the associative law holds:

$$(ab)c = a(bc).$$

By a *sub-semigroup* of S we mean a non-vacuous subset A of S with the property $A^2 \subset A$, i.e. $ab \in A$ for every a, b in A. By a *left ideal* of S we mean a non-vacuous subset L of S such that $SL \subset L$. Analogously, we can define a *right ideal* R of S. If M is both a left and a right ideal of S, then M is called a (*two-sided*) *ideal* of S.

An element e of S is called an *idempotent*, if $e^2 = e$. An element 0 is termed *zero*, if $0x = 0 = x0$ for all x in S. Then it will easily be seen that the zero of S, if it exists, is uniquely defined and is an idempotent. An element 1 is termed the *identity* of S, if $1x = x = x1$ for all x in S. Then the identity of S, if it exists, is uniquely defined, and is an idempotent.

2. If S is a semigroup and at the same time it is a topological space (in this note a topological space means a Hausdorff space), and moreover, the multiplicative operation in the semigroup S is continuous in the topological space S, then S is called a *topological semigroup*. If a sub-semigroup T of S is closed (open) in the space S, then we shall call T a *closed* (*open*) *sub-semigroup* etc.

It is clearly to be seen that a sub-semigroup T of S is itself a topological semigroup under the relative topology.

If a topological semigroup S is bicompact as a topological space, S is called a *bicompact semigroup.*

Let T_1, T_2 be two bicompact subsets of a topological semigroup S, then T_1T_2 is also bicompact. For let us consider the product space $T_1 \times T_2$, then by Tychonoff's theorem $T_1 \times T_2$ is a bicompact topological space, and T_1T_2 is a continuous image of $T_1 \times T_2$. From this it follows that T_1T_2 is bicompact. In particular, if S is bicompact, and $a \in S$, then aS, Sa, S^2, ... are all bicompact and closed in S.

3. **Lemma 1.** *Let S be a topological semigroup and B^* be a bicompact subset of S. Let $\Lambda = \{\lambda\}$ be an index system and $A = \{a_\lambda ;\ a_\lambda \in S,\ \lambda \in \Lambda\}$, $B = \{b_\lambda ;\ b_\lambda \in S,\ \lambda \in \Lambda\}$ be subsets of S whose elements correspond to the same index system Λ. Moreover, we suppose that $B \subset B^*$ and $a \in \bar{A}$. Then there exists $b \in \bar{B}$ such that $ab \in \bar{C}$, where $C = \{a_\lambda b_\lambda ;\ \lambda \in \Lambda\}$.*

Proof. We denote by $\sum_a = \{V_\tau(a) ;\ \tau \in T\}$ a complete system of neighborhoods of the element a. And put $A_\tau = V_\tau(a) \cap A$, then since $a \in \bar{A}$, A_τ is not empty. By B_τ we denote the set of elements of B, whose elements have the same indices with those of elements of A_τ. Let $\mathfrak{B} = \{B_\tau ;\ \tau \in T\}$, then $\mathfrak{B}$ is a family of subsets of B^* with the finite intersection property. For, let B_{τ_1}, ..., B_{τ_n} be any finite number of sets in $\mathfrak{B}$, and A_{τ_i}, $i = 1, \ldots\ldots, n$, are the corresponding subsets of A. Then, since $V_{\tau_i}(a)$, $i = 1, 2, \ldots\ldots, n$, are neighborhoods of the element a and $\sum_a = \{V_\tau(a) ;\ \tau \in T\}$ is a complete system of neighborhoods of a, there exists a neighborhood $V_{\tau_0}(\in \sum_a,\ \tau_0 \in T)$ of a such that $V_{\tau_0}(a) \subset \bigcap_{i=1}^{n} V_{\tau_i}(a)$. Let $A_{\tau_0} = V_{\tau_0}(a) \cap A$ and denote by B_{τ_0} the subset in $\mathfrak{B}$ which corresponds to A_{τ_0}, then it is clear that

$$\phi \neq B_{\tau_0} \subset \bigcap_{i=1}^{n} B_{\tau_i}.$$

Thus, $\mathfrak{B}$ has the finite intersection property. And since B^* is bicompact we have $\bigcap_{\tau \in T} \bar{B}_\tau \neq \phi$. Let $b \in \bigcap_{\tau \in T} \bar{B}_\tau$ and $V(ab)$ any neighborhood of ab, then there exist neighborhoods $V_\kappa(a)$ of a and $V(b)$ of b such that $V_\kappa(a)\, V(b) \subset V(ab)$, where $V_\kappa(a) \in \sum_a$, $\kappa \in T$. Since $b \in \bigcap_{\tau \in T} \bar{B}_\tau$, $b \in \bar{B}_\kappa$ and $V(b) \cap B_\kappa \neq \phi$. Let b_{κ_0} be any element of $V(b) \cap B_\kappa$, then, since

$b_{\kappa_0} \in B_\kappa$, there exists an element a_{κ_0} such that $a_{\kappa_0} \in A_{\kappa_0} = V_\kappa(a) \cap A$. Hence $a_{\kappa_0} b_{\kappa_0} \in V_\kappa(a)\, V(b) \subset V(ab)$. On the other hand $a_{\kappa_0} b_{\kappa_0} \in C$. Hence $V(ab) \cap C \neq \phi$. Thus, $ab \in \bar{C}$.

In a semigroup S, if $ax = ay\,(xa = ya)$ implies $x = y$ for every a, x, y in S, then S is called a semigroup *satisfying the left (right) cancellation law*. S is called a semigroup *satisfying the cancellation law*, if it satisfies both the left and the right cancellation law.

Lemma 2L. *Let S be a bicompact semigroup satisfying the left cancellation law, and B be a closed subset of S. If $p \in S$, $pB \subset B$, then $pB = B$.*

Proof. From the assumption, we have $B \supset pB \supset p^2B \supset \cdots\cdots$. Put $P_\nu = \{p^i;\ i \geqq \nu\}$ and $\mathfrak{P} = \{P_\nu;\ \nu = 1, 2, \cdots\cdots\}$, then it is clear that $\mathfrak{P}$ is a family of subsets of a bicompact space S with the finite intersection property. Hence

$$\bigcap_{\nu=1}^{\infty} \bar{P}_\nu \neq \phi.$$

Let q be any element of $\bigcap_{\nu=1}^{\infty} \bar{P}_\nu$. Then we shall show that $\bigcap_{i=1}^{\infty} p^iB = qB$. Let $qx\,(x \in B)$ any element of qB and $V(qx)$ be an arbitrary neighborhood of qx, then there exists a neighborhood $V(q)$ of q such that $V(q)x \subset V(qx)$. Since $q \in \bigcap_{\nu=1}^{\infty} \bar{P}_\nu$, $V(q) \cap P_\nu \neq \phi$ for $\nu = 1, 2, \cdots\cdots$. Therefore, an integer i_ν exists so that $p^{\nu+i_\nu} \in V(q)$ for $\nu = 1, 2, \cdots\cdots$. Hence, $p^{\nu+i_\nu}x \in V(q)x \in V(qx)$. On the other hand, $p^{\nu+i_\nu}x \in p^{\nu+i_\nu}B \subset p^\nu B$. Hence $V(qx) \cap p^\nu B \neq \phi$ for $\nu = 1, 2, \cdots\cdots$, and so $qx \in \overline{p^\nu B} = p^\nu B$ for $\nu = 1, 2, \cdots\cdots$. This shows that $qB \subset \bigcap_{i=1}^{\infty} p^iB$.

Conversely, if p' be any element of $\bigcap_{i=1}^{\infty} p^iB$, then p' can be written in the form $p' = p^ib_i$, $i = 1, 2, \cdots\cdots$, where $b_i \in B$. Now, let $B' = \{b_i;\ i = 1, 2, \cdots\cdots\}$ and $P = \{p^n;\ n = 1, 2, \cdots\cdots\}$, then by Lemma 1 there exists an element $b \in \bar{B}' \subset B$ such that $qb \in \overline{\{p^ib_i;\ i = 1, 2, \cdots\cdots\}} = \bar{p}' = p'$. Thus $p' = qb \in qB$. This shows that $\bigcap_{i=1}^{\infty} p^iB \subset qB$. And we have $\bigcap_{i=1} p^iB = qB$.

Analogously, if we replace B by pB, we can conclude $\bigcap_{i=1}^{\infty} p^i(pB) = q(pB)$. Since $\bigcap_{i=1}^{\infty} p^iB = \bigcap_{i=1}^{\infty} p^i(pB)$, we have $qB = qpB$. Applying the left cancellation law, if follows that $pB = B$. This proves Lemma 2L.

Similarly, we obtain the following two lemmas:

Lemma 2R. *Let S be a bicompact semigroup satisfying the right cancellation law, and B be a closed subset of S. If $p \in S$, $Bp \subset B$, then $Bp = B$.*

Lemma 2. *Let S be a bicompact semigroup satisfying the cancellation law, and B be a closed subset of S. If $p \in S$, $Bp \subset B$ and $pB \subset B$, then $Bp = B = pB$.*

From Lemma 2 it follows

Theorem 1. *A bicompact semigroup satisfying the cancellation law is a group.*

4. **Lemma 3.** *Let S be a bicompact semigroup and a an element of S and let $A = \{a^n;\ n = 1, 2, \cdots\cdots\}$. Then $\bar{A}$ contains a commutative closed group D.*

Proof. Let $A_\nu = \{a^i;\ i \geqq \nu\}$ and $\mathfrak{A} = \{A_\nu;\ \nu = 1, 2, \cdots\cdots\}$. Then in the same way with Lemma 2L $D = \bigcap_{\nu=1}^{\infty} \bar{A}_\nu \neq \phi$.

Now we shall show that D is a commutative closed group. It is easy to see that D is a commutative closed sub-semigroup of S. It remains to show that D forms a group. To prove this it is sufficient to show that $xD = D$ for all x in D. Suppose that there exists y in D such that $yD \subsetneqq D$, then there is z in D so that $z \bar{\in} yD$, that is, $z \neq yx_\lambda$ for every x_λ in D. Therefore, there exist neighborhoods $V_\lambda(y)$ of y, $V(x_\lambda)$ of x_λ and $V_\lambda(z)$ of z such that $V_\lambda(z) \cap V_\lambda(y) V(x_\lambda) = \phi$. Since $\bigcup_{x_\lambda \in D} V(x_\lambda) \supset D$ and D is bicompact as a closed subset of a bicompact space S, we can choose a finite covering $V(x_{\lambda_i})$ $(i = 1, 2, \cdots\cdots, k)$ of D, i.e. $\bigcup_{i=1}^{k} V(x_{\lambda_i}) \supset D$. Let $V(y)$, $V(z)$ be neighborhoods of y, z, respectively, such that $V(y) \subset \bigcap_{i=1}^{k} V_{\lambda_i}(y)$, $V(z) \subset \bigcap_{i=1}^{k} V_{\lambda_i}(z)$, and put $\bigcup_{i=1}^{k} V(x_{\nu_i}) = Q$, then Q is an open set containg D, and

$$V(z) \cap (V(y)Q) = \phi.$$

Since $y \in D$, there is an integer $\mu \geqq 1$ so that $a^\mu \in V(y)$, and since $z \in D$, there exist integers ν_i such that $\nu_i > \mu$, $\nu_{i+1} > \nu_i$ $(i = 1, 2, \cdots\cdots)$ and $a^{\nu_i} \in V(z)$ for every ν_i. Putting $\tau_i = \nu_i - \mu \geqq 1$ and $A^{(\tau_t)} = \{a^{\tau_j};\ j = t, t+1, \cdots\cdots\}$, then, by the above method, $\bigcap_{t=1}^{\infty} \overline{A^{(\tau_t)}} \neq \phi$. And it is easily shown that $\bigcap_{t=1}^{\infty} \overline{A^{(\tau_t)}} \subset D$. Choose an element u from $\bigcap_{t=1}^{\infty} \overline{A^{(\tau_t)}}$, then since Q is an open set containg D, there exists a neighborhood $V(u)$ of u contained in Q, and since $u \in \bigcap_{t=1}^{\infty} \overline{A^{(\tau_t)}}$, there exists an integer τ_k so that $a^{\tau_k} \in V(u)$. Then, $a^{\nu_k} \in V(z)$ and $a^{\nu_k} = a^{\mu+\tau_k} = a^{\mu} a^{\tau_k} \in V(y) V(u) \subset V(y)Q$. This contradicts to $V(z) \cap V(y)Q = \phi$. Hence, we obtain $xD = D$ for all x in D, and D is a group.

As a consequence of Lemma 3 we have

Lemma 4. *Every bicompact semigroup has at least one idempotent.*

5. If a semigroup S contains no proper ideal at all, it is called a *simple* semigroup. An idempotent f is said to be *under* another one e if $ef = f = fe$. An idempotent e is *primitive* if there is no non-zero idempotent under e. A simple semigroup S is said to be *completely simple* if every idempotent element of S is primitive, and for each $a \in S$ there exist idempotents e and f such that $ea = a = af$.

Lemma 5. *A necessary and sufficient condition for semigroup S to be simple is that $SxS = S$ for all x of S.*

Lemma 6. *If S is a simple semigroup and e is an idempotent of S, then eSe is also a simple semigroup.*

The above two lemmas can be proved in the same way with [2].

Lemma 7. *A bicompact simple semigroup S is completely simple.*

Proof. Let a be any element of S. Then, since S is simple, there exist b, c such that $bac = a$. Then, by simple induction, $b^n ac^n = a$ for all integers n. Let $B = \{b^n;\ n = 1, 2, \cdots\cdots\}$, $B_\nu = \{b^i;\ i \geqq \nu\}$ and $D_b = \bigcap_{\nu=1}^{\infty} \overline{B_\nu}$, then by Lemma 3, D_b is a commutative closed sub-group of S. Analogously, let $C = \{c^n;\ n = 1, 2, \cdots\cdots\}$, $C_\nu = \{c^i;\ i \geqq \nu\}$ and $D_c = \bigcap_{\nu=1}^{\infty} \overline{C_\nu}$, then D_c is also a commutative closed sub-group of S. We denote by e_b and e_c the identities of the groups D_b and D_c, respectively. Then, we shall show first that $e_b ac' = a$, where $c' \in \overline{C}$.

Let $H = \{b^n a;\ n = 1, 2, \cdots\cdots\}$. In Lemma 1, if we replace A by H, B by C, Λ by $\{1, 2, \cdots\cdots\}$, B^* by S and a by $e_b a$, then it is easy to see that all the conditions of Lemma 1 are satisfied for H, C and $e_b a$. Hence from Lemma 1 we can conclude that there exists $c' \in \overline{C}$ such that $e_b ac' \in \overline{\{b^n ac^n;\ n = 1, 2, \cdots\cdots\}} = \overline{\{a\}} = a$. This shows that a can be written in the form $a = e_b ac'$, where $c' \in \overline{C}$. Analogously, a can be written in the form $a = b'ae_c$, where $b' \in \overline{B}$. Hence

$$e_b a = e_b(e_b ac') = (e_b e_b)(ac') = e_b ac' = a$$
$$ae_c = (b'ae_c)e_c = (b'a)(e_c e_c) = b'ae_c = a$$

and e_b, e_c are idempotent.

Let e be any idempotent of S, and f be an idempotent under e. Then $f = efe \in eSe$, and by Lemma 6 eSe is simple, so that there exist x, x' in eSe such that $xfx' = e$. Putting $xf = y$ and $fx' = y'$, we obtain $yfy' = e$, $yf = y$ and $fy' = y'$. Then $yy' = yfy' = e$. Then, by induction, since $yfy' = e$, we have $y^n fy'^n = e$ for all integers n. Then as above proved, we can choose idempotents g, h in eSe such that $gfh' = e$ and $g'fh = e$, where g' and h' are contained in $\overline{\{y^n;\ n = 1, 2, \cdots\cdots\}}$ and $\overline{\{y'^n;\ n = 1, 2, \cdots\cdots\}}$, respectively. Then, since e is the identity of eSe, we have

$$g = ge = g \cdot gfh' = gfh' = e,$$
$$fh' = e \cdot fh' = gfh' = e,$$

henceforth

$$f = f \cdot e = f \cdot fh' = fh' = e.$$

Thus, e is the only idempotent contained in eSe, and e must be primitive. (In the latter half of the proof of this lemma, we limited ourselves in a bicompact semigroup eSe). Hence S is completely simple.

Theorem 2. *A bicompact semigroup S has the unique minimal two-sided ideal K which is completely simple and bicompact.*

Proof. Let E be the set of all idempotents in S, then by Lemma 4, E is not empty. We denote by e_λ, e_μ, $\cdots\cdots$ the elements of E. Then, it is clear that the set $K = \bigcap_{e_\lambda \in E} Se_\lambda S$ is a closed bicompact ideal of S, if $K \neq \phi$. Now, let $Se_{\lambda_i}S$ $(i = 1, 2, \cdots\cdots, m)$ be any finite number of subsets in the family $\{Se_\lambda S;\ e_\lambda \in E\}$. Then each $Se_{\lambda_j}S$ $(1 \leqq j \leqq m)$ contains the element

$$e_{\lambda_1} e_{\lambda_2} \cdots\cdots e_{\lambda_m} = (e_{\lambda_1} \cdots\cdots e_{\lambda_{j-1}}) e_{\lambda_j} (e_{\lambda_{j+1}} \cdots\cdots e_{\lambda_m})$$

so that $\bigcap_{i=1}^{m} Se_{\lambda_i}S \neq \phi$. Thus, $\{Se_\lambda S;\ e_\lambda \in E\}$ is a family of subsets of a bicompact space S with the finite intersection property. Since each $Se_\lambda S$ is closed, it holds

$$K = \bigcap_{e_\lambda \in E} Se_\lambda S \neq \phi.$$

Suppose that K' is a bicompact ideal of S. Then, by Lemma 4, K' contains an idempotent e' and it follows easily that

$$K \subset Se'S \subset K'.$$

If M is a minimal ideal of S then SaS is a bicompact ideal of S contained in M where a is an element M. SaS being bicompact, it follows

$$K \subset SaS \subset M,$$

Therefore, $K = M$ because M is a minimal ideal of S.

Now, let x be any element of K. Then, KxK is obviously a bicompact ideal of S contained in K. On the other hand, KxK must contain K. Hence, $KxK = K$. This shows that K is simple and consequently, it may contain no other ideal of S than itself. Therefore K is the minimal ideal of S.

In the following, the minimal ideal K which is completely simple and bicompact is called the "kernel (Suschkewitsch's Kerngruppe)" of S.

Since the kernel K of S is a bicompact semigroup it contains an idempotent. One can easily see that for any idempotent e of K the relation

$$SeS = K.$$

Especially, if S contains 0 then, by definition, K contains also 0. Since 0 is an idempotent, it follows immediately

$$K = S0S = 0.$$

6. **Lemma 8.** *A completely simple semigroup S with the identity 1 is a group.*

Proof. Let x be an element of S. Then, since S is simple, there exist elements b, c such that $bxc = 1$. Then, xcb, cbx are idempotents. But, by the definition of a completely simple semigroup, 1 is a primitive idempotent and so must be the only idempotent of S. Hence $xcb = cbx = 1$, and every element x has the inverse $x^{-1} = cb$. Therefore, S is a group.

Lemma 9. *Let K be the kernel of a bicompact semigroup S. Then, for any idempotent e of K, eK and Ke satisfy the left and right* *cancellation law respectively. Moreover, for arbitrary idempotents e, f, eKf is a group.*

Proof. Let x_1, x_2, y be elements of eK, and assume that

$$yx_1 = yx_2.$$

Then, we can determine the elements k_1, k_2, k of K such that $x_1 = ek_1$, $x_2 = ek_2$ and $y = ek$. Since eKe is a completely simple semigroup with e as the identity, then eKe is a group. Therefore, the element eke has the inverse element $(eke)^{-1}$ in eKe. From the relation $yx_1 = yx_2$, it follows immediately that

$$x_1 = ek_1 = (eke)^{-1}(ekek_1) = (eke)^{-1}(ekek_2) = ek_2 = x_2.$$

This shows that the left cancellation law holds in eK. Similarly, we may prove that in Ke the right cancellation law holds.

Now, it is not hard to show that eKf satisfies the left and right cancellation law. Since eKf is a bicompact semigroup, it must be a group by Theorem 1.

Theorem 3. *Let K be the kernel of a bicompact semigroup S, then K is decomposed into join of groups which have no element in common.*

Proof. Let E' be the set of all idempotents in K, and we denote by $G_{\lambda\mu}$ a group $e_\lambda K e_\mu$, where e_λ, e_μ belong to E'. Since K is a completely simple semigroup, then by the definition of a completely simple semigroup, every a of K is contained in one of the groups $G_{\lambda\mu}$. Hence, it follows that

$$K = \bigcup_{\lambda, \mu} G_{\lambda\mu}.$$

If two groups $G_{\lambda\mu}$ and $G_{\kappa\tau}$ have an element c in common, then

$$e_1 c = c = e_2 c,$$

where e_1, e_2 are the identities of $G_{\lambda\mu}$, $G_{\kappa\tau}$ respectively. By multiplication with the inverse c_1^{-1} of c in the group $G_{\lambda\mu}$ from the right side, we have

$$e_1 = e_2 e_1,$$

Analogously,

$$e_2 = e_2 e_1.$$

Henceforth, $e_1 = e_2 = e$.

Then, $G_{\lambda\mu} = eG_{\lambda\mu}e \subset eKe = e_\lambda e_\mu(eKe)e_\lambda e_\mu \subset e_\lambda K e_\mu = G_{\lambda\mu}(e_\lambda e_\mu \in eKe$ and eKe is a group), hence $G_{\lambda\mu} = eKe$. Similarly, $G_{\kappa\tau} = eKe$. Thus, $G_{\lambda\mu} = eKe = G_{\kappa\tau}$. Hence, either $G_{\lambda\mu} = G_{\kappa\tau}$ or $G_{\lambda\mu} \cap G_{\kappa\tau} = \phi$.

Theorem 4L. *Let K be the kernel of a bicompact semigroup S, then*

(1) *$L = Ke$ is a minimal left ideal of S, where e is an idempotent in K.*

(2) *every minimal left ideal L of S can be expressed in the form $L = Ke$, where e is an idempotent in K.*

Proof. (1) Since L is bicompact and satisfies the right cancellation law by Lemma 9, then we obtain from Lemma 2R $Lp = L$ for every p in L.

Now, L' be a left ideal of S contained in L, then $L = Lp \subset L'$ for $p \in L'$, and then $L = L'$. Thus, L is minimal.

(2) Let L be a minimal left ideal of S. Then for every element a of L, Ka is a left ideal of S contained in L so that $L = Ka$. Henceforth, L is bicompact, and by Lemma 4, L has an idempotent e. Thus, $L = Ke$.

Analogously, we have

Theorem 4R. *Let K be the kernel of a bicompact semigroup S, then*

(1) *$R = eK$ is a minimal right ideal of S, where e is an idempotent in K.*

(2) *every minimal right ideal R of S can be expressed in the form $R = eK$, where e is an idempotent in K.*

Corollary. *A bicompact semigroup has at least one left and one right minimal ideals.*

Corollary. *Every minimal ideal (left, right and two-sided) of a bicompact semigroup is closed and bicompact.*

Theorem 5. *Let R and L be a right and a left minimal ideals of a bicompact semigroup S, respectively, and K be the kernel of S. Then $LR = K$ and RL is a group.*

This theorem is clear, by Theorems 4R and 4L.

Lemma 10L. *If e and f are two idempotents in K, then either $Ke = Kf$ or $Ke \cap Kf = \phi$.*

By Theorem 4L, Ke, Kf are minimal left ideals of S. Hence, it is easy to see that $Ke = Kf$ or $Ke \cap Kf = \phi$.

Similarly, we have

Lemma 10R. *If e and f are two idempotents in K then either $eK = fK$ or $eK \cap fK = \phi$.*

Theorem 6. *The kernel K of a bicompact semigroup S is the set theoretical join of all minimal left (or right) ideals of S.*

Proof. By Theorem 4L, K contains all minimal left (right) ideals of S. Now, let a be an element of K. Then by Theorem 3, there exist idempotents e_λ, e_μ of K such that $e_\mu K e_\lambda$ contains a. Since $e_\mu K e_\lambda \subset K e_\lambda$, then by Theorem 4L, Ke_λ is a minimal left ideal of S containing a. Thus, K is the set theoretical join of all minimal left ideals of S.

Similarly, one can prove that K is the set theoretical join of all minimal right ideals of S.

Theorem 7. *The groups $G_{\lambda\mu} = e_\lambda K e_\mu$ are isomorphic one another.*

Proof. By Lemmas 10L, 10R and Theorem 6, each group $G_{\lambda\mu}$ is contained in one and only one minimal left ideal Ke_μ and right ideal $e_\lambda K$. Then the idempotents in Ke_μ are right identities of $G_{\lambda\mu}$ and the idempotents in $e_\lambda K$ are left identities of $G_{\lambda\mu}$, and isomorphisms of the groups $G_{\lambda\mu}$ can be established as the same way with that of Suschkewitsch.

REFERENCES.

[1] A. SUSCHKEWITSCH, Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit, Math. Ann., 99 (1928), S. 30–50.

[2] D. REES, On Semi-groups, Proc. of the Camb. Phil. Soc., Vol. 36 (1940), pp. 387–400.

DEPARTMENT OF MATHEMATICS,
HOKKAIDO UNIVERSITY

(*Received January 15, 1951*)