ON THE SPACES WITH NORMAL PROJECTIVE CONNEXIONS AND SOME IMBEDDING PROBLEM OF RIEMANNIAN SPACES

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Introduction.

In a previous paper[1], the author obtained a theorem as follows: *If the group of holonomy of a space with a normal projective connexion fixes a hyperquadric, the space is one corresponding to the class of Riemannian spaces projective to each other including an Einstein space,* in the domain of such points that they are not the images of the points on the hyperquadric into the space. Such a space has also been studied by S. Sasaki and K. Yano[2], but the properties in the neighborhood of the image of the hyperquadric invariant under the group of holonomy of the space into it have never been investigated by any one. One of the most interesting problems which arrises in connection with this space is the determination of the conditions under which a given Riemannian space V_n can be imbedded into a Riemannian space V_{n+1} as a hypersurface F_n such that the group of holonomy of the space with a normal projective connexion corresponding to V_{n+1} fixes a hyperquadric Q_n and the image of Q_n into V_{n+1} is F_n.

On the other hand, *any Riemannian space V_n can be imbedded into a suitable Einstein space A_{n+1}* whose scalar curvature is a given constant (from now on, we call this *Campbell's theorem*)[3]. As regards the signification of the theorem by means of the groups of holonomy, the author has obtained some results in connection with the space with a normal conformal connexion[4].

1) T. Ōtsuki, On projectively connected spaces whose groups of holonomy fix a hyperquadric, Jour. of the Math. Soc. of Japan, Vol. 1, No. 4, 1950, pp. 251—263.

2) S. Sasaki and K. Yano, On the structure of spaces with normal projective connexions whose groups of holonomy fix a hyperquadric, Tōhoku Math. Jour., 2nd Se., Vol. 1, No. 1, 1949, pp. 31—39.

3) J. E. Campbell, A course of differential geometry, 1926.

4) T. Ōtsuki, On the spaces with normal conformal connexions and some imbedding problem of Riemannian spaces, I, Tōhoku Math. Jour., 2nd Se., Vol. 1, No. 2, 1950, pp. 194—224.

In the present paper, we shall investigate the same problem by means of the groups of holonomy of the spaces with normal projective connecxions, considering the above-mentioned imbedding problem of V_n into V_{n+1}.

§1. Fundamental equations.

Let there be given a space with a normal projective connexion X_n corresponding to a given Riemannian space V_n with positive definite line element

$$ds^2 = g_{ij}(x)dx^i dx^j \qquad (i, j = 1, 2, \cdots\cdots, n)^{5)} \tag{1,1}$$

in each of its coordinate neighborhoods. If we take suitable semi-natural frames $R(A, A_i)$, the projective connexion of the space X_n is given, as is well known, by means of Christoffel symbols Γ^j_{ik} made by g_{ij} by the following equations:

$$dA = dx^i A_i, \qquad dA_i = \omega^0_i A + \omega^j_i A_j, \tag{1,2}$$

where

$$\omega^j_i = \Gamma^j_{ik} dx^k, \qquad \omega^0_i = \Gamma^0_{ij} dx^j = -\frac{1}{n-1} K_{ij} dx^j \tag{1,3}$$

and we put

$$\begin{cases} K^i_{j\,hk} = \dfrac{\partial \Gamma^i_{jh}}{\partial x^k} - \dfrac{\partial \Gamma^i_{jk}}{\partial x^h} + \Gamma^m_{jh}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{mh}, \\ K_{jh} = K^k_{j\,hk}, \qquad K = g^{ij}K_{ij}. \end{cases} \tag{1,4}$$

If the group of holonomy H of the given space fixes a hyperquadric Q_{n-1} (we may not always assume that Q_{n-1} is non-degerate), Q_{n-1} can be represented in the tangent projective space of each point A with respect to the natural frame by

$$Q_{n-1} \ : \ G_{\lambda\mu} z^\lambda z^\mu = 0 \qquad (\lambda, \mu = 0, 1, 2, \cdots\cdots, n),$$

where $G_{\lambda\mu}$ satisfies the relation[6)]

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5) E. Cartan, Leçon sur la théorie des espaces à connexion projective, Paris, Gauthier-Villars, 1937.

6) See Note 1), pp. 251—255.

$$
(1,5)\qquad \begin{cases} \dfrac{\partial}{\partial x^k} G_{00} - 2G_{k0} = 0, \\ \dfrac{\partial}{\partial x^k} G_{i0} - \Pi^{\rho}_{ik} G_{\rho 0} - G_{ik} = 0, \\ \dfrac{\partial}{\partial x^k} G_{ij} - \Pi^{\rho}_{ik} G_{\rho j} - \Pi^{\rho}_{jk} G_{i\rho} = 0 \end{cases}
$$

and

$$
(1,6)\qquad \begin{cases} \Pi^i_{jk} = \Gamma^i_{jk} + \delta^i_j \tau_k + \delta^i_k \tau_j, \\ \Pi^0_{jk} = -\dfrac{1}{n-1} K_{jk} + \tau_{j;k} - \tau_j \tau_k, \\ \tau = -\dfrac{1}{2(n+1)} \log g, \qquad \tau_j = -\dfrac{\partial \tau}{\partial x^j}, \\ \tau_{j;k} = \dfrac{\partial \tau_j}{\partial x^k} - \Gamma^h_{jh} \tau_h. \end{cases}
$$

We have obtained (1,5) at the points where $G_{00} \neq 0$, that is, at the points which do not belong to the surface of image F_{n-1} of Q_{n-1} into X_n. But, by virtue of the continuity of $G_{\lambda\mu}$, we may consider that (1,5) is satisfied at each point of X_n. Conversely, if (1,5) is integrable, the group of holonomy of X_n fixes a hyperquadric Q_{n-1}.

Now, in order to represent (1,5) by the quantities of V_n, let us put

$$
(1,7)\qquad G_{00} = 2\varphi, \qquad G_{i0} = p_i
$$

and putting the relation and (1,6) into (1,5), we get

$$
(1,5')\qquad \begin{cases} \dfrac{\partial \varphi}{\partial x^i} = p_i \\ p_{i;j} = p_i \tau_j + p_j \tau_i - \dfrac{2}{n-1} \varphi K_{ij} + 2\varphi(\tau_{i;j} - \tau_i \tau_j) + G_{ij}, \\ G_{ij;k} = 2\tau_k G_{ij} + \tau_j G_{ik} + \tau_i G_{kj} \\ \qquad + p_i \Big(-\dfrac{1}{n-1} K_{kj} + \tau_{k;j} - \tau_k \tau_j \Big) \\ \qquad + p_j \Big(-\dfrac{1}{n-1} K_{ik} + \tau_{i;k} - \tau_i \tau_k \Big) \end{cases}
$$

where a semicolon ";" denotes the covariant differentiation of V_n.

Now, let us consider a change of coordinate system: $(x) \rightarrow (\bar{x})$, and $R(A, A_i)$ and $\bar{R}(\bar{A}, \bar{A}_i)$ be the natural frames in the coordinate systems respectively. Putting

$$\bar{A}_\alpha = a^\mu_\alpha A_\mu, \qquad a^i_0 = 0, \qquad a^0_0 = \rho,$$
$$A_\mu = b^\alpha_\mu \bar{A}_\alpha \quad (A_0 = A),$$

and

$$dA_\lambda = \omega^\mu_\lambda A_\mu, \qquad d\bar{A}_\alpha = \bar{\omega}^\beta_\alpha A_\beta,$$

where ω^μ_λ are different from those of (1,3), we get

$$d\bar{A} = d\rho A + \rho dx^i A_i$$
$$= d \log \rho \bar{A} + \rho \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x} A_i,$$

and hence

$$a^i_j = \rho \frac{\partial x^i}{\partial \bar{x}^j}.$$

Accordingly, we get

$$b^0_0 = \rho^{-1}, \qquad b^j_i = \rho^{-1} \frac{\partial \bar{x}^j}{\partial x^i}, \qquad b^i_0 = 0,$$
$$b^0_i = -\rho^{-2} \frac{\partial \bar{x}^j}{\partial x^i} a^0_j$$

and

$$\bar{\omega}^0_0 = d \log \rho - \rho^{-1} a^0_j d\bar{x}^j.$$

We get likewise the relation

$$\bar{\omega}^i_i = da^\mu_i b^i_\mu + a^\mu_i \omega^\lambda_\mu b^i_\lambda$$
$$= nd \log \rho + d\left(\frac{\partial x^j}{\partial \bar{x}^i}\right) \frac{\partial \bar{x}^i}{\partial x^j} + \omega^i_i - \rho b^0_i dx^i,$$

and hence

$$\bar{\omega}^i_i = nd \log \rho + \rho^{-1} a^0_i d\bar{x}^i + d\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \frac{\partial \bar{x}^i}{\partial x^j}$$

since $\omega^i_i = 0$. Accordingly, from the relations above and that $\bar{R}(\bar{A}, \bar{A}_i)$ is the natural frame, we obtain

$$0 = \bar{\omega}^i_i - n\bar{\omega}^0_0 = (n+1)\rho^{-1} a^0_j d\bar{x} + d\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \frac{\partial \bar{x}^j}{\partial x^i},$$

from which we get the relation

$$a^0_i = -\frac{\rho}{n+1} \frac{\partial}{\partial \bar{x}^i} \log \left| \frac{\partial x^k}{\partial \bar{x}^h} \right|.$$

Furthermore, putting $\bar{\omega}_0^0 = 0$, we get

$$d \log \rho = \rho^{-1} a_j^0 d\bar{x}^j = -\frac{1}{n+1} d \log \left| \frac{\partial x^k}{\partial \bar{x}^h} \right|,$$

that is

$$\rho = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|^{-\frac{1}{n+1}} \tag{1,8}$$

Now, in the coordinate system $(\bar{x})$, we have

$$\bar{G}_{\alpha\beta} = a_\alpha^\lambda a_\beta^\mu G_{\lambda\mu},$$

especially

$$\bar{G}_{00} = \rho^2 G_{00} = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|^{-\frac{2}{n+1}} G_{00}.$$

Accordingly, $|g_{ij}|^{\frac{1}{n+1}} G_{00}$ is a scalar. Let us denote this by $y(x)$. Then, we get from (1,5′) the relations

$$\frac{1}{2} G_{00} = \varphi = g^{-\frac{1}{n+1}} y(x), \tag{1,9}$$

$$G_{i0} = p_i = g^{-\frac{1}{n+1}} (q_i + 2y\tau_i) \tag{1,10}$$

and

$$\begin{aligned} g^{\frac{1}{n+1}} p_{i;j} &\equiv q_{i;j} + 2y\tau_{i;j} + 2q_j\tau_i + 2\tau_j(q_i + 2y\tau_i) \\ &= (q_i + 2y\tau_i)\tau_j + (q_j + 2y\tau_j)\tau_i \\ &\quad + 2y(\tau_{i;j} - \tau_i\tau_j) - \frac{2y}{n-1} K_{ij} + g^{\frac{1}{n+1}} G_{ij}, \end{aligned}$$

where $g = |g_{ij}|$ and

$$q_i = \frac{\partial y}{\partial x^i}. \tag{1,11}$$

Now, if we define a tensor by

$$T_{ij} = g^{\frac{1}{n+1}} G_{ij} - q_i\tau_j - q_j\tau_i - 2y\tau_i\tau_j, \tag{1,12}$$

which is a symmetric covariant tensor of V_n, then the last relation becomes

$$q_{i;j} = -\frac{2y}{n-1} K_{ij} + T_{ij}. \tag{1,13}$$

Putting (1,10), (1,12) into the last relation of (1,5′), we get

$$g^{\frac{1}{n+1}} G_{ij;k} \equiv T_{ij;k} + q_{i;k}\tau_j + q_i\tau_{j;k} + q_{j;k}\tau_i + q_j\tau_{i;k}$$
$$+ 2y(\tau_{i;k} + \tau_i\tau_{j;k}) + 2q_k\tau_i\tau_j + 2g^{\frac{1}{n+1}}\tau_k G_{ij}$$
$$= 2g^{\frac{1}{n+1}}\tau_k G_{ij} + \tau_i\{T_{kj} + q_k\tau_j + \tau_k(q_j + 2y\tau_j)$$
$$+ \tau_j\{T_{ik} + q_k\tau_i + \tau_k(q_i + 2y\tau_i)\}$$
$$+ (q_i + 2y\tau_i)\left(-\frac{1}{n-1}K_{kj} + \tau_{k;j} - \tau_k\tau_j\right)$$
$$+ (q_j + 2y\tau_j)\left(-\frac{1}{n-1}K_{ik} + \tau_{i;k} - \tau_i\tau_k\right),$$

that is

$$T_{ij;k} = -\frac{1}{n-1}(q_i K_{kj} + q_j K_{ik})$$
$$- \tau_i\left(q_{j;k} + \frac{2y}{n-1}K_{kj} - T_{kj}\right) - \tau_j\left(q_{i;k} + \frac{2y}{n-1}K_{ik} - T_{ik}\right).$$

Hence, using the relation (1,13), the last one becomes

$$T_{ij;k} = -\frac{1}{n-1}(q_i T_{kj} + q_j T_{ik})$$

Thus, we see that the fundamental equations (1,5) characterising the space can be represented by means of the quantities of the Riemannian space V_n as follows:

$$\left.\begin{array}{lll} (1,11) & \dfrac{\partial y}{\partial x^i} = q_i, & \\ (1,13) & q_{i;j} = -\dfrac{2y}{n-1}K_{ij} + T_{ij}, & \\ (1,14) & T_{ij;k} = -\dfrac{1}{n-1}(q_i T_{kj} + q_j T_{ik}) & \end{array}\right\} \quad (\alpha)$$

and $G_{\lambda\mu}$ are determined by y, q_i, T_{ij} so that

$$\left.\begin{array}{ll} (1,9) & G_{00} = 2g^{-\frac{1}{n+1}}y, \\ (1,10) & G_{i0} = G_{0i} = g^{-\frac{1}{n+1}}(q_i + 2y\tau_i), \\ (1,12) & G_{ij} = g^{-\frac{1}{n+1}}(T_{ij} + q_i\tau_j + q_j\tau_i + 2y\tau_i\tau_j), \end{array}\right\} \quad (\beta)$$

where

$$\tau = -\frac{1}{2(n+1)}\log g, \qquad \tau_i = \frac{\partial \tau}{\partial x^i}.$$

Consequently, we get the following

Theorem 1. *In order that the group of holonomy of the space with a normal projective connexion corresponding to a given Riemannian space V_n fixes a hyperquadric, that the system of equations (α) is integrable is necessary and sufficient.*

§2. Relations between the spaces in which (α) is integrable and the Einstein spaces.

In this paragragh, we shall give a proof of the first theorem described in Induction.

For a given Riemannian space V_n, let the system of equations (α) be integrable. On the region of points where $y(x) \neq 0$, let us consider the following tensor

$$\bar{g}_{ij} = \frac{1}{2y} T_{ij} - \frac{1}{4y^2} q_i q_j. \tag{2,1}$$

Then, by means of (α), we get

$$\begin{aligned} \bar{g}_{ij;k} &= \frac{1}{2y} T_{ij;k} - \frac{1}{2y^2} T_{ij} q_k - \frac{1}{4y^2}(q_{i;k} q_j + q_i q_{j;k}) \\ &\quad + \frac{1}{2y^3} q_i q_j q_k \\ &= -\frac{1}{2(n-1)y}(q_i K_{kj} + q_j K_{ik}) - \frac{1}{2y^2} T_{ij} q_k \\ &\quad - \frac{1}{4y^2}\left(-\frac{2y}{n-1} K_{ik} + T_{ik}\right) q_j - \frac{1}{4y^2}\left(-\frac{2y}{n-1} K_{kj} + T_{kj}\right) q_i \\ &\quad + \frac{1}{2y^3} q_i q_j q_k \\ &= -\frac{1}{2y^2} T_{ij} q_k - \frac{1}{4y^2} T_{ik} q_j - \frac{1}{4y^2} T_{kj} q_i + \frac{1}{2y^3} q_i q_j q_k, \end{aligned}$$

that is

$$\bar{g}_{ij;k} = -\frac{1}{y}\left(q_k \bar{g}_{ij} + \frac{1}{2} q_i \bar{g}_{kj} + \frac{1}{2} q_j \bar{g}_{ik}\right). \tag{2,2}$$

Accordingly, if $|\bar{g}_{ij}| \neq 0$, the Riemannian space $\bar{V}_n$ with line element (not always positive definite)

$$ds^2 = \bar{g}_{ij}\, dx^i dx^j$$

is projective to V_n, in other words, Christoffel symbols $\bar{\Gamma}^i_{jk}$ made by $\bar{g}_{ij}$ satisfy the relation

$$(2,3)\qquad \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \frac{\partial}{\partial x^k} \log y^{-\frac{1}{2}} + \delta^i_k \frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}}.$$

For we get easily from (2,3)

$$\begin{aligned} \bar{g}_{ij;k} &= \frac{\partial \bar{g}_{ij}}{\partial x^k} - \bar{g}_{hj}\Gamma^h_{ik} - \bar{g}_{ih}\Gamma^h_{jk} \\ &= \bar{g}_{hj}(\bar{\Gamma}^h_{ik} - \Gamma^h_{ik}) + \bar{g}_{ih}(\bar{\Gamma}^h_{jk} - \Gamma^h_{jk}) \\ &= -\frac{1}{y}\left(q_k \bar{g}_{ij} + \frac{1}{2} q_i \bar{g}_{kj} + \frac{1}{2} q_j \bar{g}_{ik}\right) \end{aligned}$$

or

$$\bar{g}_{hk}(\bar{\Gamma}^k_{ij} - \Gamma^k_{ij}) = -\frac{1}{2y}(q_i \bar{g}_{hj} + q_j \bar{g}_{ih}),$$

which becomes (2,3)

Now, regarding the curvature tensor $\bar{K}^i_{jkh}$ of V_n, we get easily

$$\begin{aligned} \bar{K}_j{}^i{}_{kh} = K_j{}^i{}_{kh} &+ \delta^i_k\left(\frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}}\right)_{;h} - \delta^i_h\left(\frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}}\right)_{;k} \\ &+ \delta^i_k \frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}} \frac{\partial}{\partial x^h} \log y^{-\frac{1}{2}} - \delta^i_h \frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}} \frac{\partial}{\partial x^k} \log y^{-\frac{1}{2}}. \end{aligned}$$

Accordingly, by contraction we get

$$\bar{K}_{jk} = K_{jk} - (n-1)\left\{\left(\frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}}\right)_{;k} - \frac{\partial}{\partial x^j} \log y^{-\frac{1}{2}} \frac{\partial}{\partial x^k} \log y^{-\frac{1}{2}}\right\},$$

and hence, by means of (α), we get

$$\begin{aligned} \bar{K}_{jk} &= K_{jk} + \frac{n-1}{2y} q_{j;k} - \frac{n-1}{4y^2} q_j q_k \\ &= (n-1)\left(\frac{1}{2y} T_{jk} - \frac{1}{4y^2} q_j q_k\right) = (n-1)\bar{g}_{jk}. \end{aligned}$$

From the last relation we see that V_n is an Einstein space with non-vanishing curvature $(n>2)$ or a surface with constant curvature.

Conversely, if V_n is an Einstein space, then, since we have

$$K_{ij} = \frac{K}{n} g_{ij},$$

(α) becomes readily

$$\begin{aligned} \frac{\partial y}{\partial x^i} &= q^i, \\ q_{i;j} &= 2y\left(\bar{g}_{ij} - \frac{K}{n(n-1)} g_{ij}\right) + \frac{1}{2y} q_i q_j, \\ \bar{g}_{ij;k} &= -\frac{1}{y}\left(q_k \bar{g}_{ij} + \frac{1}{2} q_i \bar{g}_{kj} + \frac{1}{2} q_j \bar{g}_{ik}\right), \end{aligned}$$

which are satisfied by

$$y = \text{const.} \neq 0, \qquad q_i = 0, \qquad \bar{g}_{ij} = \frac{K}{n(n-1)} g_{ij}.$$

On the other hand, by means of the relations (β) and (2,1), we have

$$|G_{\lambda\mu}| = g^{-1}\begin{vmatrix} 2y & q_j + 2y\tau_j \\ q_i + 2y\tau_i & T_{ij} + q_i\tau_j + q_j\tau_i + 2y\tau_i\tau_j \end{vmatrix}$$

$$= g^{-1}\begin{vmatrix} 2y & 0 \\ q_i + 2y\tau_i & T_{ij} - \frac{1}{2y} q_i q_j \end{vmatrix} = (2y)^{n+1} g^{-1} |\bar{g}_{ij}|.$$

From the relation above, we see that the condition $|\bar{g}_{ij}| \neq 0$ is equivalent to that the invariant hyperquadric Q_{n-1} is non-degenerate.

Accordingly, we obtain the following theorem.

Theorem 2. *If the group of holonomy of the space with a normal projective connexion corresponding to a given Riemannian space V_n fixes a non-degenerate hyperquadric Q_{n-1}, the space is projective to an Einstein space with non-vanishing curvature in the region of points which do not belong to the image of Q_{n-1} into V_n. The converse is also true.*

§3. The image of Q_{n-1}.

Let F_{n-1} be the image surface of Q_{n-1} into V_n, then it is given by the equation $y = 0$. If y is not constant, we take a coordinate system $(x^1, x^2, \cdots\cdots, x^n)$ such that

(3,1) $$x^n = y, \qquad g_{an} = g^{bn} = 0$$
$$(a, b, c \cdots\cdots = 1, 2, \cdots\cdots, n-1)$$

and denote the Riemannian spaces given by the hypersurfaces $F_{n-1}(y)$ on which $y = \text{const.}$ by $V_{n-1}(y)$ whose fundamental tensors are $g_{ab}(x, y)$. Furthermore, we denote Christoffel symbols of $V_{n-1}(y)$ determined by $g_{ab}(x; y)$ by $\{^{a}_{bc}\}$ and the covariant differentiation with respect to $\{^{a}_{bc}\}$ by a comma.

Now, if we put

(3,2) $$\sqrt{g_{nn}} = \psi(x, y),$$

we can easily obtain the relation

(3,3) $$\frac{\partial}{\partial y} g_{ab} = -2\psi h_{ab}$$

where h_{ab} is the second fundamental tensor of $F_{n-1}(y)$, and

(3,4) $$\Gamma^a_{bc} = \{{}^{\,a}_{bc}\}, \qquad \Gamma^n_{ab} = \frac{1}{\psi} h_{ab}, \qquad \Gamma^a_{bn} = -\psi h^a_b,$$
$$\Gamma^n_{an} = \frac{1}{\psi}\psi_{,a}, \qquad \Gamma^a_{nn} = -\psi g^{ab}\psi_{,b}, \qquad \Gamma^n_{nn} = \frac{1}{\psi}\frac{\partial \psi}{\partial y}.$$

Furthermore, making use of Gauss-Codazzi equations[7]

(3,5) $$\begin{cases} K_{acbd} = R_{acbd} - h_{ab}h_{cd} + h_{ad}h_{cb}, \\ \psi K_a{}^n{}_{bc} = h_{ab,c} - h_{ac,b}, \end{cases}$$

and the relation

$$K_a{}^n{}_{bn} = \frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} + h^c_a h_{bc} - \frac{1}{\psi}\psi_{,ab},$$

we can obtain the relation

(3,6) $$\begin{cases} K_{ab} = \dfrac{1}{\psi}\dfrac{\partial}{\partial y} h_{ab} - h h_{ab} + 2h^c_a h_{bc} + R_{ab} - \dfrac{1}{\psi}\psi_{,ab}, \\ K_{na} = \psi(h_{,a} - h^b_{a,b}), \\ K_{nn} = \psi\dfrac{\partial h}{\partial y} - \psi^2 h^b_a h^a_b - \psi g^{ab}\psi_{,ab}, \qquad h = h^a_a. \end{cases}$$

Now, since we have the relation

$$q_a = 0, \qquad q_n = 1$$

in this coordinate system, we obtain by means of (3,4) the relation

$$q_{a;b} = -\Gamma^n_{ab} = -\frac{1}{\psi} h_{ab}.$$

Hence, by virtue of (3,6), (1,13) becomes

$$-\frac{1}{\psi} h_{ab} = -\frac{2y}{n-1}\left(\frac{1}{\psi}\frac{\partial}{\partial y} h_{ab} - h h_{ab} + 2h^c_a h_{bc} + R_{ab} - \frac{1}{\psi}\psi_{,ab}\right)$$
$$+ T_{ab}$$

or

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7) J. A. Schouten and D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, 1935.

$$(3,7)\qquad \frac{\partial}{\partial y} h_{ab} = \frac{n-1}{2y}(h_{ab} + \psi T_{ab}) + \psi(hh_{ab} - 2h_a^c h_{bc} - R_{ab}) + \psi_{,ab}.$$

From the relation above, we get easily

$$(3,7')\qquad \frac{\partial}{\partial y} h_a^b = \frac{n-1}{2y}(h_a^b + \psi T_a^b) + \psi(hh_a^b - R_a^b) + g^{bc}\psi_{,ac}$$

and by contraction

$$(3,8)\qquad \frac{\partial}{\partial y} h = \frac{n-1}{2y}(h + \psi T) + \psi(h^2 - R) + g^{ab}\psi_{,ab}.$$

We get likewise

$$\begin{aligned} q_{a;n} &= -\Gamma_{an}^n = -\frac{1}{\psi}\psi_{,a} \\ &= -\frac{2y\psi}{n-1}(h_{,a} - h_{a;b}^b) + T_{an}. \end{aligned}$$

If we put

$$(3,9)\qquad L_a = \frac{1}{\psi} T_{an},$$

which is a covariant tensor of $V_{n-1}(y)$, then the relation above becomes

$$(3,10)\qquad L_a + \frac{1}{\psi^2}\psi_{,a} - \frac{2y}{n-1}(h_{,a} - h_{a,b}^b) = 0.$$

Lastly, putting (3,8) into the relation

$$\begin{aligned} q_{n;n} &= -\Gamma_{nn}^n = -\frac{1}{\psi}\frac{\partial\psi}{\partial y} \\ &= -\frac{2y}{n-1}\left(\psi\frac{\partial h}{\partial y} - \psi^2 h_a^b h_b^a - \psi g^{ab}\psi_{,ab}\right) + T_{nn}, \end{aligned}$$

we get

$$\frac{1}{\psi}\frac{\partial\psi}{\partial y} = \psi(h + \psi T) + \frac{2y\psi^2}{n-1}(h^2 - h_a^b h_b^a - R) - T_{nn}.$$

Introducing a scalar of $V_{n-1}(y)$ such that

$$(3,11)\qquad S = \frac{1}{\psi^2} T_{nn},$$

the relation above becomes

$$(3,12)\qquad \frac{\partial\psi}{\partial y} = \psi^2 h + \psi^3(T - S) + \frac{2y\psi^3}{n-1}(h^2 - h_a^b h_b^a - R).$$

Now, let us represent (1,14) by means of the quantities of $V_{n-1}(y)$. We get by (3,4) the relation

$$T_{ab;n} = \frac{\partial T_{ab}}{\partial y} - \Gamma^t_{an}T_{tb} - \Gamma^t_{bn}T_{at}$$
$$= \frac{\partial T_{ab}}{\partial y} + \psi h^c_a T_{cb} + \psi h^c_b T_{ac} - \psi_{,a}L_b - \psi_{,b}L_a$$
$$= -\frac{1}{n-1}(q_a T_{bn} + q_b T_{an}) = 0,$$

or

$$\text{(3,13)} \qquad \frac{\partial T_{ab}}{\partial y} = -\psi(h^c_a T_{cb} + h^c_b T_{ac}) + \psi_{,a}L_b + \psi_{,b}L_a,$$

from which we get by (3,3) the relation

$$\frac{\partial T}{\partial y} = 2\psi_{,a}L^a$$

Furthermore, we get by (3,4), (3,6) the relation

$$T_{an;n} = \frac{\partial T_{an}}{\partial y} - \Gamma^t_{an}T_{tn} - \Gamma^t_{nn}T_{at}$$
$$= \psi\frac{\partial L_a}{\partial y} + \psi^2 h^b_a L_b - \psi\psi_{,a}S + \psi g^{bc}\psi_{,c}T_{ab}$$
$$= -\frac{1}{n-1}K_{an} = -\frac{\psi}{n-1}(h_{,a} - h^b_{a,b}),$$

that is

$$\text{(3,14)} \qquad \frac{\partial L_a}{\partial y} = -\psi h^b_a L_b - \psi_{,b}T^b_a + \psi_{,a}S - \frac{1}{n-1}(h_{,a} - h^b_{a,b}).$$

We get likewise by (3,8) the relation

$$T_{nn;n} = \frac{\partial T_{nn}}{\partial y} - 2\Gamma^t_{nn}T_{nt} = \psi^2\frac{\partial S}{\partial y} + 2\psi^2\psi_{,a}L^a$$
$$= -\frac{2\psi}{n-1}\left\{\frac{n-1}{2y}(h + \psi T) + \psi(h^2 - h^b_a h^a_b - R)\right\},$$

that is

$$\text{(3,15)} \qquad \frac{\partial S}{\partial y} = -2\psi_{,a}L^a - \frac{1}{y\psi}(h + \psi T) - \frac{2}{n-1}(h^2 - h^b_a h^a_b - R).$$

We get readily the following relations

$$T_{ab;c} = T_{ab,c} - h_{ac}L_b - h_{cb}L_a$$
$$= -\frac{1}{n-1}(q_a K_{cb} + q_b K_{ac}) = 0,$$
$$T_{an;b} = \psi L_{a,b} - \psi h_{ab} S + \psi h^c_b T_{ac}$$
$$= -\frac{1}{n-1}\left(\frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} - h h_{ab} + 2h^c_a h_{bc} + R_{ab} - \frac{1}{\psi}\psi_{,ab}\right),$$

whose last side becomes by means of (3,7)

$$= -\frac{1}{2y\psi}(h_{ab} + \psi T_{ab}),$$

hence we have

$$T_{ab,c} - L_a h_{bc} - L_b h_{ac} = 0, \tag{3,16}$$

$$L_{a,b} + T^c_a h_{bc} - h_{ab} S + \frac{1}{2y\psi^2}(h_{ab} + \psi T_{ab}) = 0, \tag{3,17}$$

Lastly, we get by (3,4), (3,6) the relation

$$T_{nn;a} = \psi^2 S_{,a} + 2\psi^2 h^b_a L_b$$
$$= -\frac{2}{n-1}K_{an} = -\frac{2\psi}{n-1}(h_{,a} - h^b_{a,b}),$$

that is

$$S_{,a} + 2h^b_a L_b + \frac{2}{(n-1)\psi}(h_{,a} - h^b_{a,b}) = 0. \tag{3,18}$$

Hence, if we replace n with $n+1$, we obtain the following

Theorem 3. *In order that we can imbed a given Riemannian space V_n with line element*

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu \qquad (\lambda, \mu = 1, 2, \cdots\cdots, n)^{8)}$$

into a Riemannian space V_{n+1} as a hypersurface so that the group of holonomy of the space X_{n+1} with a normal projective connexion corresponding to V_{n+1} fixes a hyperquadric Q_n and the image of Q_n into V_{n+1} is the hypersurface, a necessary and sufficient condition is that the following system of equations with respect to g_{ab}, h_{ab}, ψ, L_a, T_{ab}, S

---

8) From now on we assume that the indices take the following values:
$$a, b, c, d, \cdots\cdots, \lambda, \mu, \nu, \rho, \cdots\cdots = 1, 2, \cdots\cdots, n.$$

$$(\mathrm{I}_1)\qquad \frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},$$

$$(\mathrm{I}_2)\qquad \frac{\partial h_{ab}}{\partial y} = \frac{n}{2y}(h_{ab}+\psi T_{ab}) + \psi(hh_{ab} - 2h_a^\lambda h_{b\lambda} - R_{ab}) + \psi_{,ab},$$

$$(\mathrm{I}_3)\qquad \frac{\partial \psi}{\partial y} = \psi^2 h + \psi^3(T-S) + \frac{2y\psi^3}{n}(h^2 - h_\lambda^\mu h_\mu^\lambda - R),$$

$$(\mathrm{I}_4)\qquad \frac{\partial T_{ab}}{\partial y} = -\psi(h_a^\lambda T_{\lambda b} + h_b^\lambda T_{a\lambda}) + \psi_{,a}L_b + \psi_{,b}L_a,$$

$$(\mathrm{I}_5)\qquad \frac{\partial L_a}{\partial y} = -\psi h_a^\lambda L_\lambda - T_a^\lambda \psi_{,\lambda} + \psi_{,a}S - \frac{1}{n}(h_{,a} - h_{a,\lambda}^\lambda),$$

$$(\mathrm{I}_6)\qquad \frac{\partial S}{\partial y} = -2\psi_{,\lambda}L^\lambda - \frac{1}{y\psi}(h+\psi T) - \frac{2}{n}(h^2 - h_\lambda^\mu h_\mu^\lambda - R)$$

is integrable under the conditions

$$(\mathrm{II}_1)\qquad L_a + \frac{\psi_{,a}}{\psi^2} - \frac{2y}{n}(h_{,a} - h_{a\ ,\lambda}^\lambda) = 0,$$

$$(\mathrm{II}_2)\qquad L_{a,b} + T_a^\lambda h_{b\lambda} - h_{ab}S + \frac{1}{2y\psi^2}(h_{ab}+\psi T_{ab}) = 0,$$

$$(\mathrm{II}_3)\qquad T_{ab,c} - L_a h_{bc} - L_b h_{ac} = 0,$$

$$(\mathrm{II}_4)\qquad S_{,a} + 2h_a^\lambda L_\lambda + \frac{2}{n\psi}(h_{,a} - h_{a,\lambda}^\lambda) = 0$$

and under the initial condition

$$[g_{ab}(x,y)]_{y=0} = g_{ab}(x).$$

Then, in the coordinate neighborhood $x^1, \cdots\cdots, x^n, y$, the line element of V_{n+1} is given by

$$ds^2 = g_{\lambda\mu}(x,y)dx^\lambda dx^\mu + (\psi(x,y)dy)^2.$$

§4. Invariant hyperquadric and Campbell's theorem.

In this paragraph, we shall investigate the problem to imbed a given V_n in an Einstein space A_{n+1} so that the relation between V_n and A_{n+1} is the one stated in Theorem 3.

Now, if a Riemannian space V_{n+1} with line element

$$ds^2 = g_{\lambda\mu}(x,y)dx^\lambda dx^\mu + (\psi(x,y)dy)^2$$

is an Einstein space with scalar curvature $(n+1)k$, the following relations hold good:

$$\frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},$$

(4,1) $$\frac{\partial h_{ab}}{\partial y} = k\psi g_{ab} + \psi(h h_{ab} - 2h_a^\lambda h_{b\lambda} - R_{ab}) + \psi_{,ab},$$

(4,2) $$V_a \equiv h_{,a} - h^\lambda_{a,\lambda} = 0,$$

(4,3) $$z \equiv (n-1)k + h^2 - h^\mu_\lambda h^\lambda_\mu - R = 0.$$

The converse is also true. The proof is easy by means of (3,6).

Accordingly, in order that V_{n+1} in Theorem 3 is an Einstein space, besides (I), (II), the above relations (4,1), (4,2), (4,3) are necessary and sufficient. Therefore we shall replace these relations by other ones such that we can easily treat our problem.

From (I_2) and (4,1) we obtain

(4,4) $$T_{ab} = -\frac{1}{\psi} h_{ab} + \frac{2yk}{n} g_{ab},$$
$$T = -\frac{h}{\psi} + 2yk$$

and from (II_1) and (4,2) we obtain the relation

(4,5) $$L_a = -\frac{1}{\psi^2}\psi_{,a} = \left(\frac{1}{\psi}\right)_{,a}.$$

Conversely, if $L_a = \left(\frac{1}{\psi}\right)_{,a}$ and (II_1) hold, then we get

$$yV_a = 0.$$

from which we get $V_a = 0$ when V_a is continuous at $y = 0$.

Then, putting (4,4) into (I_3) and using (4,3), we get

$$\frac{\partial \psi}{\partial y} = \left(2yk - S + \frac{2yQ}{n}\right)\psi^3 = -\left(S - \frac{2yk}{n}\right)\psi^3,$$

where

(4,6) $$Q_{ab} = h h_{ab} - h_a^\lambda h_{b\lambda} - R_{ab}, \qquad Q = g^{\lambda\mu} Q_{\lambda\mu}.$$

By (I_2), (I_3), (I_4), (4,4), (4,5), we obtain the relation

$$\frac{\partial T_{ab}}{\partial y} = -\frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} + \frac{h_{ab}}{\psi^2}\frac{\partial \psi}{\partial y} + \frac{2k}{n} g_{ab} - \frac{4yk\psi}{n} h_{ab}$$
$$= -k g_{ab} - (Q_{ab} - h_a^\lambda h_{b\lambda}) - \frac{1}{\psi}\psi_{,ab}$$

$$+ h_{ab}\left\{-S + 2y\left(k + \frac{1}{n}Q\right)\right\}\psi + \frac{2k}{n}g_{ab} - \frac{4yk}{n}\psi h_{ab}$$
$$= -\psi h_a^\lambda\left(-\frac{1}{\psi}h_{b\lambda} + \frac{2yk}{n}g_{b\lambda}\right)$$
$$-\psi h_b^\lambda\left(-\frac{1}{\psi}h_{a\lambda} + \frac{2yk}{n}g_{a\lambda}\right) - \frac{2}{\psi^2}\psi_{,a}\psi_{,b},$$

that is

$$\left(1 - \frac{2}{n}\right)kg_{ab} + \frac{1}{\psi}\psi_{,ab} - \frac{2}{\psi^2}\psi_{,a}\psi_{,b} - R_{ab}$$
$$+ h_{ab}\left\{h + \psi S - 2y\psi\left(k + \frac{Q}{n}\right)\right\} = 0.$$

Furthermore, by (I_5) and (4,2) we get

$$\frac{\partial}{\partial y}L_a = \left\{S - 2y\left(k + \frac{Q}{n}\right)\right\}\psi_{,a} + \psi\left(S_{,a} - \frac{2y}{n}Q_{,a}\right)$$
$$= \frac{1}{\psi}h_a^\lambda\psi_{,\lambda} - \psi_{,\lambda}\left(-\frac{1}{\psi}h_a^\lambda + \frac{2yk}{n}\delta_a^\lambda\right) + \psi_{,a}S,$$

that is

$$S_{,a} - \frac{2y}{n}Q_{,a} - \frac{2}{\psi^2}h_a^\lambda\psi_{,\lambda} = 0.$$

By means of (I_6) and (4,3), we get

$$\frac{\partial S}{\partial y} = \frac{2}{\psi}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu} - \frac{2k}{n}$$

and from (II_2) we get

$$-\frac{1}{\psi^2}\psi_{,ab} + \frac{2}{\psi^3}\psi_{,a}\psi_{,b} + \left(-\frac{1}{\psi}h_a^\lambda + \frac{2yk}{n}\delta_a^\lambda\right)h_{b\lambda}$$
$$- h_{ab}S + \frac{k}{n\psi}g_{ab} = 0.$$

Lastly, from (II_3) and (II_4) we obtain the relations

$$h_{ab,c} = \frac{1}{\psi}(\psi_{,c}h_{ab} + \psi_{,a}h_{cb} + \psi_{,b}h_{ac}),$$
$$S_{,a} - \frac{2}{\psi^2}h_a^\lambda\psi_{,\lambda} = 0.$$

Hence we obtain the following

Theorem 4. *A necessary and sufficient condition in order that we can imbed a given Riemannian space V_n with line element*

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu$$

into an Einstein space A_{n+1} so that the space A_{n+1} has the property of V_{n+1} in Theorem 3 is that the following system of equations

$$(\mathrm{III}_1) \qquad \frac{\partial g_{ab}}{\partial y} = -2\psi h_{ab},$$

$$(\mathrm{III}_2) \qquad \frac{\partial h_{ab}}{\partial y} = \psi(kg_{ab} + hh_{ab} - 2h_a^\lambda h_{b\lambda} - R_{ab}) + \psi_{,ab},$$

$$(\mathrm{III}_3) \qquad \frac{\partial \psi}{\partial y} = -\left(S - \frac{2yk}{n}\right)\psi^3,$$

$$(\mathrm{III}_4) \qquad \frac{\partial S}{\partial y} = \frac{2}{\psi^2} g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu} - \frac{2}{n}k$$

is integrable under the conditions

$$(\mathrm{IV}_1) \qquad \xi_{ab} \equiv \frac{1}{\psi}\psi_{,ab} - \frac{2}{\psi^2}\psi_{,a}\psi_{,b} - \frac{k}{n}g_{ab} + h_a^\lambda h_{b\lambda} + \psi h_{ab}\left(S - \frac{2yk}{n}\right) = 0,$$

$$(\mathrm{IV}_2) \qquad \eta_{ab} \equiv \left(1 - \frac{1}{n}\right)kg_{ab} + hh_{ab} - h_a^\lambda h_{b\lambda} - R_{ab} = 0,$$

$$(\mathrm{IV}_3) \qquad \zeta_{abc} \equiv h_{ab,c} - \frac{1}{\psi}(\psi_{,a}h_{bc} + \psi_{,b}h_{ca} + \psi_{,c}h_{ab}) = 0,$$

$$(\mathrm{IV}_4) \qquad \sigma_a \equiv S_{,a} - \frac{2}{\psi^2}h_a^\lambda\psi_{,\lambda} = 0$$

and under the initial condition

$$[g_{ab}(x, y)]_{y=0} = g_{ab}(x).$$

The proof is evident from the computation above and the relation

$$z \equiv g^{\lambda\mu}\eta_{\lambda\mu}, \qquad V_a \equiv \zeta_\lambda{}^\lambda{}_a - \zeta_a{}^\lambda{}_\lambda.$$

§5. Some properties of ξ_{ab}, η_{ab}, ζ_{abc} and σ_a.

In this paragraph, we shall investigate properties of the tensors ξ_{ab}, η_{ab}, ζ_{abc} and the vector σ_a made by any solutions g_{ab}, h_{ab}, ψ, S of the system of equations (III).

Let us denote the Riemannian spaces with line elements

$$ds^2 = g_{\lambda\mu}(x, y)dx_\lambda dx^\mu$$

by $V_n(y)$. Then, we get by (III$_1$)

$$\frac{\partial \Gamma^a_{bc}}{\partial y} = g^{a\lambda}(\psi h_{bc})_{,\lambda} - (\psi h^a_b)_{,c} - (\psi h^a_c)_{,b},$$

which becomes by means of (IV$_3$)

$$\text{(5,1)} \qquad \frac{\partial \Gamma^a_{bc}}{\partial y} = \psi(\zeta_{bc}{}^a - \zeta_b{}^a{}_c - \zeta_c{}^a{}_b) - 2(\psi_{,b}h^a_c + \psi_{,c}h^a_b).$$

We get likewise by means of (IV$_4$) the relation

$$\begin{aligned} \text{(5,2)} \qquad S_{,ab} &= \sigma_{a,b} - \frac{4}{\psi^3}\psi_{,b}h^\lambda_a\psi_{,\lambda} + \frac{2}{\psi^2}(h^\lambda_{a,b}\psi_{,\lambda} + h^\lambda_a\psi_{,\lambda b}) \\ &= \sigma_{a,b} + \frac{2}{\psi^2}\zeta_a{}^\lambda{}_b\psi_{,\lambda} \\ &\quad + \frac{2}{\psi^3}\Big\{h_{ab}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu} + h^\lambda_b\psi_{,a}\psi_{,\lambda} \\ &\qquad + \psi^2 h^\lambda_a\Big(\frac{1}{\psi}\psi_{,\lambda b} - \frac{1}{\psi^2}\psi_{,\lambda}\psi_{,b}\Big)\Big\} \end{aligned}$$

Furthermore, using (IV$_2$), (III$_2$) may be replaced by the relation

$$\text{(III}_2'\text{)} \qquad \frac{\partial h_{ab}}{\partial y} = \psi\Big(\frac{k}{n}g_{ab} - h^\lambda_a h_{b\lambda} + \eta_{ab}\Big) + \psi_{,ab},$$

from which we get easily

$$\frac{\partial h}{\partial y} = \psi(k + h^\mu_\lambda h^\lambda_\mu + \eta^\lambda_\lambda) + g^{\lambda\mu}\psi_{,\lambda\mu}.$$

Then, we obtain from (IV$_1$)

$$\begin{aligned} \frac{\partial \xi_{ab}}{\partial y} = &-\psi\Big(S - \frac{2yk}{n}\Big)\Big\{\psi_{,ab} - \frac{2}{\psi}\psi_{,a}\psi_{,b} + \psi^2\Big(S - \frac{2yk}{n}\Big)h_{ab}\Big\} \\ &- \psi\Big\{\psi_{,ab}\Big(S - \frac{2yk}{n}\Big) + \psi_{,a}S_{,b} + \psi_{,b}S_{,a} + \frac{\psi}{2}(S_{,ab} + S_{,ba})\Big\} \\ &- \frac{1}{\psi}\psi_{,\lambda}\frac{\partial \Gamma^\lambda_{ab}}{\partial y} + \psi\Big(\frac{2}{\psi^2}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu} - \frac{4k}{n}\Big)h_{ab} \\ &+ \psi^2\Big(S - \frac{2yk}{n}\Big)\Big(\frac{k}{n}g_{ab} - h^\lambda_a h_{b\lambda} + \eta_{ab} + \frac{1}{\psi}\psi_{,ab}\Big) \\ &+ \frac{2k\psi}{n}h_{ab} + \psi h^\lambda_a\Big(\frac{k}{n}g_{\lambda b} - h^\mu_\lambda h_{\mu b} + \eta_{\lambda b} + \frac{1}{\psi}\psi_{,\lambda b}\Big) \\ &+ \psi h_{\lambda b}\Big(\frac{k}{n}\delta^\lambda_a + h^\mu_a h^\lambda_\mu + \eta_a{}^\lambda + \frac{1}{\psi}g^{\lambda\mu}\psi_{,a\mu}\Big) \end{aligned}$$

and hence by means of (IV), (5,1), (5,2)

$$
\begin{aligned}
= & -\psi^2\left(S-\frac{2yk}{n}\right)(\xi_{ab}-\eta_{ab}) \\
& -\psi\psi_{,a}\left(\sigma_b+\frac{2}{\psi^2}h^\lambda_b\psi_{,\lambda}\right)-\psi\psi_{,b}\left(\sigma_a+\frac{2}{\psi^2}h^\lambda_a\psi_{,\lambda}\right) \\
& -\frac{\psi^2}{2}\Big[\sigma_{a,b}+\sigma_{b,a}+\frac{2}{\psi^2}(\zeta_a{}^\lambda{}_b+\zeta_b{}^\lambda{}_a)\psi_{,\lambda} \\
& \qquad +\frac{2}{\psi^3}\{2h_{ab}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu}+\psi(h^\lambda_a\psi_{,\lambda b}+h^\lambda_b\psi_{,\lambda a})\}\Big] \\
& -\psi_{,\lambda}\left\{\zeta_{ab}{}^\lambda-\zeta_a{}^\lambda{}_b-\zeta_b{}^\lambda{}_a-\frac{2}{\psi}(\psi_{,a}h^\lambda_b+\psi_{,b}h^\lambda_a)\right\} \\
& +\frac{2}{\psi}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu}h_{ab}+\psi(h^\lambda_a\eta_{\lambda b}+h^\lambda_b\eta_{\lambda a})+h^\lambda_a\psi_{,\lambda b}+h^\lambda_b\psi_{,\lambda a},
\end{aligned}
$$

that is

$$
\begin{aligned}
(5,3)\qquad \frac{\partial\xi_{ab}}{\partial y} = & -\psi^2\left(S-\frac{2yk}{n}\right)(\xi_{ab}-\eta_{ab})-\psi(\psi_{,a}\sigma_b+\psi_{,b}\sigma_a) \\
& -\frac{\psi^2}{2}(\sigma_{a,b}+\sigma_{b,a})-\psi_{,\lambda}\zeta_{ab}{}^\lambda+\psi(h^\lambda_a\eta_{\lambda b}+h^\lambda_b\eta_{\lambda a}).
\end{aligned}
$$

Now, we have generally the relation

$$
\begin{aligned}
\frac{\partial R_{ab}}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial\Gamma^\lambda_{ab}}{\partial x^\lambda}-\frac{\partial\Gamma^\lambda_{a\lambda}}{\partial x^b}+\Gamma^\mu_{ab}\Gamma^\lambda_{\mu\lambda}-\Gamma^\mu_{a\lambda}\Gamma^\lambda_{\mu b}\right) \\
&= \left(\frac{\partial}{\partial y}\Gamma^\lambda_{ab}\right)_{,\lambda}-\left(\frac{\partial}{\partial y}\Gamma^\lambda_{a\lambda}\right)_{,b} \\
&= \left(\frac{\partial}{\partial y}\Gamma^\lambda_{ab}\right)_{,\lambda}-\frac{1}{2}\left(\frac{\partial}{\partial y}\Gamma^\lambda_{a\lambda}\right)_{,b}-\frac{1}{2}\left(\frac{\partial}{\partial y}\Gamma^\lambda_{b\lambda}\right)_{,a}.
\end{aligned}
$$

Putting (5,1) into the relation, it becomes

$$
\begin{aligned}
\frac{\partial R_{ab}}{\partial y} = & \{\psi(\zeta_{ab}{}^\lambda-\zeta_a{}^\lambda{}_b-\zeta_b{}^\lambda{}_a)-2(\psi_{,a}h^\lambda_b+\psi_{,b}h^\lambda_a)\}_{,\lambda} \\
& +\left\{\frac{\psi}{2}\zeta_\lambda{}^\lambda{}_a+\psi_{,a}h+\psi_{,\lambda}h^\lambda_a\right\}_{,b}+\left\{\frac{\psi}{2}\zeta_\lambda{}^\lambda{}_b+\psi_{,b}h+\psi_{,\lambda}h^\lambda_a\right\}_{,a} \\
= & \psi(\zeta_{ab}{}^\lambda{}_{,\lambda}-\zeta_a{}^\lambda{}_{b,\lambda}-\zeta_b{}^\lambda{}_{a,\lambda})+\psi_{,\lambda}(\zeta_{ab}{}^\lambda-\zeta_a{}^\lambda{}_b-\zeta_b{}^\lambda{}_a) \\
& -\psi_{,a\lambda}h^\lambda_b-\psi_{,b\lambda}h^\lambda_a+2\psi_{,ab}h \\
& -2\psi_{,a}\left(\zeta_b{}^\lambda{}_\lambda+\frac{1}{\psi}\psi_{,b}h+\frac{2}{\psi}\psi_{,\lambda}h^\lambda_b\right) \\
& \qquad\qquad -2\psi_{,b}\left(\zeta_a{}^\lambda{}_\lambda+\frac{1}{\psi}\psi_{,a}h+\frac{2}{\psi}\psi_{,\lambda}h^\lambda_a\right) \\
& +\frac{\psi}{2}(\zeta_\lambda{}^\lambda{}_{a,b}+\zeta_\lambda{}^\lambda{}_{b,a})+\frac{1}{2}(\psi_{,b}\zeta_\lambda{}^\lambda{}_a+\psi_{,a}\zeta_\lambda{}^\lambda{}_b)
\end{aligned}
$$

$$+ \psi_{,a}\Big(\zeta_\lambda{}^\lambda{}_b + \frac{1}{\psi}\psi_{,b}h + \frac{2}{\psi}\psi_{,\lambda}h^\lambda_b\Big)$$
$$+ \psi_{,b}\Big(\zeta_\lambda{}^\lambda{}_a + \frac{1}{\psi}\psi_{;a}h + \frac{2}{\psi}\psi_{,\lambda}h^\lambda_a\Big)$$
$$+ \psi_{,\lambda}\Big\{\zeta_a{}^\lambda{}_b + \zeta_b{}^\lambda{}_a + \frac{2}{\psi}(\psi_{,a}h^\lambda_b + \psi_{,b}h^\lambda_a + g^{\lambda\mu}\psi_{,\mu}h_{ab})\Big\},$$

that is

$$\frac{\partial R_{ab}}{\partial y} = \psi(\zeta_{ab}{}^\lambda{}_{,\lambda} - \zeta_a{}^\lambda{}_{b,\lambda} - \zeta_b{}^\lambda{}_{a,\lambda}) + \frac{\psi}{2}(\zeta_\lambda{}^\lambda{}_{a,b} + \zeta_\lambda{}^\lambda{}_{b,a})$$

(5,4)
$$+ \psi_{,\lambda}\zeta_{ab}{}^\lambda + \psi_{,a}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_b - 2\zeta_b{}^\lambda{}_\lambda\Big) + \psi_{,b}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_a - 2\zeta_a{}^\lambda{}_\lambda\Big)$$
$$- \psi_{,a\lambda}h^\lambda_b - \psi_{,b\lambda}h^\lambda_a + 2\Big(\psi_{,ab} - \frac{1}{\psi}\psi_{,a}\psi_{,b}\Big)h$$
$$+ \frac{2}{\psi}g^{\lambda\mu}\psi_{;\lambda}\psi_{;\mu}h_{ab}.$$

Then, by means of (III$_2$), (III$_5$), (5,4) we get the relation

$$\frac{\partial \eta_{ab}}{\partial y} = -2\Big(1-\frac{1}{n}\Big)\psi k h_{ab} + \psi\Big(k + h^\mu_\lambda h^\lambda_\mu + \eta_\lambda{}^\lambda + \frac{1}{\psi}g^{\lambda\mu}\psi_{,\lambda\mu}\Big)h_{ab}$$
$$+ \psi h\Big(\frac{k}{n}g_{ab} - h^\lambda_a h_{b\lambda} + \eta_{ab} + \frac{1}{\psi}\psi_{,ab}\Big)$$
$$- \psi\Big(\frac{k}{n}g_{a\lambda} + \eta_{a\lambda} + \frac{1}{\psi}\psi_{,a\lambda}\Big)h^\lambda_b - \psi\Big(\frac{k}{n}g_{b\lambda} + \eta_{b\lambda} + \frac{1}{\psi}\psi_{,b\lambda}\Big)h^\lambda_a$$
$$- \frac{\partial R_{ab}}{\partial y}$$
$$= \psi h_{ab}\Big(\frac{1}{\psi}g^{\lambda\mu}\psi_{,\lambda\mu} - \frac{2}{\psi^2}g^{\lambda\mu}\psi_{,\lambda}\psi_{,\mu} - k + h^\mu_\lambda h^\lambda_\mu + \eta_\lambda{}^\lambda\Big)$$
$$- \psi h\Big(\frac{1}{\psi}\psi_{,ab} - \frac{2}{\psi^2}\psi_{,a}\psi_{,b} - \frac{k}{n}g_{ab} + h^\lambda_a h_{b\lambda} - \eta_{ab}\Big)$$
$$- \psi(h^\lambda_a\eta_{b\lambda} + h^\lambda_b\eta_{a\lambda}) - \psi(\zeta_{ab}{}^\lambda{}_{,\lambda} - \zeta_a{}^\lambda{}_{b,\lambda} - \zeta_b{}^\lambda{}_{a,\lambda})$$
$$- \frac{\psi}{2}(\zeta_\lambda{}^\lambda{}_{a,b} + \zeta_\lambda{}^\lambda{}_{b,a}) - \psi_{,\lambda}\zeta_{ab}{}^\lambda$$
$$- \psi_{,a}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_b - 2\zeta_b{}^\lambda{}_\lambda\Big) - \psi_{,b}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_a - 2\zeta_a{}^\lambda{}_\lambda\Big),$$

that is

$$\frac{\partial \eta_{ab}}{\partial y} = \psi h_{ab}(\xi_\lambda{}^\lambda + \eta_\lambda{}^\lambda) - \psi h(\xi_{ab} - \eta_{ab})$$

(5,5)
$$- \psi(h^\lambda_a\eta_{b\lambda} + h^\lambda_b\eta_{a\lambda}) - \psi(\zeta_{ab}{}^\lambda{}_{,\lambda} - \zeta_a{}^\lambda{}_{b,\lambda} - \zeta_b{}^\lambda{}_{a,\lambda})$$

$$-\frac{\psi}{2}(\zeta_\lambda{}^\lambda{}_{a,b}+\zeta_\lambda{}^\lambda{}_{b,a})-\psi_{,\lambda}\zeta_{ab}{}^\lambda$$
$$-\psi_{,a}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_b-2\zeta_b{}^\lambda{}_\lambda\Big)-\psi_{,b}\Big(\frac{3}{2}\zeta_\lambda{}^\lambda{}_a-2\zeta_a{}^\lambda{}_\lambda\Big).$$

Now, using (IV_1), we have

$$\begin{aligned}\psi_{,abc} = {} & (\psi\xi_{ab})_{,c}+\frac{2}{\psi}(\psi_{,a}\psi_{,bc}+\psi_{,b}\psi_{,ac})-\frac{2}{\psi^2}\psi_{,a}\psi_{,b}\psi_{,c}\\ & +\frac{k}{n}g_{ab}\psi_{,c}-h_a^\lambda h_{b\lambda}\psi_{,c}-2\psi\Big(S-\frac{2yk}{n}\Big)h_{ab}\psi_{,c}\\ & -(\psi\zeta_a{}^\lambda{}_c+\psi_{,c}h_a^\lambda+\psi_{,a}h_c^\lambda+g^{\lambda\mu}\psi_{,\mu}h_{ac})h_{b\lambda}\\ & -(\psi\zeta_{b\lambda c}+\psi_{,c}h_{b\lambda}+\psi_{,b}h_{c\lambda}+\psi_{,\lambda}h_{bc})h_a^\lambda\\ & -\psi^2h_{ab}\Big(\sigma_{,c}+\frac{2}{\psi^2}h_c^\lambda\psi_{,\lambda}\Big)\\ & -\psi\Big(S-\frac{2yk}{n}\Big)(\psi\zeta_{abc}+\psi_{,c}h_{ab}+\psi_{,a}h_{bc}+\psi_{,b}h_{ac})\end{aligned}$$

Hence we have by the relation above

$$\begin{aligned}\frac{\partial\zeta_{abc}}{\partial y} = {} & \Big(\frac{\partial h_{ab}}{\partial y}\Big)_{,c}-h_{\lambda b}\frac{\partial\Gamma^\lambda_{ac}}{\partial y}-h_{a\lambda}\frac{\partial\Gamma^\lambda_{bc}}{\partial y}\\ & -\frac{\partial}{\partial y}\Big\{\frac{1}{\psi}(\psi_{,c}h_{ab}+\psi_{,a}h_{bc}+\psi_{,b}h_{ac})\Big\}\\ = {} & \psi_{,c}\Big(\frac{k}{n}g_{ab}-h_a^\lambda h_{b\lambda}+\eta_{ab}\Big)\\ & -(\psi\zeta_a{}^\lambda{}_c+\psi_{,c}h_a^\lambda+\psi_{,a}h_c^\lambda+g^{\lambda\mu}\psi_{,\mu}h_{ac})h_{b\lambda}\\ & -(\psi\zeta_{b\lambda c}+\psi_{,c}h_{b\lambda}+\psi_{,b}h_c^\lambda+\psi_{,\lambda}h_{bc})h_a^\lambda\\ & +\psi\eta_{ab,c}+\psi_{,abc}\\ & -h_{\lambda b}\{\psi(\zeta_{ac}{}^\lambda-\zeta_a{}^\lambda{}_c-\zeta_c{}^\lambda{}_a)-2(\psi_{,a}h_c^\lambda+\psi_{,c}h_a^\lambda)\}\\ & -h_{a\lambda}\{\psi(\zeta_{bc}{}^\lambda-\zeta_b{}^\lambda{}_c-\zeta_c{}^\lambda{}_b)-2(\psi_{,b}h_c^\lambda+\psi_{,c}h_b^\lambda)\}\\ & +\sum_{a,b,c=\text{Cyclic}}\Big[h_{ab}\Big\{2\psi\psi_{,c}\Big(S-\frac{2yk}{n}\Big)+\psi^2\sigma_c+2h_c^\lambda\psi_{,\lambda}\Big\}\\ & \qquad -\psi_{,c}\Big(\frac{k}{n}g_{ab}-h_a^\lambda h_{b\lambda}+\eta_{ab}+\frac{1}{\psi}\psi_{,ab}\Big)\Big]\\ = {} & -\psi h_{a\lambda}(\zeta_{bc}{}^\lambda+\zeta_b{}^\lambda{}_c-\zeta_c{}^\lambda{}_b)-\psi h_{b\lambda}(\zeta_{ac}{}^\lambda+\zeta_a{}^\lambda{}_c-\zeta_c{}^\lambda{}_a)\\ & -\psi^2\Big(S-\frac{2}{n}yk\Big)\zeta_{abc}+(\psi\xi_{ab}+\psi\eta_{ab})_{,c}\\ & +\psi^2(h_{ac}\sigma_b+h_{bc}\sigma_a)\\ & -\psi_{,c}\Big\{\frac{1}{\psi}\psi_{,ab}-\frac{2}{\psi^2}\psi_{,a}\psi_{,b}-\frac{k}{n}g_{ab}\\ & \qquad +h_a^\lambda h_{b\lambda}+\psi h_{ab}\Big(S-\frac{2}{n}yk\Big)+\eta_{ab}\Big\}\end{aligned}$$

$$+\sum_{a,b=\text{Cyclic}} \psi_{,a}\left\{\frac{1}{\psi}\psi_{,bc} - \frac{2}{\psi^2}\psi_{,b}\psi_{,c} - \frac{k}{n}g_{bc} + h_b^\lambda h_{c\lambda} + \psi h_{bc}\left(S - \frac{2}{n}yk\right) - \eta_{bc}\right\},$$

which becomes by (IV_1)

$$(5,6)\qquad \begin{aligned}\frac{\partial \zeta_{abc}}{\partial y} = & -\psi h_a(\zeta_{bc}{}^\lambda + \zeta_b{}^\lambda{}_c - \zeta_c{}^\lambda{}_b) \\ & - \psi h_{b\lambda}(\zeta_{ac}{}^\lambda + \zeta_a{}^\lambda{}_c - \zeta_c{}^\lambda{}_a) \\ & - \psi^2\left(S - \frac{2}{n}yk\right)\zeta_{abc} + \psi\xi_{ab,c} + \psi\eta_{ab,c} \\ & + \psi^2(h_{ac}\sigma_b + h_{bc}\sigma_a) + \psi_{,a}(\xi_{bc} - \eta_{bc}) + \psi_{,b}(\xi_{ac} - \eta_{ac}).\end{aligned}$$

Lastly, regarding σ_a we get by (IV_4) relation

$$\begin{aligned}\frac{\partial \sigma_a}{\partial y} &= \left(\frac{\partial S}{\partial y}\right)_{,a} - \frac{2}{\psi^2}\psi_{,\lambda}\frac{\partial h_a^\lambda}{\partial y} + 2h_a^\lambda\left(\frac{\partial}{\partial y}\frac{1}{\psi}\right)_{,\lambda} \\ &= \frac{4}{\psi}g^{\lambda\mu}\psi_{,\mu}\left(\frac{1}{\psi}\psi_{,\lambda a} - \frac{1}{\psi^2}\psi_{,\lambda}\psi_{,a}\right) \\ &\quad - \frac{2}{\psi}\psi_{,\lambda}\left(\frac{k}{n}\delta_a^\lambda + h_a^\mu h_\mu^\lambda + \eta_a{}^\lambda + \frac{1}{\psi}g^{\lambda\mu}\psi_{,a\mu}\right) \\ &\quad + 2h_a^\lambda\left\{\left(S - \frac{2}{n}yk\right)\psi_{,\lambda} + \psi\left(\sigma_\lambda + \frac{2}{\psi^2}h_\lambda^\mu\psi_{,\mu}\right)\right\} \\ &= 2\psi h_a^\lambda\sigma_\lambda \\ &\quad + \frac{2}{\psi}g^{\lambda\mu}\psi_{,\mu}\left\{\frac{1}{\psi}\psi_{,\lambda a} - \frac{2}{\psi^2}\psi_{,\lambda}\psi_{,a} - \frac{k}{n}g_{\lambda a} + h_\lambda^\mu h_{\mu a} + \psi\left(S - \frac{2}{n}yk\right)h_{\lambda a} - \eta_{a\lambda}\right\},\end{aligned}$$

that is

$$(5,7)\qquad \frac{\partial \sigma_a}{\partial y} = 2\psi h_a^\lambda\sigma_\lambda + \frac{2}{\psi}g^{\lambda\mu}\psi_{,\lambda}(\xi_{\mu a} - \eta_{\mu a})$$

Thus we see that ξ_{ab}, η_{ab}, ζ_{abc}, σ_a made by any solutions of the system of equations (III) satisfy a system of equations (5.3), (5,5), (5,6), (5,7) linear with respect to these quantities and their derivatives. Therefore, if we have at $y = 0$

$$\xi_{ab} = 0, \qquad \eta_{ab} = 0, \qquad \zeta_{abc} = 0, \qquad \sigma_a = 0,$$

the relation holds good in a proper neighborhood of $y = 0$. Hence we obtain a more exactly theorem as follows:

Theorem 5. *In order that we can imbed a given Riemannian space V_n with line element*

$$ds^2 = g_{\lambda\mu}(x)dx^\lambda dx^\mu$$

into an Einstein space in the sense as stated in Theorem 4, a necessary and sufficient condition is that the following equations with respect to h_{ab} $(= h_{bc})$, ψ, S is integrable for the space V_n:

$$(5,8)\qquad \begin{cases} \dfrac{1}{\psi}\psi_{,ab} - \dfrac{2}{\psi^2}\psi_{,a}\psi_{,c} - \dfrac{k}{n}g_{ab} + h_a^\lambda h_{b\lambda} + \psi h_{ab}S = 0, \\ h_{ab,c} - \dfrac{1}{\psi}(\psi_{,a}h_{bc} + \psi_{,b}h_{ac} + \psi_{,c}h_{ab}) = 0, \\ S_{,a} - \dfrac{2}{\psi^2}h_a^\lambda\psi_{,\lambda} = 0 \end{cases}$$

under the condition

$$(5,9)\qquad \left(1 - \frac{1}{n}\right)kg_{ab} + hh_{ab} - h_a^\lambda h_{b\lambda} - R_{ab} = 0.$$

§6. Integrability conditions of (5,8), (5,9) $(n > 2)$.

In order to investigate the integrability of (5,8), (5,7), let us replace them by the following equivalent system of equations

$$(6,1)\qquad \psi_{,a} = \psi\rho_a,$$

$$(6,2)\qquad \rho_{a,b} = \rho_a\rho_b + \frac{k}{n}g_{ab} - h_a^\lambda h_{b\lambda} - \psi S h_{ab},$$

$$(6,3)\qquad h_{ab,c} = \rho_c h_{ab} + \rho_a h_{bc} + \rho_b h_{ac},$$

$$(6,4)\qquad S_{,a} = \frac{2}{\psi}h_a^\lambda\rho_\lambda$$

and

$$(\mathrm{IV}_2)\qquad \eta_{ab} \equiv \left(1 - \frac{1}{n}\right)kg_{ab} + hh_{ab} - h_a^\lambda h_{b\lambda} - R_{ab} = 0.$$

Now, we get from (6,1) the relation

$$\psi_{,[ab]} = \psi\rho_{[a,b]} + \psi_{,[b}\rho_{a]} = \psi\{\rho_{[a,b]} + \rho_{[a}\rho_{b]}\} = 0$$

and from (6,2) the relation

$$\rho_{a,[bc]} = \rho_{a[c}\rho_{b]} - h_{a,[c}^\lambda h_{b]\lambda} - \psi S h_{a[b}\rho_{c]} - \psi h_{a[b}S_{,c]}$$

$$
\begin{aligned}
&= \left(\frac{k}{n} g_{a[c} - h_a^\lambda h_{\lambda[c} - \psi S h_{a[c}\right)\rho_{b]} \\
&\quad - (h_a^\lambda \rho_{[c} + \rho^\lambda h_{a[c} + \rho_a h_{[c}^\lambda) h_{b]\lambda} \\
&\quad - \psi S h_{a[b}\rho_{c]} - 2h_{a[b} h_{c]}^\lambda \rho_\lambda \\
&= \frac{k}{n} g_{a[c}\rho_{b]} - h_{a[b} h_{c]}^\lambda \rho_\lambda = -\frac{1}{2} R_a{}^\lambda{}_{bc}\rho_\lambda
\end{aligned}
$$

that is

$$
\left\{R_a{}^\lambda{}_{bc} - h_{ab} h_c^\lambda + h_{ac} h_b^\lambda - \frac{k}{n}(g_{ab}\delta_c^\lambda - g_{ac}\delta_b^\lambda)\right\}\rho_\lambda = 0.
$$

We get from (6,3) the relation

$$
\begin{aligned}
h_{ab,[cd]} &= -\frac{1}{2} R_a{}^\lambda{}_{cd} h_{\lambda b} - \frac{1}{2} R_b{}^\lambda{}_{cd} h_{a\lambda} \\
&= h_{ab,[d}\rho_{c]} + \rho_{a,[d} h_{b]b} + \rho_{b,[d} h_{c]a} \\
&= \rho_a h_{b[d}\rho_{c]} + \rho_b h_{a[d}\rho_{c]} \\
&\quad + \left(\rho_a \rho_{[d} + \frac{k}{n} g_{a[d} - h_{a\lambda} h_{[d}^\lambda - \psi S h_{a[d}\right) h_{c]b} \\
&\quad + \left(\rho_b \rho_{[d} + \frac{k}{n} g_{b[d} - h_{b\lambda} h_{[d}^\lambda - \psi S h_{b]d}\right) h_{c]a} \\
&= h_{a\lambda}\left(h_{[c}^\lambda h_{d]b} + \frac{k}{n}\delta_{[c}^\lambda g_{d]b}\right) + h_{b\lambda}\left(h_{[c}^\lambda h_{d]a} + \frac{k}{n}\delta_{[c}^\lambda g_{d]a}\right),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\left\{R_a{}^\lambda{}_{cd} - h_{ac} h_d^\lambda + h_{ad} h_c^\lambda - \frac{k}{n}(g_{ac}\delta_d^\lambda - g_{ad}\delta_c^\lambda)\right\} h_{b\lambda} \\
+ &\left\{R_b{}^\lambda{}_{cd} - h_{bc} h_d^\lambda + h_{bd} h_c^\lambda - \frac{k}{n}(g_{bc}\delta_d^\lambda - g_{bd}\delta_c^\lambda)\right\} h_{a\lambda} = 0.
\end{aligned}
$$

We get lastly from (6,4) the relation

$$
\begin{aligned}
S_{,[ab]} &= -\frac{2}{\psi}\rho_{[b} h_{a]}^\lambda \rho_\lambda + \frac{2}{\psi}\rho_{\lambda,[b} h_{a]}^\lambda \\
&= -\frac{2}{\psi}\rho_{[b} h_{a]}^\lambda \rho_\lambda + \frac{2}{\psi}\left(\rho_\lambda \rho_{[b} + \frac{k}{n} g_{\lambda[b} - h_\lambda^\mu h_{\mu[b} - \psi S h_{\lambda[b}\right) h_{a]}^\lambda \\
&= 0.
\end{aligned}
$$

Hence, if we put

$$
(6,5) \qquad F_{abcd} = R_{abcd} - h_{ac} h_{bd} + h_{ad} h_{bc} - \frac{k}{n}(g_{ac} g_{bd} - g_{ad} g_{bc}),
$$

the results above are represented by

$$F_{a\ bc}^{\ \lambda}\rho_{\lambda} = 0, \tag{6,6}$$

$$F_{a\lambda cd}h_{b}^{\lambda} + F_{b\lambda cd}h_{a}^{\lambda} = 0, \tag{6,7}$$

and

$$F_{abcd} = -F_{bacd} = -F_{abdc} = F_{cdab}. \tag{6,8}$$

Accordingly, we obtain the following theorem.

Theorem 6. *A condition of integrability of the system of equations* (6,1) – (6,4), (IV_2) *is that the system of algebraic relations with respect to* ρ_a, h_{ab}, ψ, S *derived successively from* (IV_2), (6,6), (6,7) *by differentiation and by substitution of* (6,1) – (6,4) *is compatible.*

Now, if V_n is an Einstein space, then by definition we have the relation

$$R_{ab} = \frac{R}{n}g_{ab}.$$

Then if we put

$$h_{ab} = \psi g_{ab},$$

(IV_2) becomes

$$\eta_{ab} \equiv \left\{\left(1-\frac{1}{n}\right)k + (n-1)\varphi^2 - \frac{R}{n}\right\}g_{ab} = 0,$$

hence we have the relation

$$\varphi = \left(\frac{R}{n(n-1)} - \frac{k}{n}\right)^{\frac{1}{2}}.$$

Furthermore, (6,7) is satisfied by means of (6,8). If we put ψ = const., S = const., then we get from (6,2) the relation

$$\frac{k}{n} - \varphi^2 - \psi\varphi S = 0.$$

We can easily determine ψ, S so that the last relation holds good. Thus we obtain the following corollary.

Corollary. *Any Einstein space* A_n *can be imbedded into an Einstein space* A_{n+1} *in the sense as stated in Theorem 4 and so that* A_n *is totally geodesic or umbilical in* A_{n+1}.

As easily seen, the spaces V_n which can be imbedded in an Einstein space A_{n+1} and are totally geodesic or umbilical in it are Einstein spaces.

§7. Integrability conditions of (5,8), (5,9) ($n = 2$).

In the case $n = 2$, let us denote the Gaussian total curvature by

$$K = \frac{R_{1212}}{g}.$$

Then, since we have the relations

$$h h_{ab} - h_a^\lambda h_{b\lambda} = \frac{1}{g} g_{ab} \mid h_{\lambda\mu} \mid$$
$$R_{ab} = K g_{ab},$$

(IV_2) becomes

$$\gamma_{ab} \equiv \frac{k}{2} g_{ab} + h h_{ab} - h_a^\lambda h_{b\lambda} - R_{ab}$$
$$= \left\{ \frac{\mid h_{\lambda\mu} \mid}{g} - \left(K - \frac{k}{2}\right)\right\} g_{ab} = 0.$$

Hence we have a equivalent condition

(7,1) $$K - \frac{k}{2} - \frac{1}{g} \mid h_{\lambda\mu} \mid = 0.$$

On the other hand, as regards F_{abcd} we have

$$F_{1212} \equiv R_{1212} - \mid h_{\lambda\mu} \mid - \frac{k}{2} g = \left(K - \frac{k}{2}\right) g - \mid h_{\lambda\mu} \mid .$$

Accordingly, we see that (6,6), (6,7) are identically satisfied if (7,1) holds good.

By differentiation, we get from (7,1) the relation

$$K_{,a} + \frac{1}{g^2} \mid h_{\lambda\mu} \mid \frac{\partial g}{\partial x^a} - \frac{1}{g}\left(h_{11} \frac{\partial h_{22}}{\partial x^a} + h_{22} \frac{\partial h_{11}}{\partial x^a} - 2h_{12} \frac{\partial h_{12}}{\partial x^a}\right)$$
$$= K_{,a} - \frac{1}{g} (h_{11} h_{22,a} + h_{11,a} h_{22} - 2h_{12} h_{12,a}) = 0.$$

Putting (6,3) into the last relation, we get

$$K_{,a} - \frac{2}{g}\{ \mid h_{\lambda\mu} \mid \rho_a + \rho_1(h_{1a} h_{22} - h_{12} h_{2a}) + \rho_2(h_{11} h_{2a} - h_{12} h_{1a})\} = 0,$$

that is

$$K_{,a} - \frac{4}{g} \mid h_{\lambda\mu} \mid \rho_a = 0.$$

Furthermore, putting (7,1) in the relation, we get

$$K_{,a} - 4\left(K - \frac{k}{2}\right)\rho_a = 0. \tag{7,2}$$

i) *The case $K =$ constant.*

If we put $k = 2K$, we have the sole condition (7,1), since (7,2) becomes a trivial one. Then the system of equations (6,1) — (6,4), (IV.) is clearly integrable.

ii) *The case $K \neq$ constant.*

Furthermore, differentiating (7,2), we get the relation

$$K_{,ab} - 4K_{,b}\rho_a - 4\left(K - \frac{k}{2}\right)\rho_{a,b} = 0,$$

into which we put (6,2), we get the relation

$$K_{,ab} - 4K_{,b}\rho_a - 4\left(K - \frac{k}{2}\right)\left\{\rho_a\rho_b + \frac{k}{2}g_{ab} + \frac{\mid h_{\lambda\mu} \mid}{g}g_{ab} - (h + \psi S)h_{ab}\right\} = 0.$$

Putting (7,1), (7,2) into the last relation, we obtain the relation

$$K_{,ab} - \frac{5}{4\left(K - \frac{k}{2}\right)}K_{,a}K_{,b} - 4\left(K - \frac{k}{2}\right)\{Kg_{ab} - (h + \psi S)h_{ab}\} = 0. \tag{7,3}$$

On the other hand, we get by (6,3)

$$h_{,a} = \rho_a h + 2h_a^\lambda \rho_\lambda,$$

that is

$$\left(\frac{h}{\psi}\right)_{,a} = \frac{2}{\psi}h_a^\lambda \rho_\lambda.$$

Comparing this with (6,4), we have the relation

$$S = \frac{h}{\psi} + 2C \qquad (C = \text{constant}). \tag{7,4}$$

If we put (7,4) into (7,3), we obtain

$$K_{,ab} - \frac{5}{4\left(K - \frac{k}{2}\right)}K_{,a}K_{,b} - 4\left(K - \frac{k}{2}\right)Kg_{ab} + 8\left(K - \frac{k}{2}\right)(h + \psi C)h_{ab} = 0. \tag{7,5}$$

Let us define a tensor of V_2 depending on k such that

$$L_{ab}(k) = \frac{K_{,ab}}{4\left(K-\frac{k}{2}\right)} - \frac{5K_{,a}K_{,b}}{16\left(K-\frac{k}{2}\right)^2} - Kg_{ab},$$

then (7,5) is represented by

$$L_{ab}(k) + 2(h+\psi C)h_{ab} = 0. \tag{7,5'}$$

Now, we divide the case into the two following cases.

ii$_1$) *The case* $L_{ab}(k) = 0$.

Then, we have $h_{ab} = 0$ or $h + \psi C = 0$. In the first case, we get from (7,1) the relation $K =$ const. which is contradictory to our assumption. In the second case, we get easily

$$h_{,a} + C\psi_{,a} = 2h_a^\lambda \rho_\lambda + (h + C\psi)\rho_a = 2h_a^\lambda \rho_\lambda = 0.$$

Hence, solving these relations with respect to h_{ab}, we get

$$h_{11} = -\psi Cg \frac{\rho^2\rho^2}{g_{\lambda\mu}\rho^\lambda\rho^\mu},$$
$$h_{12} = \psi Cg \frac{\rho^1\rho^2}{g_{\lambda\mu}\rho^\lambda\rho^\mu},$$
$$h_{22} = -\psi Cg \frac{\rho^1\rho^1}{g_{\lambda\mu}\rho^\lambda\rho^\mu},$$

from which we get the relation

$$|h_{\lambda\mu}| = 0.$$

Accordingly we get also $K =$ const., which is contradictory to our assumption.

ii$_2$) *The case* $L_{ab}(k) \neq 0$.

Then, we have

$$|L_{ab}| = 4(h+\psi C)^2 |h_{\lambda\mu}|,$$

into which putting (7,1), we get

$$|L_{ab}| = 4(h+\psi C)^2 g\left(K - \frac{k}{2}\right),$$

that is

$$h + \psi C = \pm\frac{1}{2}\sqrt{\frac{|L_{ab}(k)|}{\left(K-\frac{k}{2}\right)g}}.$$

If we put

$$F(k) = \frac{1}{2}\sqrt{\frac{|L_{ab}(k)|}{4\left(K-\frac{k}{2}\right)g}},$$

(7,5′) becomes

$$L_{ab} \pm 2Fh_{ab} = 0. \tag{7,5''}$$

Then, if $F(k)=0$, it leads to a contradiction as (ii_1). Hence, by virtue of the above calculation, we get the relation

$$\begin{aligned} \rho_a &= \frac{K_{,a}}{4\left(K-\frac{k}{2}\right)}, \qquad h_{ab} = \mp\frac{L_{ab}(k)}{2F(k)}, \\ \psi C &= \pm F(k) \pm \frac{g^{\lambda\mu}L_{\lambda\mu}(k)}{2F(k)}, \\ S &= \pm\frac{F(k)}{\psi} + C \qquad\qquad (k,\ C = \text{constant}). \end{aligned} \tag{7,6}$$

Accordingly, in order that our system is integrable, it is necessary and sufficient that the relation derived from (7,5) by differentiation is satisfied for the space.

By means of (6,3), we get from (7,5″)

$$\begin{aligned} &L_{ab,c} \pm 2F_{,c}h_{ab} \pm 2Fh_{ab,c} \\ &= L_{ab,c} \pm 2F_{,c}h_{ab} \pm 2F(\rho_c h_{ab} + \rho_a h_{bc} + \rho_b h_{ac}) = 0, \end{aligned}$$

into which putting (7,6), we obtain the relation

$$L_{ab,c} - \frac{F_{,c}}{F}L_{ab} - \frac{1}{4\left(K-\frac{k}{2}\right)}(K_{,c}L_{ab} + K_{,a}L_{bc} + K_{,b}L_{ac}) = 0. \tag{7,7}$$

Accordingly we obtain the following theorem.

Theorem 7. *In order that we can imbed an two-dimensional Riemannian space V_2 into an Einstein space A_3 as a surface so that it is the image of the quadric which the group of holonomy of the space with a normal projective connexion corresponding to A_3 fixes, the following condition is necessary and sufficient:*

V_2 is a surface with constant curvature, or for the tensor $L_{ab}(k)$ and the scalar $F(k)$ depending on a constant k the following relation holds good

$$L_{ab,c} - L_{ab}\left\{\log F\left(K - \frac{k}{2}\right)^{\frac{1}{4}}\right\}_{,c} - \frac{1}{4\left(K - \frac{k}{2}\right)}(K_{,a}L_{bc} + K_{,b}L_{ac}) = 0.$$

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