ON THE REPRESENTATIONS OF GROUPS OF FINITE ORDER

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Introduction.

The representations of a group $\mathfrak{G}$ of finite order g were first studied by G. Frobenius[1] in his theory of group characters. The coefficients of the linear transformations are taken as complex numbers, but we may take them as the elements of an algebraically closed field of characteristic 0. Recently the modular representations of $\mathfrak{G}$ (i.e. representations of $\mathfrak{G}$ by matrices with coefficients in a modular field) which were first treated by L. E. Dickson[2], has been studied by R. Brauer and C. Nesbitt jointly and very interesting results have been obtained[3]. In the present paper, we shall give a new method to the theory of group representations which enables us in particular to prove the orthogonality relations for group characters in a quite natural way.

In Part I, we study the properties of the regular representations of algebras. Let A be an algebra with unit element. Let A' be an algebra anti-isomorphic to A and $a \rightarrow a'$ an anti-isomorphism between A and A'. If we denote by $S(a)$ and $R(a)$ the left and the right regular representations of A, then $a \times b' \rightarrow S(a)R'(b)$ is a representation of the direct product $A \times A'$ where $R'(b)$ is the transpose of $R(b)$. We can derive the properties of the regular representations of A by studying the structure of the representation $S(a)R'(b)$ of $A \times A'$. Theorem 1 and Theorem 2 play a principal role in our theory. Applying Theorem 1 to the group ring of $\mathfrak{G}$ we can obtain the orthogonality relations for group characters. The relations for the induced characters of $\mathfrak{G}$ are derived from Theorem 2. In Parts II and III, we study the ordinary representations and the modular representations of $\mathfrak{G}$ respectively. In particular, we can obtain a simple proof of the

1) All of Frobenius' papers were published in the Sitzungsber. Preuss. Akad. A complete list of titles is to be found in Speiser (18). Three treatments of the theory were given by Burnside (7), Schur (16) and Noeter (14). Cf. also the accounts in Dickson (10), Speiser (18) and Waerden (20).

2) Dickson (8), (9).

3) Brauer (2). Brauer-Nesbitt (4), (6).

fundamental relation between the Cartan invariants and the decomposition numbers of $\mathfrak{G}$[1]. The last Part deals with the representations of $\mathfrak{G}$ by collineations.

I. Regular representations of algebras.[2]

1. Let A be an (associative) algebra with unit element 1 over an algebraically closed field K, and N be the radical of A. Let

$$\bar{A} = A/N = \bar{A}_1 + \bar{A}_2 + \cdots\cdots + \bar{A}_n$$

be a decomposition of residue class algebra $\bar{A} = A/N$ into a direct sum of simple two-sided ideals $\bar{A}_\lambda$. Denote by $\bar{E}_1, \bar{E}_2, \cdots\cdots, \bar{E}_n$ the unit elements of $\bar{A}_1, \bar{A}_2, \cdots\cdots, \bar{A}_n$. Each $\bar{E}_\lambda$ can be decomposed into a sum of mutually orthogonal idempotent elements $\bar{e}_{\lambda,1}, \bar{e}_{\lambda,2}, \cdots\cdots, \bar{e}_{\lambda,f(\lambda)}$ such that left ideals $\bar{A}\bar{e}_{\lambda,i}$ as well as right ideals $\bar{e}_{\lambda,i}\bar{A}$ are simple. There exist mutually orthogonal idempotent elements $e_{\lambda,i}$ in A such that $e_{\lambda,i}$ (mod N) $= \bar{e}_{\lambda,i}$ ($\lambda = 1, 2, \cdots\cdots, n$; $i = 1, 2, \cdots\cdots, f(\lambda)$). If we put $E_\lambda = e_{\lambda,1} + e_{\lambda,2} + \cdots\cdots + e_{\lambda,f(\lambda)}$, then $E_1 + E_2 + \cdots\cdots + E_n = 1$. A is a direct sum:

$$A = Ae_{1,1} + \cdots\cdots + Ae_{1,f(1)} + Ae_{2,1} + \cdots\cdots + Ae_{n,f(n)}$$
$$[A = e_{1,1}A + \cdots\cdots + e_{1,f(1)}A + e_{2,1}A + \cdots\cdots + e_{n,f(n)}A].$$

The idempotent elements $e_{\lambda,i}$ are primitive and the left ideals $Ae_{\lambda,i}$ as well as the right ideals $e_{\lambda,i}A$ are directly indecomposable. Further $Ae_{\lambda,i}$ $[e_{\lambda,i}A]$ with one and the same first suffix λ, and only those are (operator-) isomorphic to each other:

$$Ae_{\lambda,1} \cong Ae_{\lambda,2} \cong \cdots\cdots \cong Ae_{\lambda,f(\lambda)} \; [e_{\lambda,1}A \cong e_{\lambda,2}A \cong \cdots\cdots \cong e_{\lambda,f(\lambda)}A].$$

For the sake of simplicity, let us denote one of $e_{\lambda,i}$ ($i = 1, 2, \cdots\cdots, f(\lambda)$), say $e_{\lambda,1}$, by e_λ. Incidentally we denote $\bar{e}_{\lambda,1}$ by $\bar{e}_\lambda$. Let U_λ and V_λ be the indecomposable representations of A belonging to the left ideal Ae_λ and the right ideal $e_\lambda A$ respectively. Then

$$U_\lambda \cong \begin{pmatrix} F_\lambda & \\ * & M_\lambda \end{pmatrix}, \qquad V_\lambda \cong \begin{pmatrix} F_\lambda & * \\ & N_\lambda \end{pmatrix}$$

where M_λ and N_λ are (reducible or irreducible) representations of A

1) See Brauer-Nesbitt (4), Nakayama (11) and Brauer (3).
2) Cf. Brauer-Nesbitt (5), Nakayama (11) and Nesbitt (13).

and F_λ is the irreducible representation of A belonging to $\bar{A}\bar{e}_\lambda$ [$\bar{e}_\lambda \bar{A}$]. All these are well known.

Let $m_1, m_2, \cdots\cdots, m_t$ be a basis of A. For every a in A, we have equations

$$(1.1) \qquad \begin{aligned} am_\lambda &= \sum_\kappa s_{\kappa\lambda} m_\kappa \\ m_\lambda a &= \sum_\kappa r_{\kappa\lambda} m_\kappa \end{aligned}$$

where the coefficients $s_{\kappa\lambda}$ and $r_{\kappa\lambda}$ lie in K. We then obtain two representations of A by associating the matrices $S(a) = (s_{\kappa\lambda})$, $R(a) = (r_{\lambda\kappa})$ with a. These representations $a \rightarrow S(a)$ and $a \rightarrow R(a)$ are called the left and the right regular representations of A respectively. Since equations (1.1) become in matrix form

$$(1.2) \qquad \begin{aligned} a(m_1\, m_2 \cdots\cdots m_t) &= (m_1\, m_2 \cdots\cdots m_t)\, S(a) \\ (m_1\, m_2 \cdots\cdots m_t)a &= (m_1\, m_2 \cdots\cdots m_t)\, R'(a) \end{aligned}$$

we have

$$(1.3) \qquad a(m_1\, m_2 \cdots\cdots m_t)b = (m_1\, m_2 \cdots\cdots m_t)\, S(a)\, R'(b).$$

Let A' be an algebra anti-isomorphic to A and $a \rightarrow a'$ an anti-isomorphism between A and A'. Then $a' \rightarrow R'(a)$ is the left regular representation of A'. Since $S(a)R'(b) = R'(b)S(a)$ for any $a, b \in A$, $a \times b \rightarrow S(a)R'(b)$ is a representation of the direct product $A \times A'$. (1.3) *shows that the representation* $S(a)R'(b)$ *of* $A \times A'$ *belongs to the A-two-sided module* A. Since $a' \rightarrow F'_\lambda(a)$ ($\lambda = 1, 2, \cdots\cdots, n$) are the irreducible representations of A', the distinct irreducible representations of $A \times A'$ are given by $F_\kappa(a) \times F'_\lambda(b)$ ($\kappa, \lambda = 1, 2, \cdots\cdots, n$) according to our assumption concerning K. Let $c_{\kappa\lambda}$ denote the multiplicity of $F_\kappa(a) \times F'_\lambda(b)$ as irreducible constituent of $S(a)R'(b)$;

$$(1.4) \qquad S(a)R'(b) \leftrightarrow \sum_{\kappa, \lambda} c_{\kappa\lambda}(F_\kappa(a) \times F'_\lambda(b))$$

(the sign $\leftrightarrow$ indicates that two representations have the same irreducible constituents).

Lemma 1. *Let* $A \supset A_1 \supset A_2 \supset \cdots\cdots \supset A_r = 0$ *be a composition series of two-sided ideals of* A.

1) *If* $A_i e_\lambda \supset A_{i+1} e_\lambda$, *then the A-left-module* $A_i e_\lambda / A_{i+1} e_\lambda$ *is simple and* $A_i e_\mu = A_{i+1} e_\mu$ *for* $\mu \neq \lambda$.

2) *If* $e_\kappa A_\iota \supset e_\kappa A_{\iota+1}$, *then the A-right-module* $e_\kappa A_\iota / e_\kappa A_{\iota+1}$ *is simple and* $e_\nu A_\iota = e_\nu A_{\iota+1}$ *for* $\nu \neq \kappa$.

3) *If* $A_\iota e_\lambda / A_{\iota+1} e_\lambda \cong \bar{A}\bar{e}_\kappa$, *then* $e_\kappa A_\iota / e_\kappa A_{\iota+1} \cong \bar{e}_\lambda \bar{A}$, *and conversely.*

Proof. If $F_\kappa(a) \times F'_\lambda(b)$ belongs to the A-two-sided module $A_\iota / A_{\iota+1}$, then the A-left-module $A_\iota / A_{\iota+1}$ is a direct sum of $f(\lambda)$ simple left-moduli isomorphic to $\bar{A}\bar{e}_\kappa$, and the A-right-module $A_\iota / A_{\iota+1}$ is a direct sum of $f(\kappa)$ simple right-moduli isomorphic to $\bar{e}_\lambda \bar{A}$:

$$A_\iota / A_{\iota+1} \cong \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots\cdots + \mathfrak{M}_{f(\lambda)}, \qquad \mathfrak{M}_s \cong \bar{A}\bar{e}_\kappa$$
$$A_\iota / A_{\iota+1} \cong \mathfrak{N}_1 + \mathfrak{N}_2 + \cdots\cdots + \mathfrak{N}_{f(\kappa)}, \qquad \mathfrak{N}_t \cong \bar{e}_\lambda \bar{A}.$$

Since A_ι is a direct sum: $A_\iota = A_\iota E_1 + A_\iota E_2 + \cdots\cdots + A_\iota E_n$, we have for the A-left-module $A_\iota / A_{\iota+1}$

$$\begin{aligned} A_\iota / A_{\iota+1} &= A_\iota E_1 / A_{\iota+1} E_1 + A_\iota E_2 / A_{\iota+2} E_2 + \cdots\cdots + A_\iota E_n / A_{\iota+1} E_n \\ &= (A_\iota / A_{\iota+1}) E_1 + (A_\iota / A_{\iota+1}) E_2 + \cdots\cdots + (A_\iota / A_{\iota+1}) E_n. \end{aligned}$$

While we have

$$(A_\iota / A_{\iota+1}) E_\mu \cong (\mathfrak{N}_1 + \mathfrak{N}_2 + \cdots\cdots + \mathfrak{N}_{f(\kappa)}) E_\mu = 0 \qquad (\mu \neq \lambda)$$

since $\mathfrak{N}_t E_\mu \cong \bar{e}_\lambda \bar{A} E_\mu = 0$. This shows that $A_\iota E_\mu = A_{\iota+1} E_\mu$ for $\mu \neq \lambda$ and

$$\begin{aligned} A_\iota / A_{\iota+1} &= A_\iota E_\lambda / A_{\iota+1} E_\lambda \\ &= A_\iota e_{\lambda,1} / A_{\iota+1} e_{\lambda,1} + \cdots\cdots + A_\iota e_{\lambda, f(\lambda)} / A_{\iota+1} e_{\lambda, f(\lambda)}. \end{aligned}$$

We then have $A_\iota e_\lambda / A_{\iota+1} e_\lambda \cong \bar{A}\bar{e}_\kappa$ and $A_\iota e_\mu = A_{\iota+1} e_\mu$ for $\mu \neq \lambda$. Similarly, we have for the A-right-module $A_\iota / A_{\iota+1}$

$$A_\iota / A_{\iota+1} = E_\kappa A_\iota / E_\kappa A_{\iota+1}, \qquad E_\nu A_\iota / E_\nu A_{\iota+1} = 0 \qquad (\nu \neq \kappa).$$

Hence $e_\kappa A_\iota / e_\kappa A_{\iota+1} \cong \bar{e}_\lambda \bar{A}$ and $e_\nu A_\iota = e_\nu A_{\iota+1}$ for $\nu \neq \kappa$. This completes the proof.

From the composition series of A in Lemma 1, we have

$$Ae_\lambda \supseteqq A_1 e_\lambda \supseteqq \cdots\cdots \supseteqq A_r e_\lambda = 0.$$

We can choose a subsequence $Ae_\lambda = B_0 e_\lambda, B_1 e_\lambda, \cdots\cdots, B_{m(\lambda)} e_\lambda = 0$ from this sequence such that $B_i e_\lambda \supset B_{i+1} e_\lambda$ and every $A_j e_\lambda$ is equal to one of them. According to Lemma 1

$$Ae_\lambda \supset B_1 e_\lambda \supset \cdots\cdots \supset B_{m(\lambda)} e_\lambda = 0$$

is a composition series of the left ideal Ae_λ. We obtain readily from Lemma 1 the following

Theorem 1. *Let* $c_{\kappa\lambda}$ *denote multiplicity of* $F_\kappa(a) \times F'_\lambda(b)$ *in* $S(a)R'(b)$, *Then*

1) $$\begin{cases} U_\lambda(a) \leftrightarrow \sum_\kappa c_{\kappa\lambda} F_\kappa(a) \\ V_\kappa(a) \leftrightarrow \sum_\lambda c_{\kappa\lambda} F_\lambda(a) \end{cases}$$

2) $$S(a)R'(b) \leftrightarrow \sum_\lambda U_\lambda(a) \times F'_\lambda(b) \leftrightarrow \sum_\kappa F_\kappa(a) \times V'_\kappa(b).$$

Theorem 1 shows that the $c_{\kappa\lambda}$ are the *Cartan invariants* of A. Let $m(\lambda)$ and $n(\kappa)$ be the lengths of composition series of Ae_λ and $e_\lambda A$. Then

$$\sum_\lambda m(\lambda) = \sum_\kappa n(\kappa) = \sum_{\kappa,\lambda} c_{\kappa\lambda} = r.$$

Corollary. *If A is semi-simple, then*

$$S(a)R'(b) \cong \sum_\kappa F_\kappa(a) \times F'_\kappa(b). \tag{1.5}$$

As one can easily see, we can replace in Lemma 1 A by any two-sided ideal $\mathfrak{A}$ of A. Hence, if U^*_λ and V^*_κ are the representations of A belonging to $\mathfrak{A}e_\lambda$ and $e_\kappa\mathfrak{A}$, then

$$U_\lambda \cong \begin{pmatrix} G_\lambda & \\ * & U^*_\lambda \end{pmatrix}, \qquad V_\kappa \cong \begin{pmatrix} H_\kappa & * \\ & V^*_\kappa \end{pmatrix}$$

and

$$\begin{cases} U^*_\lambda(a) \leftrightarrow \sum_\kappa h_{\kappa\lambda} F_\kappa(a) \\ V^*_\kappa(a) \leftrightarrow \sum_\lambda h_{\kappa\lambda} F_\lambda(a). \end{cases} \tag{1.6}$$

2. Let B and C be two subalgebras of A having the unit element 1 in common with A. Define $E^{(1)}_\kappa$ and $e^{(1)}_\kappa$ of B in the same way as we defined E_λ and e_λ of A. Let $\bar{B}$ be the residue class algebra of B with respect to its radical, and let $\bar{e}^{(1)}_\kappa$ be the residue class containing $e^{(1)}_\kappa$. Let $F^{(1)}_1, F^{(1)}_2, \cdots\cdots, F^{(1)}_l$ be the distinct irreducible representations of B. We denote by $U^{(1)}_1, U^{(1)}_2, \cdots\cdots, U^{(1)}_l$ and $V^{(1)}_1, V^{(1)}_2, \cdots\cdots, V^{(1)}_l$ the indecomposable constituents of the left and the right regular representations ot B respectively, where $U^{(1)}_\kappa$ and $V^{(1)}_\kappa$ belong to $Be^{(1)}_\kappa$ and $e^{(1)}_\kappa B$. We call the representations of A belonging to $Ae^{(1)}_\kappa$ and $e^{(1)}_\kappa A$, the induced representations of A from $U^{(1)}_\kappa$ and $V^{(1)}_\kappa$, and denote by $\widetilde{U}_\kappa^{(1)}$ and $\widetilde{V}_\kappa^{(1)}$. Similarly we can define $F_\lambda^{(2)}$, $U_\lambda^{(2)}$, $V_\lambda^{(2)}$, $\widetilde{U}_\lambda^{(2)}$ and $\widetilde{V}_\lambda^{(2)}$ $(\lambda = 1, 2, \cdots\cdots, m)$ with respect to C. Let $S(a)$ and $R(a)$ have the same meaning as in section 1. Then $b \times c' \rightarrow S(b)R'(c)$ for $b \in B$ and $c \in C$, is a representation of the direct product $B \times C'$ and belongs to the B-C-double-module A. The distinct irreducible representations of $B \times C'$ are given by $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$ $(\kappa = 1, 2, \cdots\cdots, l;\ \lambda = 1, 2, \cdots\cdots, m)$. Corresponding to Lemma 1, we have

Lemma 2. *Let $A \supset M_1 \supset M_2 \supset \cdots\cdots \supset M_s = 0$ be a composition series of B-C-double-module A. Then*

1) *If $M_i e_\lambda^{(2)} \supset M_{i+1} e_\lambda^{(2)}$, then the B-left-module $M_i e_\lambda^{(2)} / M_{i+1} e_\lambda^{(2)}$ is simple and $M_i e_\mu^{(2)} = M_{i+1} e_\mu^{(2)}$ for $\mu \neq \lambda$.*

2) *If $e_\kappa^{(1)} M_i \supset e_\kappa^{(1)} M_{i+1}$, then the C-right-module $e_\kappa^{(1)} M_i / e_\kappa^{(1)} M_{i+1}$ is simple and $e_\nu^{(1)} M_i = e_\nu^{(1)} M_{i+1}$ for $\nu \neq \kappa$.*

3) *If $M_i e_\lambda^{(2)} / M_{i+1} e_\lambda^{(2)} \cong \bar{B}\bar{e}_\kappa^{(1)}$, then $e_\kappa^{(1)} M_i / e_\kappa^{(1)} M_{i+1} \cong \bar{e}_\lambda^{(2)} \bar{C}$, and conversely.*

Proof. If $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$ belongs to the B-C-double-module M_i / M_{i+1} then the B-left-module M_i / M_{i+1} is a direct sum of $f_2(\lambda)$ simple left-moduli isomorphic to $\bar{B}\bar{e}_\kappa^{(1)}$, and the C-right-module M_i / M_{i+1} is a direct sum of $f_1(\kappa)$ simple right-moduli isomorphic to $\bar{e}_\lambda^{(2)}\bar{C}$:

$$M_i / M_{i+1} \cong \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots\cdots + \mathfrak{S}_{f_2(\lambda)}, \qquad \mathfrak{S}_u \cong \bar{B}\bar{e}_\kappa^{(1)}$$
$$M_i / M_{i+1} \cong \mathfrak{T}_1 + \mathfrak{T}_2 + \cdots\cdots + \mathfrak{T}_{f_1(\kappa)}, \qquad \mathfrak{T}_v \cong \bar{e}_\lambda^{(2)}\bar{C}.$$

Since M_i is a direct sum:

$$\begin{aligned} M_i &= M_i E_1^{(2)} + M_i E_2^{(2)} + \cdots\cdots + M_i E_m^{(2)} \\ &= E_1^{(1)} M_i + E_2^{(1)} M_i + \cdots\cdots + E_l^{(1)} M_i, \end{aligned}$$

we have in a quite similar manner as Lemma 1

$$M_i / M_{i+1} = M_i E_\lambda^{(2)} / M_{i+1} E_\lambda^{(2)}, \quad M_i E_\mu^{(2)} / M_{i+1} E_\mu^{(2)} = 0 \qquad (\mu \neq \lambda)$$
$$M_i / M_{i+1} = E_\kappa^{(1)} M_i / E_\kappa^{(1)} M_{i+1}, \quad E_\nu^{(1)} M_i / E_\nu^{(1)} M_{i+1} = 0 \qquad (\nu \neq \kappa).$$

Hence we obtain readily our assertions.

As an immediate consequence we have

Theorem 2. *Let $\sigma_{\kappa\lambda}$ denote the multiplicity of $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$ in $S(b)R'(c)$. Then*

1)
$$\begin{cases} \widetilde{U}_\lambda^{(2)}(b) \leftrightarrow \sum_\kappa \sigma_{\kappa\lambda} F_\kappa^{(1)}(b) & (\textit{for } b \in B) \\ \widetilde{V}_\kappa^{(1)}(c) \leftrightarrow \sum_\lambda \sigma_{\kappa\lambda} F_\lambda^{(2)}(c) & (\textit{for } c \in C) \end{cases}$$

2) $S(b)R'(c) \leftrightarrow \sum_{\kappa} F_{\kappa}^{(1)}(b) \times (\tilde{V}_{\kappa}^{(1)}(c))' \leftrightarrow \sum_{\lambda} \tilde{U}_{\lambda}^{(2)}(b) \times (F_{\lambda}^{(2)}(c))'$.

Corollary. *Let $\sigma_{\lambda\kappa}^{*}$ denote the multiplicity of $F_{\lambda}^{(2)}(c) \times (F_{\kappa}^{(1)}(b))'$ in $S(c)R'(b)$. Then*

1)
$$\begin{cases} \tilde{U}_{\kappa}^{(1)}(c) \leftrightarrow \sum_{\lambda} \sigma_{\lambda\kappa}^{*} F_{\lambda}^{(2)}(c) & (\text{for } c \in C) \\ \tilde{V}_{\lambda}^{(2)}(b) \leftrightarrow \sum_{\kappa} \sigma_{\lambda\kappa}^{*} F_{\kappa}^{(2)}(b) & (\text{for } b \in B) \end{cases}$$

2) $S(c)R'(b) \leftrightarrow \sum_{\lambda} F_{\lambda}^{(2)}(c) \times (\tilde{V}_{\lambda}^{(2)}(b))' \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(c) \times (F_{\kappa}^{(1)}(b))'$.

In particular, for $C = A$, we have the following relations[1]

$$(2.1)\quad \begin{cases} U_{\lambda}(b) \leftrightarrow \sum_{\kappa} \pi_{\kappa\lambda} F_{\kappa}^{(1)}(b) & (\text{for } b \in B) \\ \tilde{V}_{\kappa}^{(1)}(a) \leftrightarrow \sum_{\lambda} \pi_{\kappa\lambda} F_{\lambda}(a) & (\text{for } a \in A) \end{cases}$$

$$(2.2)\quad \begin{cases} \tilde{U}_{\kappa}^{(1)}(a) \leftrightarrow \sum_{\lambda} \pi_{\lambda\kappa}^{*} F_{\lambda}(a) & (\text{for } a \in A) \\ V_{\lambda}(b) \leftrightarrow \sum_{\kappa} \pi_{\lambda\kappa}^{*} F_{\kappa}^{(1)}(b) & (\text{for } b \in B) \end{cases}$$

$$(2.3)\quad \begin{cases} S(b)R'(a) \leftrightarrow \sum_{\kappa} F_{\kappa}^{(1)}(b) \times (\tilde{V}_{\kappa}^{(1)}(a))' \leftrightarrow \sum_{\lambda} U_{\lambda}(b) \times F_{\lambda}'(a) \\ S(a)R'(b) \leftrightarrow \sum_{\lambda} F_{\lambda}(a) \times V_{\lambda}'(b) \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(a) \times (F_{\kappa}^{(1)}(b))'. \end{cases}$$

Further, for $C = B$, we have

$$(2.4)\quad \begin{cases} \tilde{U}_{\lambda}^{(1)}(b) \leftrightarrow \sum_{\kappa} \omega_{\kappa\lambda} F_{\kappa}^{(1)}(b) \\ \tilde{V}_{\kappa}^{(1)}(b) \leftrightarrow \sum_{\lambda} \omega_{\kappa\lambda} F_{\lambda}^{(1)}(b) \end{cases} \quad (\text{for } b \in B).$$

Theorem 1 and (2.3) yield

$$(2.5)\quad S(a)R'(b) \leftrightarrow \sum_{\lambda} U_{\lambda}(a) \times F_{\lambda}'(b) \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(a) \times (F_{\kappa}^{(1)}(b))'.$$

If we set $B^{*} = B/(B \cap N)$, then B^{*} is a subalgebra of $\bar{A} = A/N$. Since $B \cap N$ is contained in the radical of B, B^{*} has the same irreducible representations $F_{\kappa}^{(1)}$ ($\kappa = 1, 2, \cdots\cdots, l$) with B. Let U_{κ}^{*} be the indecomposable constituent of the left regular representation of B^{*} corresponding to $F_{\kappa}^{(1)}$. Then we get

$$U_{\kappa}^{(1)} \cong \begin{pmatrix} U_{\kappa}^{*} & \\ * & W_{\kappa} \end{pmatrix}.$$

1) Cf. Nakayama [11] p. 335.

Further, let us denote by $\tilde{U}_\kappa^*$ the representation of $\bar{A}$ induced from U_κ^*. If $\tilde{U}_\kappa^{(1)}(a) \cong \sum_\lambda \alpha_{\kappa\lambda} U_\lambda(a)$ for $a \in A$, then

$$\tilde{U}_\kappa^*(a) \cong \sum_\lambda \alpha_{\kappa\lambda} F_\lambda(a). \tag{2.6}$$

(2.5), applied to $\bar{A}$ and its subalgebra B^*, gives

$$\bar{S}(a)\bar{R}'(b) \leftrightarrow \sum_\lambda F_\lambda(a) \times F'_\lambda(b) \leftrightarrow \sum_\kappa \tilde{U}_\kappa^*(a) \times (F_\kappa^{(1)}(b))'$$

where $\bar{S}(\bar{a})$ and $\bar{R}(\bar{a})$ (for $\bar{a} \in \bar{A}$) are the regular representations of $\bar{A}$. We then have from (2.6)

$$F_\lambda(b) \leftrightarrow \sum_\kappa \alpha_{\kappa\lambda} F_\kappa^{(1)}(b).$$

Hence we have formulas

$$\begin{cases} F_\lambda(b) \leftrightarrow \sum_\kappa \alpha_{\kappa\lambda} F_\kappa^{(1)}(b) & (\text{for } b \in B) \\ \tilde{U}_\kappa^{(1)}(a) \leftrightarrow \sum_\lambda \alpha_{\kappa\lambda} U_\lambda(a) & (\text{for } a \in A). \end{cases} \tag{2.7}$$

Similarly we obtain $\tilde{V}_\kappa^{(1)}(a) \cong \sum_\lambda \alpha_{\kappa\lambda} V_\lambda(a)$ with the same $\alpha_{\kappa\lambda}$.

II. Ordinary representations of groups.

3. Let $\Gamma(\mathfrak{G})$ be the group ring of a group $\mathfrak{G}$ over an algebraically closed field K of characteristic 0:

$$\Gamma(\mathfrak{G}) = G_1K + G_2K + \cdots\cdots + G_gK, \qquad G_1 = 1$$

where $G_1, G_2, \cdots\cdots, G_g$ are the elements of $\mathfrak{G}$. Instead of considering representations of $\mathfrak{G}$, we may consider representations of $\Gamma(\mathfrak{G})$. Let $Z_1, Z_2, \cdots\cdots, Z_n$ be the distinct irreducible representations of $\mathfrak{G}$. To each Z_i there corresponds a contragredient (irreducible) representation $G \to Z_i(G^{-1})$ $(G \in \mathfrak{G})$ which we denote by $Z_{i'}$. Let $S(G)$ and $R(G)$ be the left and the right regular representations of $\Gamma(\mathfrak{G})$ defined by a basis $G_1, G_2, \cdots\cdots, G_g$. Then

$$G_s \times G_t \to S(G_s)R'(G_t^{-1})$$

is a representation of the direct product $\mathfrak{G} \times \mathfrak{G}$. Since $\Gamma(\mathfrak{G})$ is semisimple, we have from (1.5)

$$S(G_s)R'(G_t^{-1}) \cong \sum_i Z_i(G_s) \times Z_{i'}(G_t). \tag{3.1}$$

Let $C_1, C_2, \cdots\cdots, C_m$ be the classes of conjugate elements in $\mathfrak{G}$ and let n_ν be the order of the normalizer $\mathfrak{N}(G)$ of an element G contained in C_ν. Then $g_\nu = g/n_\nu$ denotes the number of elements in C_ν. Denote by C_{ν^*} the class containing the elements reciprocal to those of C_ν.

Theorem 3. *Let $\psi(G_s \times G_t)$ be the character of the representation $S(G_s)R'(G_t^{-1})$ of $\mathfrak{G} \times \mathfrak{G}$. Then*

$$\psi(G_s \times G_t) = n_\nu \delta_{\nu\mu} \qquad (\text{for } G_s \in G_\nu, \ G_t \in C_\mu).$$

Proof. From $G_s(G_1 G_2 \cdots\cdots G_g)G_t^{-1} = (G_1 G_2 \cdots\cdots G_g)S(G_s)R'(G_t^{-1})$, we have $S(G_s)R'(G_t^{-1}) = (\alpha_{kl}(G_s \times G_t))$ where $(\alpha_{kl}(G_s \times G_t))$ possesses one 1 in each column and row. If $G_k^{-1}G_sG_k \neq G_t$ for any G_k, then $G_sG_kG_t^{-1} \neq G_k$, hence $\alpha_{kk}(G_s \times G_t) = 0$ for any k. This implies that $\psi(G_s \times G_t) = 0$ for $\nu \neq \mu$. Now we consider the case when $G_t = G_s$. $G_sG_kG^{-1} = G_k$, that is, $\alpha_{kk}(G_s \times G_s) = 1$ if and only if G_k lies in $\mathfrak{N}(G_s)$. Hence we have $\psi(G_s \times G_s) = n_\nu$. Finally suppose that G_s and G_t are conjugate in $\mathfrak{G}$. From $G_t = G_r^{-1}G_sG_r$, we find

$$\begin{aligned} S(G_s)R'(G_t^{-1}) &= S(G_s)R'(G_r)R'(G_s^{-1})R'(G_r^{-1}) \\ &= R'(G_r)S(G_s)R'(G_s^{-1})(R'(G_r))^{-1}. \end{aligned}$$

This shows that $\psi(G_s \times G_t) = \psi(G_s \times G_s) = n_\nu$.

We denote by χ_i the character of Z_i. The value of a character χ_i for the class C_ν will be indicated by $\chi_i^{(\nu)}$. From (3.1) and Theorem 3, we have the orthogonality relation for ordinary group characters:

$$\sum_i \chi_i^{(\nu)}\chi_i^{(\mu)} = n_\nu \delta_{\nu\mu^*}. \tag{3.2}$$

We arrange $\chi_i^{(\nu)}$ in matrix form $Z = (\chi_i^{(\nu)})$ (i row index, ν column index). Then (3.2) becomes

$$Z'Z = (n_\nu \delta_{\nu\mu^*}) = T. \tag{3.3}$$

Since T in (3.3) is non-singular, we obtain $u = m$ by a well known manner. The number of distinct (absolutely) irreducible representations is equal to the number of classes of conjugate elements in $\mathfrak{G}$. We can derive from (3.2)

$$\sum_\nu g_\nu \chi_i^{(\nu)}\chi_j^{(\nu^*)} = g\delta_{ij} \qquad (i, j = 1, 2, \cdots\cdots, u). \tag{3.4}$$

Further, (3.4) yields

(3.5) $$\sum_{G \in \mathfrak{G}} \chi_i(G) = \begin{cases} g & (i = 1) \\ 0 & (i \neq 1). \end{cases}$$

Here, χ_1 means the character of the 1-representation.

4. Let $\mathfrak{H}$ and $\mathfrak{J}$ be two subgroups of $\mathfrak{G}$, and denote by $\xi_1, \xi_2, \cdots\cdots, \xi_s$ the irreducible characters of $\mathfrak{H}$ and by $\zeta_1, \zeta_2, \cdots\cdots, \zeta_t$ those of $\mathfrak{J}$. Let $\tilde{\xi}_\lambda$ and $\tilde{\zeta}_\iota$ be the characters of $\mathfrak{G}$ induced from ξ_λ and ζ_ι. From Theorem 2 we have

(4.1) $$\begin{cases} \tilde{\zeta}_\iota(H) = \sum_\lambda k_{\iota\lambda}\xi_\lambda(H) & (\text{for } H \in \mathfrak{H}) \\ \tilde{\xi}_\lambda(J) = \sum_\iota k_{\iota\lambda}\zeta_\iota(J) & (\text{for } J \in \mathfrak{J}). \end{cases}$$

In particular, for $\mathfrak{J} = \mathfrak{G}$, we have following Frobenius' theorem on induced characters:

(4.2) $$\begin{cases} \chi_i(H) = \sum_\lambda l_{i\lambda}\xi_\lambda(H) & (\text{for } H \in \mathfrak{H}) \\ \tilde{\xi}_\lambda(G) = \sum_i l_{i\lambda}\chi_i(G) & (\text{for } G \in \mathfrak{G}). \end{cases}$$

Further, from (2.4) we have

(4.3) $$\begin{cases} \tilde{\xi}_\lambda(H) = \sum_\kappa q_{\kappa\lambda}\xi_\kappa(H) \\ \tilde{\xi}_\kappa(H) = \sum_\lambda q_{\kappa\lambda}\xi_\lambda(H) \end{cases} \quad (\text{for } H \in \mathfrak{H}).$$

From (4.2) it follows that

$$\tilde{\xi}_\lambda(H) = \sum_i l_{i\lambda}\chi_i(H) = \sum_\kappa (\sum_i l_{i\kappa}l_{i\lambda})\xi_\kappa(H).$$

Then (4.3) yields $q_{\kappa\lambda} = \sum_i l_{i\kappa}l_{i\lambda}$, or in matrix form

(4.4) $$Q = L'L$$

where $Q = (q_{\kappa\lambda})$, $L = (l_{i\kappa})$. Theorem 3 and (2.3) yield for $H \in \mathfrak{H}$

(4.5) $$\sum \tilde{\xi}_\kappa(G)\xi_\kappa(H^{-1}) = \begin{cases} n(H) & \text{for } C(G) = C(H) \\ 0 & \text{for } C(G) \neq C(H) \end{cases}$$

where $C(G)$ denotes the class of conjugate elements in $\mathfrak{G}$ which contains G, and where $n(G)$ denotes the order of the normalizer $\mathfrak{N}(G)$. From (4.3) we obtain

$$\sum_{H\in\mathfrak{H}}\tilde{\xi}_\kappa(H)\xi_\lambda(H^{-1}) = \sum_{H\in\mathfrak{H}}\sum_{\rho} q_{\kappa\rho}\xi_\rho(H)\xi_\lambda(H^{-1})$$
$$= \sum_{\rho} q_{\kappa\rho}\sum_{H\in\mathfrak{H}}\xi_\rho(H)\xi_\lambda(H^{-1}) = q_{\kappa\lambda}h$$

where h denotes the order of $\mathfrak{H}$. Hence

$$\sum_{\kappa}\sum_{H\in\mathfrak{H}}\tilde{\xi}_\kappa(H)\xi_\kappa(H^{-1}) = \sum_{\kappa} q_{\kappa\kappa}h = \sum_{H\in\mathfrak{H}} n(H).$$

Consequently we have

$$\mathrm{tr}(q_{\kappa\lambda}) = \sum_{H\in\mathfrak{H}} n(H)\,/\,h. \tag{4.6}$$

Let us denote by $\bar{C}_1, \bar{C}_2, \cdots\cdots, \bar{C}_k$ the classes of conjugate elements in $\mathfrak{G}$ which contain an element of $\mathfrak{H}$, and let $H_1, H_2, \cdots\cdots, H_k$ ($H_i \in \mathfrak{H}$) be a complete system of representatives for these classes.

Theorem 4. *The number of linearly independent characters of $\mathfrak{G}$ induced from the s distinct irreducible characters ξ_κ of $\mathfrak{H}$ is equal to the number k of those $\bar{C}_\nu$ which contain an element of $\mathfrak{H}$.*

Proof. If we arrange $\tilde{\xi}_\kappa(H_m)$ and $\xi_\kappa(H_m^{-1})$ in matrix form

$$W = (\tilde{\xi}_\kappa(H_m)), \qquad U = (\xi_\kappa(H_m^{-1}))$$

(κ row index; m column index). Then (4.5) becomes

$$W'U = (n(H_m)\delta_{mn}) = S.$$

Since S is non-singular, the rank of W is equal to k. But we have $\tilde{\xi}_\kappa(G) = 0$ for every $G \notin \bar{C}_\nu$ ($\nu = 1, 2, \cdots\cdots, k$), whence the number of linearly independent characters among $\tilde{\xi}_1, \tilde{\xi}_2, \cdots\cdots, \tilde{\xi}_s$ is equal to k.

5. Let $\bar{\mathfrak{G}}$ be a group isomorphic to $\mathfrak{G}$ by correspondence $G_m \rightarrow \bar{G}_m$. Then the elements $G_m \times \bar{G}_m$ ($m = 1, 2, \cdots\cdots, g$) of the direct product $\mathfrak{G} \times \bar{\mathfrak{G}}$ form the subgroup $\mathfrak{G}_0$ isomorphic to $\mathfrak{G}$. We can choose $G_1, G_2, \cdots\cdots, G_g$ as a complete residue system of $\mathfrak{G} \times \bar{\mathfrak{G}}$ (mod $\mathfrak{G}_0$):

$$\mathfrak{G} \times \bar{\mathfrak{G}} = G_1\mathfrak{G}_0 + G_2\mathfrak{G}_0 + \cdots\cdots + G_g\mathfrak{G}_0, \qquad G_1 = 1.$$

Lemma 3. *Let us denote by $\tilde{D}$ the representation of $\mathfrak{G} \times \bar{\mathfrak{G}}$ induced from a representation D of $\mathfrak{G}_0$. Then*

$$\tilde{D}(G_m \times \bar{G}_n) \cong S(G_m)R'(G_n^{-1}) \times D(G_n)$$

where $S(G)$ and $R(G)$ are the regular representations of $\mathfrak{G}$.

Proof. We have

$$\widetilde{D}(G_m \times \bar{G}_n) \cong (D(G_k(G_m \times \bar{G}_n)G_l^{-1}))_{kl} = (D(G_kG_mG_l^{-1} \times \bar{G}_n))_{kl}$$

where $D(G_s \times G_t)$ is defined to be the zero matrix for $G_s \times G_t$ not contained in $\mathfrak{G}_0$. If we set $M(G_m \times \bar{G}_n) = (D(G_kG_mG_l^{-1} \times \bar{G}_n))_{kl}$, then

$$M(G_m) = (D(G_kG_mG_l^{-1}))_{kl} = S(G_m) \times I_f$$

where f is the degree of D and I_f is the unit matrix of degree f. Further we have

$$\begin{aligned} M(\bar{G}_n) &= (D(G_kG_l^{-1} \times \bar{G}_n))_{kl} = R'(G_n^{-1}) \times D(G_n \times \bar{G}_n) \\ &= R'(G_n^{-1}) \times D(G_n) \end{aligned}$$

since we get $G_kG_n^{-1} = G_l$ from $G_kG_l = G_n$. Hence

$$\begin{aligned} \widetilde{D}(G_m \times \bar{G}_n) &\cong (S(G_m) \times I_f)(R'(G_n^{-1}) \times D(G_n)) \\ &= S(G_m)R'(G_n^{-1}) \times D(G_n). \end{aligned}$$

If we take $\bar{G}_1, \bar{G}_2, \cdots\cdots, \bar{G}_g$ as a complete residue system of $\mathfrak{G} \times \bar{\mathfrak{G}}$ (mod $\mathfrak{G}_0$), then we have in a same way, $S(G_n)R'(G_m^{-1}) \times D(G_m)$ as the representation of $\mathfrak{G} \times \bar{\mathfrak{G}}$ induced from D. Of course we find

$$S(G_m)R'(G_n^{-1}) \times D(G_n) \cong S(G_n)R'(G_m^{-1}) \times D(G_m). \tag{5.1}$$

In particular, for $G_n = 1$, we obtain

$$S(G) \times I_f \cong R'(G^{-1}) \times D(G) \qquad (G \in \mathfrak{G}). \tag{5.2}$$

We have finally $S(G) \times I_f \cong S(G) \times D(G)$[1], since $S(G) \cong R'(G^{-1})$. Further we can see that $S(G_m)R'(G_n^{-1})$ is the representation of $\mathfrak{G} \times \bar{\mathfrak{G}}$, induced from the 1-representation of $\mathfrak{G}_0$.

Let us denote the irreducible characters of $\mathfrak{G}$ by $\chi_1, \chi_2, \cdots\cdots, \chi_u$. Then the distinct irreducible characters of $\mathfrak{G} \times \bar{\mathfrak{G}}$ are given by $\chi_i(G_m)\chi_j(G_n)$ $(i, j = 1, 2, \cdots\cdots, u)$. Since $\chi_i(G)\chi_j(G)$ $(G \in \mathfrak{G})$ is a character of $\mathfrak{G}$, irreducible or reducible, we obtain formulas

$$\chi_i(G)\chi_j(G) = \sum_k a_{ijk}\chi_k(G) \tag{5.3}$$

where the a_{ijk} are rational integers, $a_{ijk} \geqq 0$, and $a_{ijk} = a_{jik}$. Let us

1) Cf. Osima (15).

denote by $\tilde{\chi}_k$ the character of $\mathfrak{G} \times \bar{\mathfrak{G}}$ induced from χ_k of $\mathfrak{G}_0$. Then from Lemma 3

$$\tilde{\chi}_k(G_m \times \bar{G}_n) = \sum_i \chi_i(G_m)\chi_{i'}(G_n)\chi_k(G_n) \tag{5.4}$$

where $\chi_{i'}$ is the character contragredient to χ_i.

Theorem 5. *If* $\chi_i(G)\chi_j(G) = \sum_k a_{ijk}\chi_k(G)$, *then* $\chi_{i'}(G)\chi_k(G) = \sum_j a_{ijk}\chi_j(G)$, *that is,* $a_{ijk} = a_{i'kj}$.

Proof. (4.2) applied to $\mathfrak{G} \times \bar{\mathfrak{G}}$ and its subgroup $\mathfrak{G}_0$, gives

$$\tilde{\chi}_k(G_m \times \bar{G}_n) = \sum_{i,j} a_{ijk}\chi_i(G_m)\chi_j(G_n).$$

Hence, by (5.4) we have

$$\sum_i \chi_i(G_m)\chi_{i'}(G_n)\chi_k(G_n) = \sum_{i,j} a_{ijk}\chi_i(G_m)\chi_j(G_n).$$

Since $\chi_1, \chi_2, \cdots\cdots, \chi_u$ are linearly independent, it follows that

$$\chi_{i'}(G_n)\chi_k(G_n) = \sum_j a_{ijk}\chi_j(G_n).$$

Theorem 6. *Let* $\chi_i^{(\nu)}$ *be the value of* χ_i *for the class* C_ν *of conjugate elements in* $\mathfrak{G}$. *Then*

$$\sum_\nu \chi_i^{(\nu)}\chi_j^{(\nu*)} = \sum_{k,l} a_{ikl}a_{jkl}.$$

Proof. From (5.3) and Theorem 5, it follows that

$$\sum_k \chi_k^{(\nu)}\chi_{k'}^{(\nu)}\chi_i^{(\nu)} = \sum_k \sum_l a_{ikl}\chi_l^{(\nu)}\chi_{k'}^{(\nu)} = \sum_{k,l} (\sum_m a_{ikl}a_{k'lm}\chi_m^{(\nu)})$$
$$= \sum_{k,l} (\sum_m a_{ikl}a_{mkl}\chi_m^{(\nu)}).$$

On the other hand, from (3.2)

$$\sum_k \chi_k^{(\nu)}\chi_{k'}^{(\nu)}\chi_i^{(\nu)} = n_\nu\chi_i^{(\nu)} = g\chi_i^{(\nu)}/g_\nu.$$

Hence

$$\sum_m (\sum_{k,l} a_{ikl}a_{mkl})g_\nu\chi_m^{(\nu)} = g\chi_i^{(\nu)}.$$

Here, we multiply by $\chi_j^{(\nu*)}$, and add over ν, and use (3.4)

$$\sum_{k,l} a_{ikl}a_{jkl} = \sum_\nu \chi_i^{(\nu)}\chi_j^{(\nu*)}.$$

We shall derive some further relations for the a_{ijk}. By Theorem 5

$$\sum_{k,l} a_{ikl}a_{jkl} = \sum_{k',l} a_{ik'l}a_{jk'l} = \sum_{k',l} a_{k'li}a_{k'lj}.$$

Thus we have

(5.5) $$\sum_{k,l} a_{ikl}a_{jkl} = \sum_{k,l} a_{kli}a_{klj}.$$

We can also show the following relations

$$\sum_{k,l} a_{ikl}a_{j'kl} = \sum_{k,l} a_{ikl}a_{jlk} = \sum_{\nu} \chi_i^{(\nu)}\chi_j^{(\nu)}.$$

In particular, from Theorem 6 we find

(5.6) $$\sum_{k,l} a_{ikl}^2 = \sum_{k,l} a_{kli}^2 = \sum_{\nu} \chi_i^{(\nu)}\chi_i^{(\nu*)}.$$

(5.7) $$\sum_{k} a_{kk'i} = \sum_{k} a_{ikk} = \sum_{\nu} \chi_i^{(\nu)}.$$

6. Let $Z_1, Z_2, \cdots\cdots, Z_u$ have the same meaning as in section 3. Denote by $X_1, X_2, \cdots\cdots, X_s$ the distinct irreducible representations of a subgroup $\mathfrak{H}$ of $\mathfrak{G}$. $H \to Z_i(H)$ $(H \in \mathfrak{H})$ is a representation of $\mathfrak{H}$ which we denote by $Z_i(\mathfrak{H})$. Now we can distribute $Z_1, Z_2, \cdots\cdots, Z_u$ into a certain number of blocks with respect to $\mathfrak{H}$ by the following manner. We say that Z_i and Z_j belong to the same block, if in the sequence

$$Z_i(\mathfrak{H}),\ Z_k(\mathfrak{H}),\ \cdots\cdots,\ Z_l(\mathfrak{H}),\ Z_j(\mathfrak{H})$$

any two consecutive $Z_m(\mathfrak{H})$ have an irreducible constituent in common. Thus $Z_1, Z_2, \cdots\cdots, Z_u$ appear distributed into r "$\mathfrak{H}$-block" $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_r$. Further we say that all the irreducible constituents of $Z_i(\mathfrak{H})$ belong to $\mathfrak{B}_\kappa$, if Z_i belongs to $\mathfrak{B}_\kappa$. Denote by $\tilde{X}_\lambda$ the representation of $\mathfrak{G}$ induced from X_λ. Then, as we can easily see, all the irreducible constituents Z_i of $\tilde{X}_\lambda$ belong to the same block. Let us set

(6.1) $$\mathfrak{M} = \bigcap_{G \in \mathfrak{G}} G^{-1}\mathfrak{H}G.$$

Then $\mathfrak{M}(\subset \mathfrak{H})$ is an invariant subgroup of $\mathfrak{G}$. Let $\bar{C}_1, \bar{C}_2, \cdots\cdots, \bar{C}_t$ be the classes of conjugate elements in $\mathfrak{G}$ which contain an element of $\mathfrak{M}$. We denote by $\bar{C}_\nu^*$ the sum of all elements in $\bar{C}_\nu$. Since $\bar{C}_\nu^*$ is a sum of complete classes of $\mathfrak{H}$, we have

(6.2) $$X_\lambda(G^{-1}\bar{C}_\nu^* G) = X_\lambda(\bar{C}_\nu^*) = \rho_\lambda^{(\nu)} I_{h(\lambda)}$$

where $h(\lambda)$ is the degree of X_λ and $I_{h(\lambda)}$ is the unit matrix of degree $h(\lambda)$. Let $\Lambda(\mathfrak{G})$ and $\Lambda(\mathfrak{H})$ be the centers of group rings $\Gamma(\mathfrak{G})$ and $\Gamma(\mathfrak{H})$, and let ω_i be the character of $\Lambda(\mathfrak{G})$ determined by χ_i. Then $Z_i(\bar{C}_\nu^*) = \omega_i(\bar{C}_\nu^*)I_{f_i}$. Since

$$Z_i(\mathfrak{H}) \cong \begin{pmatrix} X_\kappa & & \\ & \ddots & \\ & & X_\lambda \end{pmatrix}$$

we have from (6.2)

$$Z_i(\bar{C}_\nu^*) = \begin{pmatrix} X_\kappa(\bar{C}_\nu^*) & & \\ & \ddots & \\ & & X_\lambda(\bar{C}_\nu^*) \end{pmatrix} = \begin{pmatrix} \rho_\kappa^{(\nu)} I_{h(\kappa)} & & \\ & \ddots & \\ & & \rho_\lambda^{(\nu)} I_{h(\lambda)} \end{pmatrix}$$

Then it follows that

$$\omega_i(\bar{C}_\nu^*) = \rho_\kappa^{(\nu)} = \cdots\cdots = \rho_\lambda^{(\nu)} \tag{6.3}$$

Theorem 7. *The two irreducible representations Z_i and Z_j belong to the same $\mathfrak{H}$-block if and only if $\chi_i(M)/f_i = \chi_j(M)/f_j$ for all $M \in \mathfrak{M}$.*

Proof. Assume that Z_i and Z_j belong to the same block. From (6.3) it follows that $\omega_i(\bar{C}_\nu^*) = \omega_j(\bar{C}_\nu^*)$, whence $\chi_i(M)/f_i = \chi_j(M)/f_j$. Now we prove the converse. Let us denote by $\mathfrak{A}_\lambda$ the set of those elements of $\Gamma(\mathfrak{G})$ which are represented by 0 in every Z_i outside of $\mathfrak{B}_\lambda$. Then $\mathfrak{A}_\lambda (\lambda = 1, 2, \cdots\cdots, r)$ are ideals of $\Gamma(\mathfrak{G})$, and $\Gamma(\mathfrak{G})$ splits into a direct sum:

$$\Gamma(\mathfrak{G}) = \mathfrak{A}_1 + \mathfrak{A}_2 + \cdots\cdots + \mathfrak{A}_r.$$

We then have

$$\Gamma(\mathfrak{H}) = \Gamma(\mathfrak{H}) \cap \mathfrak{A}_1 + \Gamma(\mathfrak{H}) \cap \mathfrak{A}_2 + \cdots\cdots + \Gamma(\mathfrak{H}) \cap \mathfrak{A}_r.$$

Let ϵ_λ be the unit element of $\Gamma(\mathfrak{H}) \cap \mathfrak{A}_\lambda$. Then $\Gamma(\mathfrak{H}) \cap \mathfrak{A}_\lambda = \Gamma(\mathfrak{H})\epsilon_\lambda$ and $\Gamma(\mathfrak{G})\epsilon_\lambda \subseteqq \mathfrak{A}_\lambda$. But we find $\Gamma(\mathfrak{G})\epsilon_\lambda = \mathfrak{A}_\lambda$, since $\Gamma(\mathfrak{G}) = \sum \Gamma(\mathfrak{G})\epsilon_\lambda$. This implies that ϵ_λ is the unit element of $\mathfrak{A}_\lambda$, and belongs to $\Lambda(\mathfrak{G})$. Then ϵ_λ belongs to $\Gamma(G^{-1}\mathfrak{H}G)$ for any $G \in \mathfrak{G}$, whence ϵ_λ is in $\Gamma(\mathfrak{M})$. Consequently ϵ_λ belongs to

$$\Gamma(\mathfrak{M}) \cap \Lambda(\mathfrak{G}) = K\bar{C}_1^* + K\bar{C}_2^* + \cdots\cdots + K\bar{C}_t^*$$

and, hence, is expressed by $a_1\bar{C}_1^* + a_2\bar{C}_2^* + \cdots\cdots + a_t\bar{C}_t^*$ $(a_i \in K)$. Since

ϵ_λ is represented by 1 in all Z_i of $\mathfrak{B}_\lambda$, and is represented by 0 in every Z_m outside of $\mathfrak{B}_\lambda$, there exists at least a class $\bar{C}_\mu$ such that $\omega_l(\bar{C}_\mu^*) \neq \omega_m(\bar{C}_\mu^*)$, i.e. $\chi_i(M)/f_i \neq \chi_m(M)/f_m$ for $M \in \bar{C}_\mu$. This completes the proof.

Theorem 8. *The number of* $\mathfrak{H}$*-blocks of* $\mathfrak{G}$ *is equal to the number of classes of conjugate elements in* $\mathfrak{G}$ *which contain an element of* $\mathfrak{M}$.

Proof. $\mathfrak{B}_\lambda(\lambda = 1, 2, \cdots\cdots, r)$ are in $1-1$ correspondence with the irreducible representations of

$$A(\mathfrak{G}) \cap A(\mathfrak{M}) = K\bar{C}_1^* + K\bar{C}_2^* + \cdots\cdots + K\bar{C}_l^*.$$

Hence we have $r = l$.

Let θ_i and θ_j be two irreducible characters of $\mathfrak{M}$, then θ_i and θ_j are called associated in $\mathfrak{G}$, if there exists a fixed element G such that $\theta_i(M) = \theta_j(G^{-1}MG)$ $(M \in \mathfrak{M})$. We can distribute the irreducible characters of $\mathfrak{M}$ into the associated classes. From Theorem 4, the number of the associated classes is equal to l. Since the irreducible constituents of $Z_i(\mathfrak{M})$ are associated in $\mathfrak{G}$, $\mathfrak{H}$-blocks $\mathfrak{B}_\lambda(\lambda = 1, 2, \cdots\cdots, l)$ are in $1-1$ correspondence with the associated classes of $\mathfrak{M}$.

Let us denote by $Z_1, Z_2, \cdots\cdots, Z_l$ a complete system of representatives for $\mathfrak{H}$-blocks $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_l$ and let $Z_{\lambda,1} = Z_\lambda, Z_{\lambda,2}, \cdots\cdots, Z_{\lambda,s(\lambda)}$ be the irreducible representations in $\mathfrak{B}_\lambda$. We have from (3.4) for $M_i \in \bar{C}_\nu$, $M_j \in \bar{C}_\mu$,

$$\begin{aligned}\sum_{m=1}^{u} g(M_i)\chi_m(M_i)\chi_m(M_j) &= \sum_{m=1}^{u} f_m\omega_m(\bar{C}_\nu^*)\chi_m(M_j)\\ &= \sum_{\kappa=1}^{l} \omega_\kappa(\bar{C}_\nu^*)\sigma_\kappa(M_j) = g\delta_{\nu\mu^*}.\end{aligned}$$

where $\sigma_\kappa(M_j) = \sum_{\rho=1}^{s(\kappa)} f_{\kappa\rho}\chi_{\kappa,\rho}(M_j)$ and $f_{\kappa\rho}$ is the degree of $Z_{\kappa,\rho}$. From Theorem 7

$$\sigma_\kappa(M_j) = (\sum_{\rho=1}^{s(\kappa)} f_{\kappa\rho}^2/f_\kappa)\chi_\kappa(M_j) = a_\kappa\chi_\kappa(M_j).$$

Hence we have

$$\sum_\kappa b_\kappa\chi_\kappa(M_i)\chi_\kappa(M_j) = n(M_i)\delta_{\nu\mu^*} \tag{6.4}$$

where $b_\kappa = a_\kappa/f_\kappa = \sum_\rho f_{\kappa\rho}^2/f_\kappa^2$. Further (6.4) yields

$$\sum_{M \in \mathfrak{M}} b_\kappa\chi_\kappa(M)\chi_\lambda(M^{-1}) = g\delta_{\kappa\lambda} \qquad \text{(for } \chi_\kappa \text{ in } \mathfrak{B}_\kappa,\ \chi_\lambda \text{ in } \mathfrak{B}_\lambda). \tag{6.5}$$

III. Modular representations of groups.

7. We consider representations of $\mathfrak{G}$ in an algebraically closed field $\bar{K}$ of characteristic p. Let $F_1, F_2, \cdots\cdots, F_m$ be the distinct irreducible representations and let $U_1, U_2, \cdots\cdots, U_m$ be corresponding indecomposable constituents of the (left, for example) regular representation of $\mathfrak{G}$. Let us denote by φ_λ and η_λ the characters of F_λ and U_λ. We understand these characters in the sense of Brauer and Nesbitt[1]: they are complex numbers and are defined only for the p-regular elements[2]. We denote by $C_1, C_2, \cdots\cdots, C_t$ the classes of conjugate elements which contain the p-regular elements. The value of characters φ_λ and η_λ for the class C_ν will be indicated by $\varphi_\lambda^{(\nu)}$ and $\eta_\lambda^{(\nu)}$. Theorem 3, combined with (1.4), yields

$$\sum_{\kappa,\lambda} c_{\kappa\lambda}\varphi_\kappa^{(\nu)}\varphi_\lambda^{(\mu)} = n_\nu\delta_{\nu\mu^*}.$$

Since $\eta_\lambda^{(\nu)} = \sum_\kappa c_{\kappa\lambda}\varphi_\kappa^{(\nu)}$ by Theorem 1, we have

$$\sum_\lambda \eta_\lambda^{(\nu)}\varphi_\lambda^{(\mu)} = n_\nu\delta_{\nu\mu^*}. \tag{7.1}$$

We arrange $\varphi_\lambda^{(\nu)}$ and $\eta_\lambda^{(\nu)}$ in matrix form $\Phi = (\varphi_\lambda^{(\nu)})$, $H = (\eta_\lambda^{(\nu)})$ (λ row index, ν column index). Then (7.1) becomes

$$H'\Phi = (n_\nu\delta_{\nu\mu^*}) = P. \tag{7.2}$$

Since P is non-singular, we get $m = t$ in a same way as in Brauer and Nesbitt (6), and consequently

$$|H| \neq 0, \qquad |\Phi| \neq 0.$$

(7.1) yields the following

$$\sum_\nu g_\nu\varphi_\kappa^{(\nu)}\eta_\lambda^{(\nu^*)} = g\delta_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \cdots\cdots, m). \tag{7.3}$$

As is well known, the ordinary irreducible representation Z_i determines a modular representation (reducible or irreducible) $\bar{Z}_i$[3]. Let $d_{i\lambda}$ denote the multiplicity of F_λ in $\bar{Z}_i$. Brauer and Nesbitt called these $d_{i\lambda}$ the decomposition numbers of $\mathfrak{G}$.

Theorem 9. *If* $\chi_i = \sum_\lambda d_{i\lambda}\varphi_\lambda$, *then* $\eta_\lambda = \sum_i d_{i\lambda}\chi_i$.

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1) See Brauer-Nesbitt (6).
2) By a p-regular element of $\mathfrak{G}$, we understand an element whose order is prime to p.
3) See Brauer-Nesbitt (6) p. 558.

Proof. From (3.2) and (7.1), we have

$$\sum_i \chi_i^{(\nu)}\chi_i^{(\mu)} = \sum_\lambda \eta_\lambda^{(\nu)}\varphi_\lambda^{(\mu)} \qquad (\nu, \mu = 1, 2, \cdots\cdots, m).$$

Hence

$$\sum_i \chi_i^{(\nu)}\chi_i^{(\mu)} = \sum_i \chi_i^{(\nu)} \sum_\lambda d_{i\lambda}\varphi_\lambda^{(\mu)}$$
$$= \sum_\lambda (\sum_i d_{i\lambda}\chi_i^{(\nu)})\varphi_\lambda^{(\mu)} = \sum_\lambda \eta_\lambda^{(\nu)}\varphi_\lambda^{(\mu)}.$$

This implies that $\eta_\lambda = \sum_i d_{i\lambda}\chi_i$.

Theorem 10[1]. *If* $\chi_i = \sum_\kappa d_{i\kappa}\varphi_\kappa$ *and* $\eta_\lambda = \sum_\kappa c_{\kappa\lambda}\varphi_\kappa$, *then*

$$c_{\kappa\lambda} = c_{\lambda\kappa} = \sum_i d_{i\kappa}d_{i\lambda}.$$

Proof. According to Theorem 9, we have

$$\eta_\lambda = \sum_i d_{i\lambda}\chi_i = \sum_i d_{i\lambda} \sum_\kappa d_{i\kappa}\varphi_\kappa = \sum_\kappa (\sum_i d_{i\kappa}d_{i\lambda})\varphi_\kappa$$

and hence $c_{\kappa\lambda} = \sum_i d_{i\kappa}d_{i\lambda} = c_{\lambda\kappa}$.

If we set $(c_{\kappa\lambda}) = C$, $(d_{i\kappa}) = D$, then

$$C = D'D \tag{7.4}$$

where D' is the transpose of D. From Theorem 1, combined with $c_{\kappa\lambda} = c_{\lambda\kappa}$, we can see that $U_\lambda \leftrightarrow V_\lambda$, but in virtue of the fact that group ring is symmetric[2], we have certainly $U_\lambda \cong V_\lambda$.

8. Let $\mathfrak{H}$ and $\mathfrak{J}$ be two subgroups of $\mathfrak{G}$. Let us denote by φ_1^*, $\varphi_2^*, \cdots\cdots, \varphi_k^*$ the irreducible characters of $\mathfrak{H}$ and let $\eta_1^*, \eta_2^*, \cdots\cdots, \eta_k^*$ be the corresponding indecomposable characters of the regular representation of $\mathfrak{H}$. Similarly we define φ_λ' and η_λ' ($\lambda = 1, 2, \cdots\cdots, l$) for $\mathfrak{J}$. Further we denote by $\tilde{\varphi}_\rho^*$, $\tilde{\eta}_\rho^*$, $\tilde{\varphi}_\lambda'$ and $\tilde{\eta}_\lambda'$ the characters of $\mathfrak{G}$ induced from φ_ρ^*, η_ρ^*, φ_λ' and η_λ' respectively. Theorem 2, applied to the group ring of $\mathfrak{G}$, yields

$$\begin{cases} \tilde{\eta}_\lambda'(H) = \sum_\rho \sigma_{\rho\lambda}\varphi_\rho^*(H) & \text{(for } p\text{-regular elements } H \in \mathfrak{H}) \\ \tilde{\eta}_\rho^*(J) = \sum_\lambda \sigma_{\rho\lambda}\varphi_\lambda'(J) & \text{(for } p\text{-regular elements } J \in \mathfrak{J}). \end{cases} \tag{8.1}$$

In particular, for $\mathfrak{J} = \mathfrak{G}$, from (2.1) and (2.7) we have formulas[3]

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1) H. Nagao has obtained independently a simple proof for this theorem using the properties of the induced characters of $\mathfrak{G}$.

2) See Brauer-Nesbitt (5). Cf. also Nakayama-Nesbitt (12).

3) Nakayama (11) p. 366. Brauer-Nesbitt (6) p. 582.

$$(8.2)\qquad \begin{cases} \eta_\lambda(H') = \sum_\rho \pi_{\rho\lambda}\varphi_\rho^*(H') \\ \tilde{\eta}_\rho^*(G') = \sum_\lambda \pi_{\rho\lambda}\varphi_\lambda(G') \end{cases}$$

$$(8.3)\qquad \begin{cases} \varphi_\lambda(H') = \sum_\rho \alpha_{\rho\lambda}\varphi_\rho^*(H') \\ \tilde{\eta}_\rho^*(G') = \sum_\lambda \alpha_{\rho\lambda}\eta_\lambda(G') \end{cases}$$

where G' and H' mean the p-regular elements of $\mathfrak{G}$ and $\mathfrak{H}$ respectively. Finally, from

$$\sum_\sigma \tilde{\eta}_\sigma^*(G')\varphi_\sigma^*(H') = \sum_\rho \tilde{\varphi}_\rho^*(G')\eta_\rho^*(H')$$

or from (8.3) directly, we have formulas

$$(8.4)\qquad \begin{cases} \eta_\lambda(H') = \sum_\rho \beta_{\rho\lambda}\eta_\rho^*(H') \\ \tilde{\varphi}_\rho^*(G') = \sum_\lambda \beta_{\rho\lambda}\varphi_\lambda(G'). \end{cases}$$

Let χ_i and ξ_ν be the ordinary irreducible characters of $\mathfrak{G}$ and $\mathfrak{H}$. We can prove easily the following formulas

$$(8.5)\qquad \begin{cases} \chi_i(H') = \sum_\rho m_{i\rho}\varphi_\rho^*(H') \\ \tilde{\eta}_\rho^*(G') = \sum_i m_{i\rho}\chi_i(G') \end{cases}$$

$$(8.6)\qquad \begin{cases} \tilde{\xi}_\nu(H') = \sum_\rho n_{\nu\rho}\varphi_\rho^*(H') \\ \tilde{\eta}_\rho^*(H') = \sum_\nu n_{\nu\rho}\xi_\nu(H') \end{cases}$$

$$(8.7)\qquad \begin{cases} \tilde{\varphi}_\sigma^*(H') = \sum_\rho r_{\rho\sigma}\varphi_\rho^*(H') \\ \tilde{\eta}_\rho^*(H') = \sum_\sigma r_{\rho\sigma}\eta_\sigma^*(H'). \end{cases}$$

Further, from (2.4) we have

$$(8.8)\qquad \begin{cases} \tilde{\eta}_\sigma^*(H') = \sum_\rho \omega_{\rho\sigma}\varphi_\rho^*(H') \\ \eta_\rho^*(H') = \sum_\sigma \omega_{\rho\sigma}\varphi^*(H'). \end{cases}$$

If we put

$$W = (\omega_{\rho\sigma}),\quad M = (m_{i\rho}),\quad R = (r_{\rho\sigma}),\quad A = (\alpha_{\rho\lambda}),\quad B = (\beta_{\rho\lambda}),$$

then we obtain from the above formulas

$$(8.9) \qquad W = M'M = ACA' = C^*BA' = C^*R'$$

where $C^* = (c^*_{\rho\sigma})$ has the same significance for $\mathfrak{H}$ as C has for $\mathfrak{G}$.

Theorem 11[1]. *The number of linearly independent characters of $\mathfrak{G}$ induced from the k distinct irreducible characters φ^*_ρ of $\mathfrak{H}$ is equal to the number of the classes of conjugate elements which contain a p-regular element of $\mathfrak{H}$.*

Proof. We have for p-regular elements H_i, H_j of $\mathfrak{H}$

$$\sum_\rho \tilde{\varphi}^*_\rho(H_i)\eta^*_\rho(H_j^{-1}) = \begin{cases} n(H_i) & \text{for } C(H_j) = C(H_i) \\ 0 & \text{for } C(H_j) \neq C(H_i) \end{cases}$$

where $C(H)$ denotes a class of conjugate elements in $\mathfrak{G}$ which contains H. Then we can obtain our assertion in the similar way as Theorem 4.

Similarly in section 6, we can distribute the indecomposable representations $U_1, U_2, \cdots\cdots, U_m$ into a certain number of blocks with respect to $\mathfrak{H}$. We say that U_κ and U_λ belong to the same block, if in the sequence

$$U_\kappa(\mathfrak{H}),\ U_\mu(\mathfrak{H}),\ \cdots\cdots,\ U_\nu(\mathfrak{H}),\ U_\lambda(\mathfrak{H})$$

any two consequtive $U_\rho(\mathfrak{H})$ have an irreducible constituent in common. Thus $U_1, U_2, \cdots\cdots, U_m$ appear distributed in s "$\mathfrak{H}^*$-blocks" $\mathfrak{B}^*_1, \mathfrak{B}^*_2, \cdots\cdots, \mathfrak{B}^*_s$. We also say that F_κ belongs to $\mathfrak{B}^*_\sigma$ when U_κ belongs to $\mathfrak{B}^*_\sigma$. Then we can see that all the irreducible constituents F_ρ of U_κ belong to $\mathfrak{B}^*_\sigma$. Moreover all the irreducible constituents of the modular representation $\bar{Z}_i$ of $\mathfrak{G}$ which is determined by the ordinary irreducible representation Z_i belong to the same block. If $\bar{Z}_i$ contains F_κ in $\mathfrak{B}^*_\sigma$ as its irreducible constituent, then we say that Z_i also belongs to $\mathfrak{B}^*_\sigma$. Let $\mathfrak{M}$ have the same meaning as in section 6.

Theorem 12[2]. *The ordinary irreducible representations Z_i and Z_j belong to the same $\mathfrak{H}^*$-block, if and only if*

$$g(M)\chi_i(M)/f_i \equiv g(M)\chi_j(M)/f_j \qquad (\text{mod } \mathfrak{p})$$

for all $M \in \mathfrak{M}$, where $\mathfrak{p}$ is a fixed prime ideal divisor of p in K^*[3].

1) Nakayama (11) p. 366.

2) Cf. Brauer-Nesbitt (6) p. 562.

3) We choose the algebraic number field K^* so that the absolutely irreducible representations of $\mathfrak{G}$ can be written with coefficients in K^*.

By Theorem 7, we have

Corollary. *If Z_i and Z_j belong to the same $\mathfrak{H}$-block, then they belong to the same $\mathfrak{H}^*$-block.*

9. Let $\overline{\mathfrak{G}}$ and $\mathfrak{G}_0$ have the same meaning as in section 5. Since $\mathfrak{G}_0$ is isomorphic to $\mathfrak{G}$, the characters φ_λ and η_λ of $\mathfrak{G}$ may be considered as the characters of $\mathfrak{G}_0$. Denote by $\tilde{\varphi}_\lambda$ and $\tilde{\eta}_\lambda$ the characters of $\mathfrak{G} \times \overline{\mathfrak{G}}$ induced from φ_λ and η_λ of $\mathfrak{G}_0$. Lemma 3 holds also in the modular case, and hence for p-regular elements G_i, G_j of $\mathfrak{G}$, we have

$$(9.1)\qquad \begin{aligned} \tilde{\varphi}_\mu(G_i \times \overline{G}_j) &= \sum_\kappa \varphi_\kappa(G_i)\eta_{\kappa'}(G_j)\varphi_\mu(G_j) \\ \tilde{\eta}_\mu(G_i \times \overline{G}_j) &= \sum_\kappa \varphi_\kappa(G_i)\eta_{\kappa'}(G_j)\eta_\mu(G_j) \\ &= \sum_\kappa \eta_\kappa(G_i)\varphi_{\kappa'}(G_j)\eta_\mu(G_j) \end{aligned}$$

where $\varphi_{\kappa'}$ and $\eta_{\kappa'}$ are the characters contragredient to φ_κ and η_κ.

Theorem 13[1]**.** *If $\eta_\kappa(G)\eta_\lambda(G) = \sum_\mu \pi_{\kappa\lambda\mu}\varphi_\mu(G)$ for p-regular elements G of $\mathfrak{G}$, then $\eta_{\kappa'}(G)\eta_\mu(G) = \sum_\lambda \pi_{\kappa\lambda\mu}\varphi_\lambda(G)$, that is, $\pi_{\kappa\lambda\mu} = \pi_{\kappa'\mu\lambda}$.*

Proof. Applying (8.2) to $\mathfrak{G} \times \overline{\mathfrak{G}}$ and its subgroup $\mathfrak{G}_0$, we have

$$\tilde{\eta}_\mu(G_i \times \overline{G}_j) = \sum_\kappa \varphi_\kappa(G_i)\eta_{\kappa'}(G_j)\eta_\mu(G_j) = \sum_{\mu,\lambda} \pi_{\kappa\lambda\mu}\varphi_\kappa(G_i)\varphi_\lambda(G_j).$$

This implies that $\eta_{\kappa'}(G_j)\eta_\mu(G_j) = \sum_\lambda \pi_{\kappa\lambda\mu}\varphi_\lambda(G_j)$.

Further we obtain the following

Theorem 14. *For p-regular elements G of $\mathfrak{G}$*

$$1)\qquad \begin{cases} \varphi_\kappa(G)\varphi_\lambda(G) = \sum_\mu \alpha_{\kappa\lambda\mu}\varphi_\mu(G) \\ \varphi_{\kappa'}(G)\eta_\mu(G) = \sum_\lambda \alpha_{\kappa\lambda\mu}\eta_\lambda(G) \end{cases}$$

$$2)\qquad \begin{cases} \eta_\kappa(G)\eta_\lambda(G) = \sum_\mu \beta_{\kappa\lambda\mu}\eta_\mu(G) \\ \eta_{\kappa'}(G)\varphi_\mu(G) = \sum_\lambda \beta_{\kappa\lambda\mu}\varphi_\lambda(G). \end{cases}$$

Proof. Every indecomposable constituent of the regular representation of $\mathfrak{G} \times \overline{\mathfrak{G}}$ is given by $U_\kappa(G_i) \times U_\lambda(\overline{G}_j)$. Hence (8.3) and (9.1) yield

1) H. Nagao has proved independently Theorems 13 and 14 by the same manner.

$$\begin{aligned}\widetilde{\eta}_\mu(G_i \times \bar{G}_j) &= \sum_\kappa \eta_\kappa(G_i)\varphi_{\kappa'}(G_j)\eta_\mu(G_j) \\ &= \sum_{\kappa,\lambda} \alpha_{\kappa\lambda\mu}\eta_\kappa(G_i)\eta_\lambda(G_j).\end{aligned}$$

This implies that $\varphi_{\kappa'}(G_j)\eta_\mu(G_j) = \sum_\lambda \alpha_{\kappa\lambda\mu}\eta_\lambda(G_j)$. Similarly from (8.4) we can obtain 2).

In particular, we can see that $\beta_{\kappa'\lambda 1} = c_{\kappa\lambda}$. Since (5.2) shows that $U_\kappa \times D$ splits completely into $U_1, U_2, \cdots\cdots, U_m$, by Theorem 14 we find

$$F_{\kappa'}(G) \times U_\mu(G) \cong \sum_\lambda \alpha_{\kappa\lambda\mu}U_\lambda(G)^{1)}. \tag{9.2}$$

Corresponding to Theorem 6, we have the following formulas

$$\begin{aligned}\sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\beta_{\rho\kappa\mu} &= \sum_{\kappa,\mu} \beta_{\kappa\mu\lambda}\alpha_{\kappa\mu\rho} = \sum_\nu \varphi_\lambda^{(\nu)}\eta_\rho^{(\nu*)} \\ \sum_{\kappa,\mu} \alpha_{\lambda\mu\kappa}\alpha_{\rho\kappa\mu} &= \sum_\nu \varphi_\lambda^{(\nu)}\varphi_\rho^{(\nu)} \\ \sum_{\kappa,\mu} \pi_{\kappa\mu\lambda}\alpha_{\kappa\mu\rho} &= \sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\pi_{\rho\kappa\mu} = \sum_\nu \eta_\lambda^{(\nu)}\eta_\rho^{(\nu*)} \\ \sum_{\kappa,\mu} \beta_{\lambda\mu\kappa}\beta_{\rho\kappa\mu} &= \sum_\nu \eta_\lambda^{(\nu)}\eta_\rho^{(\nu)}.\end{aligned} \tag{9.3}$$

From (7.1) we have

$$\begin{aligned}\sum_\kappa g_\nu\varphi_\kappa^{(\nu)}\eta_{\kappa'}^{(\nu)}\varphi_\lambda^{(\nu)} &= \sum_\kappa\sum_\mu \alpha_{\lambda\kappa\mu}g_\nu\varphi_\mu^{(\nu)}\eta_{\kappa'}^{(\nu)} \\ &= \sum_\sigma (\sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\beta_{\sigma\kappa\mu})g_\nu\varphi_\sigma^{(\nu)} = g\varphi_\lambda^{(\nu)}.\end{aligned}$$

Here, we multiply by $\eta_\rho^{(\nu*)}$, add over ν, and use (7.3)

$$\sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\beta_{\rho\kappa\mu} = \sum_\nu \varphi_\lambda^{(\nu)}\eta_\rho^{(\nu*)}.$$

On the other hand, we have

$$\sum_\kappa \varphi_\kappa^{(\nu)}\eta_{\kappa'}^{(\nu)}\varphi_\lambda^{(\nu)} = \sum_\kappa\sum_\mu \beta_{\kappa\mu\lambda}\varphi_\mu^{(\nu)}\varphi_\kappa^{(\nu)} = \sum_\sigma (\sum_{\kappa,\mu} \beta_{\kappa\mu\lambda}\alpha_{\kappa\mu\sigma})\varphi_\sigma^{(\nu)}.$$

Consequently $\sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\beta_{\rho\kappa\mu} = \sum_{\kappa,\mu} \beta_{\kappa\mu\lambda}\alpha_{\kappa\mu\rho}$. This completes the proof of the first formula (9.3). Now we shall prove the second formula (9.3).

$$\begin{aligned}\sum_\kappa g_\nu\varphi_\kappa^{(\nu)}\eta_{\kappa'}^{(\nu)}\varphi_\lambda^{(\nu)} &= \sum_\kappa (\sum_{\mu'} \alpha_{\lambda'\mu'\kappa'}\eta_{\mu'}^{(\nu)})g_\nu\varphi_\kappa^{(\nu)} \\ &= \sum_{\kappa,\mu'} \alpha_{\lambda'\mu'\kappa'} \sum_{\sigma'} \alpha_{\kappa'\sigma'\mu'}g_\nu\eta_{\sigma'}^{(\nu)} \\ &= \sum_\sigma (\sum_{\kappa,\mu} \alpha_{\lambda\mu\kappa}\alpha_{\kappa\sigma\mu})g_\nu\eta_{\sigma'}^{(\nu)} = g\varphi_\lambda^{(\nu)}.\end{aligned}$$

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1) Cf. Osima (15).

Here, we multiply by $\varphi_\rho^{(\nu)}$, add over ν, and use (7.3)

$$\sum_{\kappa,\mu} \alpha_{\lambda\mu\kappa}\alpha_{\rho\kappa\mu} = \sum_\nu \varphi_\lambda^{(\nu)}\varphi_\rho^{(\nu)}.$$

Similarly $\sum_\kappa \varphi_\kappa^{(\nu)}\eta_{\kappa'}^{(\nu)}\eta_\lambda^{(\nu)} = n_\nu\eta_\lambda^{(\nu)}$ and $\sum_\kappa \varphi_\kappa^{(\nu)}\eta_{\kappa'}^{(\nu)}\eta_{\lambda'}^{(\nu)} = n\eta_{\lambda'}^{(\nu)}$ yield last two formulas (9.3) respectively. Furthermore from (9.3) we find

$$\sum_{\kappa,\mu} \pi_{\kappa\mu\lambda}\alpha_{\kappa\mu\rho'} = \sum_{\kappa,\mu} \alpha_{\lambda\kappa\mu}\pi_{\rho'\kappa\mu} = \sum_{\kappa,\mu} \beta_{\lambda\mu\kappa}\beta_{\rho\kappa\mu}.$$

In particular, for $\lambda = 1$ in (9.3), we have[1]

$$\sum_\kappa \beta_{\rho\kappa\kappa} = \sum_{\kappa,\mu} \beta_{\kappa\mu 1}\alpha_{\kappa\mu\rho} = \sum_{\kappa,\mu} c_{\kappa'\mu}\alpha_{\kappa\mu\rho}$$
$$= \sum_{\kappa,\mu} c_{\kappa\mu}\alpha_{\kappa'\mu\rho} = \sum_\nu \eta_\rho^{(\nu)}$$
$$\sum_\kappa \alpha_{\rho\kappa\kappa} = \sum_\nu \varphi_\rho^{(\nu)}.$$

IV. Representations of groups by collineations[2].

10. Let $\mathfrak{G}$ be a group of finite order g. We consider the algebra

(10.1) $$(r, \mathfrak{G}) = U_E K + U_P K + \cdots\cdots + U_Q K, \qquad U_E = 1$$

$\mathfrak{G} = \{E, P, \cdots\cdots, Q\}$ over an algebraically closed field K in which the multiplication is defined by $U_P U_Q = r_{P,Q} U_{PQ}$. Here, the $r_{P,Q}$ are non-zero elements from K such that $r_{P,Q} r_{PQ,R} = r_{P,QR} r_{Q,R}$. $(r, \mathfrak{G})$ is called the collinear group ring of $\mathfrak{G}$ with factor set $r = \{r_{P,Q}\}$. If $U_P \to M(P)$ is a representation of $(r, \mathfrak{G})$ by linear transformations, then $M(P)M(Q) = r_{P,Q}M(PQ)$. Hence $P \to M(P)$ is a representation of $\mathfrak{G}$ by collineations. In the sequence $P \to M(P)$ may be called the representation of $\mathfrak{G}$ with factor set r. If $P \to N(P)$ is a representation of $\mathfrak{G}$ with factor set r, then $U_P \to N(P)$ is certainly a representation of $(r, \mathfrak{G})$. Hence we may consider representations of $(r, \mathfrak{G})$ instead of considering representations of $\mathfrak{G}$ with factor set r. The two representations $P \to M(P)$ with factor set r and $P \to N(P)$ with factor set r', are called associated if $M(P) = k_P N(P)$ for all P where the k_P are non-zero elements from K. The factor sets r and r' are also called associated. If r and r' are associated, then we find

1) See Brauer-Nesbitt (6) p. 579.

2) Cf. Schur (17), Tazawa (19).

(10.2) $$r'_{P,Q} = k_P k_Q k_{PQ}^{-1} r_{P,Q}.$$

Associated representations are regarded as not essentially distinct.

Now we assume that the characteristic of K is 0. Then $(r, \mathfrak{G})$ is semi-simple. We denote by $Z_1, Z_2, \cdots\cdots, Z_k$ the distinct irreducible representations of $(r, \mathfrak{G})$. Let $C_1, C_2, \cdots\cdots, C_n$ be the classes of conjugate elements in $\mathfrak{G}$. A class C_ν is called *regular* with respect to r, if C_ν contains an element P such that $r_{P,Q} = r_{Q,P}$ for any Q of the normalizer $\mathfrak{N}(P)$. We shall denote by $C_1, C_2, \cdots\cdots, C_d$ $(d \leqq n)$ the regular classes. Since Z_i is an irreducible representation of $\mathfrak{G}$ with factor set r, we may set $\chi_i(U_s) = \chi_i(S)$ where χ_i is the character of Z_i. We then have from $U_P U_S U_P^{-1} = r_{P,S} r_{PSP^{-1},P}^{-1} U_{PSP^{-1}}$

(10.3) $$\chi_i(S) = r_{P,S} r_{PSP^{-1},P}^{-1} \chi_i(PSP^{-1}).$$

Lemma 4. *If S is contained in a non-regular class, then* $\chi_i(S) = 0$ $(i = 1, 2, \cdots\cdots, k)$.

Proof. From (10.3) we find $\chi_i(S) = r_{P,S} r_{S,P}^{-1} \chi_i(S)$ for any $P \in \mathfrak{N}(S)$. By our assumption there exists $Q \in \mathfrak{N}(S)$ such that $r_{Q,S} r_{S,Q}^{-1} \neq 0$, whence $\chi_i(S) = 0$.

Since $Z_i'(U_P^{-1}) Z_i'(U_Q^{-1}) = r_{P,Q}^{-1} Z_i'(U_{PQ}^{-1})$, $U_P \rightarrow Z_i'(U_P^{-1})$ is an irreducible representation of $\mathfrak{G}$ with factor set $r^{-1} = \{r_{P,Q}^{-1}\}$. We call this representation contragredient to Z_i and denote by $Z_{i'}$. If we denote by $\chi_{i'}$ the character of $Z_{i'}$, then from $U_P^{-1} = r_{P,P^{-1}}^{-1} U_{P^{-1}}$ we have $\chi_{i'}(P) = \chi_i(U_P^{-1}) = r_{P,P^{-1}}^{-1} \chi_i(P^{-1})$.

Let $S(U_P)$ and $R(U_P)$ be the left and the right regular representations of $(r, \mathfrak{G})$ defined by a basis $U_E, U_P, \cdots\cdots, U_Q$. Then $U_P \times U_S \rightarrow S(U_P) R'(U_S^{-1})$ is a representation of the direct product $(r, \mathfrak{G}) \times (r^{-1}, \mathfrak{G})$ with factor set $\{r_{P,Q} r_{S,T}^{-1}\}$. If we denote by $\Phi(U_P \times U_S)$ the character of $S(U_P) R'(U_S^{-1})$, then we have from (1.5) $\Phi(U_P \times U_S) = \sum_i \chi_i(U_P) \chi_i(U_S^{-1})$, that is

(10.4) $$\Phi(P \times S) = \sum_i \chi_i(P) \chi_{i'}(S).$$

Lemma 5. *If P is an element of $\mathfrak{N}(S)$, then $P \rightarrow r_{S,P} r_{P,S}^{-1}$ is a linear representation of $\mathfrak{N}(S)$.*

Proof. We have $U_S U_P U_S^{-1} = r_{S,P} r_{P,S}^{-1} U_P$, $U_S U_Q U_S^{-1} = r_{S,Q} r_{Q,S}^{-1} U_Q$ $(P, Q \in \mathfrak{N}(S))$, whence $U_S U_P U_Q U_S^{-1} = r_{S,P} r_{P,S}^{-1} r_{S,Q} r_{Q,S}^{-1} U_P U_Q$. Then $U_S U_{PQ} U_S^{-1} = r_{S,P} r_{P,S}^{-1} r_{S,Q} r_{Q,S}^{-1} U_{PQ}$. On the other hand, since $PQ \in \mathfrak{N}(S)$, we find $U_S U_{PQ} U_S^{-1} = r_{S,PQ} r_{PQ,S}^{-1} U_{PQ}$. Thus we obtain $(r_{S,P} r_{P,S}^{-1})(r_{S,Q} r_{Q,S}^{-1}) = r_{S,PQ} r_{PQ,S}^{-1}$.

Lemma 6. *Let $n(S)$ be the order of the normalizer $\mathfrak{N}(S)$. Then*

$$\sum_{P \in \mathfrak{N}(S)} r_{S,P} r_{P,S}^{-1} = \begin{cases} n(S) & \text{for } S \text{ in the regular classes} \\ 0 & \text{for } S \text{ in the non-regular classes.} \end{cases}$$

Proof. It follows readily from Lemma 5 and (3.5).

Lemma 7. $\Phi(S \times S) = \sum_{P \in \mathfrak{N}(S)} r_{S,P} r_{P,S}^{-1}$.

Proof. Since $U_S(U_E U_P \cdots\cdots U_Q)U_S^{-1} = (U_E U_P \cdots\cdots U_Q)S(U_S)R'(U_S^{-1})$, we have our assertion in the similar manner as Theorem 3.

If S_i and S_j are not conjugate in $\mathfrak{G}$, then as one can easily see $\Phi(S_i \times S_j) = 0$. Further if S is contained in a non-regular class, then from Lemmas 6 and 7, $\Phi(S \times S) = 0$. Let $S_1, S_2, \cdots\cdots, S_d$ be a complete system of representatives for the regular classes. Then we have from above consideration

$$\Phi(S_\nu \times S_\lambda) = n(S_\nu)\delta_{\nu\lambda}. \tag{10.5}$$

Consequently (10.4) yields

$$\sum_i \chi_i(S_\nu)\chi_{i'}(S_\lambda) = n(S_\nu)\delta_{\nu\lambda} \qquad (\nu, \lambda = 1, 2, \cdots\cdots, d). \tag{10.6}$$

If we set

$$Z = (\chi_i(S_\nu)), \qquad Y = (\chi_{i'}(S_\nu))$$

(ν row index: i column index). Then (10.5) becomes $Y'Z = (n(S_\nu)\delta_{\nu\lambda})' = V$. Since V is non-singular, we have $k \geqq d$. Suppose that $k > d$. Then $\chi_1(S_\nu), \chi_2(S_\nu), \cdots\cdots, \chi_k(S_\nu)$ ($\nu = 1, 2, \cdots\cdots, d$) are linearly dependent:

$$\sum_i a_i \chi_i(S_\nu) = 0.$$

From (10.3) and Lemma 4 we can see that $\sum_i a_i\chi_i(P) = 0$ for any $P \in \mathfrak{G}$. But such a relation is impossible, whence we have $k = d$. The number of the distinct irreducible representations of $\mathfrak{G}$ with factor set r is equal to the number of the regular classes of conjugate elements in $\mathfrak{G}$. From (10.6) we can derive as usual

$$\sum_{\nu=1}^{d} g_\nu \chi_i(S_\nu)\chi_{j'}(S_\nu) = g\delta_{ij} \tag{10.7}$$

where $g_\nu = g / n(S_\nu)$.

Let $(s, \mathfrak{G})$ be the group ring of $\mathfrak{G}$ with factor set $s = \{s_{P,Q}\}$:

$$(s, \mathfrak{G}) = V_E K + V_P K + \cdots\cdots + V_Q K.$$

We denote by $S_r(U_P)$, $R_r(U_P)$ and $S_s(V_P)$, $R_s(V_P)$ the regular representations of $(r, \mathfrak{G})$ and $(s, \mathfrak{G})$. Let $Z_1^{(r)}, Z_2^{(r)}, \cdots\cdots, Z_l^{(r)} : Z_1^{(s)}, Z_2^{(s)}, \cdots\cdots, Z_m^{(s)}$ and $Z_1^{(t)}, Z_2^{(t)}, \cdots\cdots, Z_q^{(t)}$ $(t = rs)$ be the distinct irreducible representations of $(r, \mathfrak{G})$, $(s, \mathfrak{G})$, and $(t, \mathfrak{G})$[1]. The characters of $Z_i^{(r)}$, $Z_j^{(s)}$ and $Z_k^{(t)}$ we denote by $\chi_i^{(r)}$, $\chi_j^{(s)}$ and $\chi_k^{(t)}$. Since $Z_i^{(r)}(U_P) \dot{\times} Z_j^{(s)}(V_P)$ is a representation of an algebra $(t, \mathfrak{G})$, we have

$$\chi_i^{(r)}(P)\chi_j^{(s)}(P) = \sum_k b_{ijk}\chi_k^{(t)}(P). \tag{10.8}$$

$$\mathfrak{A} = (U_E \times V_E)K + (U_P \times V_P)K + \cdots\cdots + (U_Q \times V_Q)K$$

is a subalgebra of $(r, \mathfrak{G}) \times (s, \mathfrak{G})$ and is isomorphic to $(t, \mathfrak{G}) = W_E K + W_P K + \cdots\cdots + W_Q K$. Hence we may denote $\mathfrak{A}$ by $(t, \mathfrak{G})$.

Lemma 8. *If $\widetilde{D}_t$ is the representation of $(r, \mathfrak{G}) \times (s, \mathfrak{G})$ induced from a representation D_t of $(t, \mathfrak{G})$, then*

$$\widetilde{D}_t(U_P \times V_S) \cong S_r(U_P)R_r'(U_S^{-1}) \times D_t(W_S).$$

Proof. We can prove in the similar manner as Lemma 3.

Since $\widetilde{D}_t$ is representation of $\mathfrak{G} \times \mathfrak{G}$ with factor set $\{r_{P,Q}s_{S,T}\}$, we may write

$$\widetilde{D}_t(P \times S) \cong S_r'(P)R_r^*(S) \times D_t(S)$$

where R_r^* is the representation of $\mathfrak{G}$ contragredient to R_r. Lemma 8 yields

$$S_r(P)R_r^*(S) \times D_t(S) \cong S_s(S)R_s^*(P) \times D_t(P).$$

In particular, for $S = E$, we have $S_r(P) \times I_f \cong R_s^*(P) \times D_t(P)$. Applying (4.2) to the direct product $(r, \mathfrak{G}) \times (s, \mathfrak{G})$ and its subalgebra $(t, \mathfrak{G})$, we obtain the following

Theorem 15. *Let*

$$\chi_i^{(r)}(P)\chi_j^{(s)}(P) = \sum_k b_{ijk}\chi_k^{(t)}(P).$$

Then

$$\chi_i^{(r)}(P)\chi_k^{(t)}(P) = \sum_j b_{ijk}\chi_j^{(s)}(P).$$

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1) If $r = \{r_{P,Q}\}$ and $s = \{s_{P,Q}\}$ are the two factor sets, then $\{t_{P,Q}\} = r_{P,Q}s_{P,Q}\}$ is also a factor set.

11. We shall study briefly the modular representations of $\mathfrak{G}$ with factor set[1]. Let $(\sigma, \mathfrak{G})$ be the collinear group ring with factor set $\sigma = \{\sigma_{P,Q}\}$ over an algebraically closed field $\bar{K}$ of characteristic p. We set

$$g = g'p^a, \qquad (g', p) = 1.$$

Any factor set $\{\sigma_{P,Q}\}$ is associated with a factor set $\{\rho_{P,Q}\}$ such that $\rho^g_{P,Q} = 1$ then necessarily $\rho^{g'}_{P,Q} = 1$ for any $P, Q \in \mathfrak{G}$. In the sequence we may only consider such factor set ρ. Corresponding to a factor set ρ, a factor set $r = \{r_{P,Q}\}$ is defined as complex numbers:

$$\bar{r}_{P,Q} = \rho_{P,Q}, \qquad r^{g'}_{P,Q} = 1$$

in the same manner as the modular characters were defined. A class C_ν is regular with respect to ρ if and only if C_ν is regular with respect to r. Let $C_1^*, C_2^*, \cdots\cdots, C_s^*$ be the regular classes which contain an element whose order is prime to p. We denote by $F_1, F_2, \cdots\cdots, F_t$ the distinct irreducible representations of $(\rho, \mathfrak{G})$ and by $U_1, U_2, \cdots\cdots, U_t$ the corresponding indecomposable constituents of the regular representation of $(\rho, \mathfrak{G})$. If φ_λ and η_λ are the characters of F_λ and $U_\lambda^{(2)}$, then the modular characters of F_λ and U_λ are $\bar{\varphi}_\lambda$ and $\bar{\eta}_\lambda$ (residue classes mod $\mathfrak{p}$). We set $\varphi_\lambda(U_Q) = \varphi_\lambda(Q)$ and $\eta_\lambda(U_Q) = \eta_\lambda(Q)$. Now we have

Lemma 9. *Let Q be an element whose order is prime to p. If Q is contained in a non-regular class, then $\varphi_\lambda(Q) = 0$ (and hence $\bar{\varphi}_\lambda(Q) = 0$).*

Proof. We can prove in the similar manner as Lemma 4.

Let $Q_1, Q_2, \cdots\cdots, Q_s$ be a complete system of representatives for the classes C_ν^* $(\nu = 1, 2, \cdots\cdots, s)$. We have from Theorem 1 and Lemma 7

$$\sum_\lambda \eta_\lambda(Q_\nu)\varphi_{\lambda'}(Q_\mu) = n(Q_\nu)\delta_{\nu\mu} \qquad (\nu, \mu = 1, 2, \cdots\cdots, s) \tag{11.1}$$

where $\varphi_{\lambda'}$ is the character contragredient to φ_λ. By (11.1) we have $t \geqq s$. Now suppose that $t > s$. Then the modular characters $\bar{\varphi}_1(Q_\nu)$, $\bar{\varphi}_2(Q_\nu), \cdots\cdots, \bar{\varphi}_t(Q_\nu)$ $(\nu = 1, 2, \cdots\cdots, s)$ are linearly dependent:

1) See Asano-Osima-Takahasi (1).

2) The value of these characters are complex numbers as in section 7 and are defined only for U_Q where Q has an order prime to p.

$$\sum_{\lambda} \alpha_{\lambda}\bar{\varphi}_{\lambda}(Q_{\nu}) = 0.$$

Then from Lemma 9 it follows that $\sum_{\lambda} \alpha_{\lambda}\bar{\varphi}_{\lambda}(Q) = 0$ for any element Q whose order is prime to p, whence we have finally $\sum_{\lambda} \alpha_{\lambda}\bar{\varphi}_{\lambda}(G) = 0$ for any element G of $\mathfrak{G}$[1]. But such a relation is impossible. Hence $t = s$. The number of the distinct irreducible representations of $\mathfrak{G}$ with factor set ρ is equal to the number of regular classes of conjugate elements in $\mathfrak{G}$ which contain elements of an order prime to p. It follows from (11.1) that

$$\sum_{\nu} g_{\nu}\eta_{\kappa}(Q_{\nu})\varphi_{\lambda'}(Q_{\nu}) = g\delta_{\kappa\lambda}. \tag{11.2}$$

Corresponding to Theorem 15, we have the following formulas for $Q \in C_{\nu}^{*}$

$$\begin{cases} \eta_{\kappa}^{(\rho)}(Q)\eta_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \pi_{\kappa\lambda\mu}^{*}\varphi_{\mu}^{(\tau)}(Q) \\ \eta_{\kappa'}^{(\rho)}(Q)\eta_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \pi_{\kappa\lambda\mu}^{*}\varphi_{\lambda}^{(\sigma)}(Q) \end{cases} \tag{11.3}$$

$$\begin{cases} \varphi_{\kappa}^{(\rho)}(Q)\varphi_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \alpha_{\kappa\lambda\mu}^{*}\varphi_{\mu}^{(\tau)}(Q) \\ \varphi_{\kappa'}^{(\rho)}(Q)\eta_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \alpha_{\kappa\lambda\mu}^{*}\eta_{\lambda}^{(\sigma)}(Q) \end{cases} \tag{11.4}$$

$$\begin{cases} \eta_{\kappa}^{(\rho)}(Q)\eta_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \beta_{\kappa\lambda\mu}^{*}\,\eta_{\mu}^{(\tau)}(Q) \\ \eta_{\kappa'}^{(\rho)}(Q)\varphi_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \beta_{\kappa\lambda\mu}^{*}\varphi_{\lambda}^{(\sigma)}(Q) \end{cases} \tag{11.5}$$

where $\tau_{P,Q} = \rho_{P,Q}\sigma_{P,Q}$. In particular, we can prove from (11.4)

$$F_{\kappa'}^{(\rho)}(G) \times U_{\mu}^{(\tau)}(G) \cong \sum_{\lambda} \alpha_{\kappa\lambda\mu}^{*}U_{\lambda}^{(\sigma)}(G). \tag{11.6}$$

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