A functorial approach to modules of G-dimension zero

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Abstract

Let $R$ be a commutative Noetherian ring and let $\mathcal{G}$ be the category of modules of G-dimension zero over $R$. We denote the associated stable category by $\mathcal{G}$. We show that the functor category $\text{mod}\mathcal{G}$ is a Frobenius category and we argue how this property could characterize $\mathcal{G}$ as a subcategory of $\text{mod}R$.

Key Words: Noetherian commutative ring, G-dimension, functor, Auslander-Reiten sequence.

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1 Introduction

In this paper $R$ always denotes a commutative Noetherian ring, and $\text{mod}R$ is the category of finitely generated $R$-modules.

We say that an object $X \in \text{mod}R$ is a module of G-dimension zero if it satisfies the conditions

$$\text{Ext}^i_R(X, R) = 0 \quad \text{and} \quad \text{Ext}^i_R(\text{Tr}X, R) = 0 \quad \text{for any} \quad i > 0, \quad (1)$$

as it is introduced in the paper [2] of Auslander and Bridger. Note that this is equivalent to saying that $X$ is a reflexive module and, in addition, it satisfies

$$\text{Ext}^i_R(X, R) = \text{Ext}^i_R(X^*, R) = 0 \quad \text{for any} \quad i > 0.$$

We should remark as well that other terminology for modules of G-dimension zero is used by other authors. Actually, Avramov and Martsinkovsky [6] call them totally reflexive modules and Enochs and Jenda [7] call them Gorenstein projective modules.

In this paper we are mainly interested in the independence of the conditions in (1) and our motivated question is the following:
• Is the condition $\text{Ext}^i_R(X, R) = 0$ for all $i > 0$ sufficient for $X$ to be a module of $G$-dimension zero?

Very recently, Jorgensen and Sega [9] constructed a counterexample to this question. However we still expect that the question will have an affirmative answer in sufficiently many cases. In fact, as one of the main theorems of this paper we shall show that it is true if there are only a finite number of isomorphism classes of indecomposable modules $X$ with $\text{Ext}^i_R(X, R) = 0$ for all $i > 0$.

In this paper we naturally introduce the two subcategories $G$ and $H$ of $\text{mod} R$, where $G$ is the full subcategory of $\text{mod} R$ consisting of all modules of $G$-dimension zero and $H$ is the full subcategory consisting of all modules $X \in \text{mod} R$ with $\text{Ext}^i_R(X, R) = 0$ for all $i > 0$. Of course, we have the natural inclusion $G \subseteq H$ and we shall discuss the problem of how $H$ is close to $G$.

To this end, in the first half of this paper, we try to make functorial characterizations of $G$ and $H$ as the subcategories of $\text{mod} R$. We need to recall several notation to make this more explicit. For any subcategory $C$ of $\text{mod} R$, we denote by $\underline{C}$ the associated stable category and denote by $\text{mod}\underline{C}$ the category of finitely presented contravariant additive functors from $\underline{C}$ to the category of Abelian groups. See §2 for the precise definitions for these associated categories. See also the papers [1], [4] and [5] for general and basic materials of categories of modules. As the first result of this paper we shall prove in §3 that the functor category $\text{mod}\underline{H}$ is a quasi-Frobenius category, on the other hand, $\text{mod}\underline{G}$ is a Frobenius category. See Theorems 3.5 and 3.7.

Looking through the argument in the proofs of these theorems, we see that Frobenius and quasi-Frobenius property of the category $\text{mod}\underline{C}$ will be essential for a general subcategory $C$ of $\text{mod} R$ to be contained in $G$ and $H$ respectively. We shall study this closely in §4. To be more precise, let $R$ be a henselian local ring, and hence the category $\text{mod} R$ is a Krull-Schmidt category. Then we shall prove in Theorem 4.2 that a resolving subcategory $C$ of $\text{mod} R$ is contained in $H$ if and only if $\text{mod}\underline{C}$ is a quasi-Frobenius category. In this sense, we succeed to characterize functorially the subcategory $H$ as the maximum subcategory $C$ for which $\text{mod}\underline{C}$ is a quasi-Frobenius category. For the subcategory $G$, it is also possible to make a similar functorial characterization using Frobenius property instead of quasi-Frobenius property, but with an additional assumption that the Auslander-Reiten conjecture is true. See Theorem 5.2 for the detail.

In the final section §5, we shall prove the main Theorem 5.5 of this paper, in which we assert that any resolving subcategory of finite type in $H$ is always contained in $G$. In particular, if $H$ itself is of finite type, then we can deduce the equality $G = H$.

## 2 Preliminary and Notation

Let $R$ be a commutative Noetherian ring, and let $\text{mod} R$ be the category of finitely generated $R$-modules as in the introduction.
When we say \( C \) is a subcategory of \( \text{mod} \, R \), we always mean the following:

- \( C \) is essential in \( \text{mod} \, R \), i.e. if \( X \cong Y \) in \( \text{mod} \, R \) and if \( X \in C \), then \( Y \in C \).
- \( C \) is full in \( \text{mod} \, R \), i.e. \( \text{Hom}_C(X,Y) = \text{Hom}_R(X,Y) \) for \( X, Y \in C \).
- \( C \) is additive and additively closed in \( \text{mod} \, R \), i.e. for any \( X, Y \in \text{mod} \, R \), \( X \oplus Y \in C \) if and only if \( X \in C \) and \( Y \in C \).

Furthermore, if all the projective modules in \( \text{mod} \, R \) are belonging to \( C \), then we say that \( C \) is a subcategory which contains the projectives.

The aim of this section is to settle the notation that will be used throughout this paper and to recall several notion of the categories associated to a given subcategory.

Let \( C \) be any subcategory of \( \text{mod} \, R \). At first, we define the associated stable category \( \mathcal{C} \) as follows:

- The objects of \( \mathcal{C} \) are the same as those of \( C \).
- For \( X, Y \in C \), the morphism set is an \( R \)-module
  \[
  \text{Hom}_R(X,Y) = \text{Hom}_R(X,Y)/P(X,Y),
  \]
  where \( P(X,Y) \) is the \( R \)-submodule of \( \text{Hom}_R(X,Y) \) consisting of all \( R \)-homomorphisms which factor through projective modules.

Of course, there is a natural functor \( C \to \mathcal{C} \). And for an object \( X \) and a morphism \( f \) in \( C \) we denote their images in \( \mathcal{C} \) under this natural functor by \( \overline{X} \) and \( \overline{f} \).

**Definition 2.1** Let \( C \) be a subcategory of \( \text{mod} \, R \). For a module \( X \) in \( C \), we take a finite presentation by finite projective modules

\[
P_1 \to P_0 \to X \to 0,
\]
and define the transpose \( \text{Tr}_X \) of \( X \) as the cokernel of \( \text{Hom}_R(P_0, R) \to \text{Hom}_R(P_1, R) \).

Similarly for a morphism \( f : X \to Y \) in \( C \), since it induces a morphism between finite presentations

\[
\begin{array}{c}
P_1 \to P_0 \to X \to 0 \\
f_1 \downarrow \quad f_0 \downarrow \quad f \\
Q_1 \to Q_0 \to Y \to 0,
\end{array}
\]
we define the morphism \( \text{Tr}_f : \text{Tr}_Y \to \text{Tr}_X \) as the morphism induced by \( \text{Hom}_R(f_1, R) \).
It is easy to see that \( \text{Tr}_X \) and \( \text{Tr}_f \) are uniquely determined as an object and a morphism in the stable category \( \mathcal{C} \), and it defines well the functor

\[
\text{Tr} : (\mathcal{C})^{op} \to \text{mod} \, R.
\]
For a module $X \in \mathcal{C}$ its syzygy module $\Omega X$ is defined by the exact sequence
\[ 0 \to \Omega X \to P_0 \to X \to 0, \]
where $P_0$ is a projective module. It is also easy to see that $\Omega$ defines a functor
\[ \Omega : \mathcal{C} \to \text{mod} \mathcal{R}. \]

We are interested in this paper two particular subcategories and their associated stable categories.

**Notation 2.2** We denote by $\mathcal{G}$ the subcategory of $\text{mod} \mathcal{R}$ consisting of all modules of \( G \)-dimension zero, that is, a module $X \in \text{mod} \mathcal{R}$ is an object in $\mathcal{G}$ if and only if
\[ \text{Ext}_R^i(X, R) = 0 \quad \text{and} \quad \text{Ext}_R^i(\text{Tr}X, R) = 0 \quad \text{for any} \quad i > 0. \]

We also denote by $\mathcal{H}$ the subcategory consisting of all modules with the first half of the above conditions, that is, a module $X \in \text{mod} \mathcal{R}$ is an object in $\mathcal{H}$ if and only if
\[ \text{Ext}_R^i(X, R) = 0 \quad \text{for any} \quad i > 0. \]

Of course we have $\mathcal{G} \subseteq \mathcal{H}$. And as a main motivation of this paper, we argue how they are different by characterizing these subcategories by a functorial method.

Note just from the definition that $\text{Tr}$ gives dualities on $\mathcal{G}$ and $\text{mod} \mathcal{R}$, that is, the first and the third vertical arrows in the following diagram are isomorphisms of categories:
\[ \begin{array}{ccc}
\mathcal{G} \leftarrow \text{Tr}\mathcal{H} \leftarrow \text{mod} \mathcal{R} & \downarrow & \\
\mathcal{G} \leftarrow \text{mod} \mathcal{R} & \downarrow & \\
\mathcal{G} \leftarrow \mathcal{H} \leftarrow \text{mod} \mathcal{R} & \downarrow & \\
\end{array} \]

Here we note that $\text{Tr}\mathcal{H}$ is the subcategory consisting of all modules $X$ satisfying $\text{Ext}_R^i(\text{Tr}X, R) = 0$ for all $i > 0$, hence we have the equality
\[ \mathcal{G} = \mathcal{H} \cap \text{Tr}\mathcal{H}. \]

Therefore $\mathcal{G} = \mathcal{H}$ is equivalent to that $\text{Tr}\mathcal{H} = \mathcal{H}$, that is, $\mathcal{H}$ is closed under $\text{Tr}$. Note also that $\mathcal{G}$ and $\mathcal{H}$ are closed under the syzygy functor, i.e. $\Omega \mathcal{G} = \mathcal{G}$ and $\Omega \mathcal{H} \subseteq \mathcal{H}$.

For an additive category $\mathcal{A}$, a contravariant additive functor from $\mathcal{A}$ to the category $(\text{Ab})$ of abelian groups is referred to as an $\mathcal{A}$-module, and a natural transform between two $\mathcal{A}$-modules is referred to as an $\mathcal{A}$-module morphism. We denote by $\text{Mod} \mathcal{A}$ the category consisting of all $\mathcal{A}$-modules and all $\mathcal{A}$-module morphisms. Note that $\text{Mod} \mathcal{A}$ is obviously an abelian category. An $\mathcal{A}$-module $F$ is called finitely presented if there is an exact sequence
\[ \text{Hom}_{\mathcal{A}}(, X_1) \to \text{Hom}_{\mathcal{A}}(, X_0) \to F \to 0, \]
for some $X_0, X_1 \in \mathcal{A}$. We denote by $\text{mod} \mathcal{A}$ the full subcategory of $\text{Mod} \mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules. For more general and basic materials about $\text{mod} \mathcal{A}$, the reader should refer to the papers [1], [4] and [5].
Lemma 2.3 (Yoneda) For any $X \in \mathcal{A}$ and any $F \in \text{Mod}\mathcal{A}$, we have the following natural isomorphism:

$$\text{Hom}_{\text{Mod}\mathcal{A}}(\text{Hom}_\mathcal{A}(, X), F) \cong F(X).$$

Corollary 2.4 An $\mathcal{A}$-module is projective in $\text{mod}\mathcal{A}$ if and only if it is isomorphic to $\text{Hom}_\mathcal{A}(, X)$ for some $X \in \mathcal{A}$.

Corollary 2.5 The functor $\mathcal{A}$ to $\text{mod}\mathcal{A}$ which sends $X$ to $\text{Hom}_\mathcal{A}(, X)$ is a full embedding.

Now let $\mathcal{C}$ be a subcategory of $\text{mod}\mathcal{R}$ and let $\mathcal{C}$ be the associated stable category. Then the category of finitely presented $\mathcal{C}$-modules $\text{mod}\mathcal{C}$ and the category of finitely presented $\mathcal{C}$-modules $\text{mod}\mathcal{C}$ are defined as in the above course. Note that for any $F \in \text{mod}\mathcal{C}$ (resp. $G \in \text{mod}\mathcal{C}$) and for any $X \in \mathcal{C}$ (resp. $X \in \mathcal{C}$), the abelian group $F(X)$ (resp. $G(X)$) has naturally an $R$-module structure, hence $F$ (resp. $G$) is in fact a contravariant additive functor from $\mathcal{C}$ (resp. $\mathcal{C}$) to $\text{mod}\mathcal{R}$.

Remark 2.6 As we stated above there is a natural functor $\mathcal{C} \to \mathcal{C}$. We can define from this the functor $\iota : \text{mod}\mathcal{C} \to \text{mod}\mathcal{C}$ by sending $F \in \text{mod}\mathcal{C}$ to the composition functor of $\mathcal{C} \to \mathcal{C}$ with $F$. Then it is well known and is easy to prove that $\iota$ gives an equivalence of categories between $\text{mod}\mathcal{C}$ and the full subcategory of $\text{mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules $F$ with $F(R) = 0$.

We prepare the following lemma for a later use. The proof is straightforward and we leave it to the reader.

Lemma 2.7 Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\text{mod}\mathcal{R}$. Then we have the following.

1. The induced sequence $\text{Hom}_\mathcal{R}(W, X) \to \text{Hom}_\mathcal{R}(W, Y) \to \text{Hom}_\mathcal{R}(W, Z)$ is exact for any $W \in \text{mod}\mathcal{R}$.
2. If $\text{Ext}^1_\mathcal{R}(Z, R) = 0$, then the induced sequence $\text{Hom}_\mathcal{R}(Z, W) \to \text{Hom}_\mathcal{R}(Y, W) \to \text{Hom}_\mathcal{R}(X, W)$ is exact for any $W \in \text{mod}\mathcal{R}$.

Corollary 2.8 Let $W$ be in $\text{mod}\mathcal{R}$. Then the covariant functor $\text{Hom}_\mathcal{R}(W, )$ is a half-exact functor on $\text{mod}\mathcal{R}$. On the other hand, the contravariant functor $\text{Hom}_\mathcal{R}(, W)$ is half-exact on $\mathcal{H}$.

3 Frobenius property of $\text{mod}\mathcal{G}$

Definition 3.1 Let $\mathcal{C}$ be a subcategory of $\text{mod}\mathcal{R}$.
We say that \( C \) is closed under kernels of epimorphisms if it satisfies the following condition:

If 0 → \( \omega \rightarrow Y \rightarrow Z \rightarrow 0 \) is an exact sequence in \( \text{mod} R \), and if \( Y, Z \in C \), then \( X \in C \).

(In Quillen’s terminology, all epimorphisms from \( \text{mod} R \) in \( C \) are admissible.)

We say that \( C \) is closed under extension or extension-closed if it satisfies the following condition:

If 0 → \( \omega \rightarrow Y \rightarrow Z \rightarrow 0 \) is an exact sequence in \( \text{mod} R \), and if \( X, Z \in C \), then \( Y \in C \).

We say that \( C \) is a resolving subcategory if \( C \) contains the projetives and if it is extension-closed and closed under kernels of epimorphisms.

We say that \( C \) is closed under \( \Omega \) if it satisfies the following condition:

If 0 → \( \omega \rightarrow P \rightarrow Z \rightarrow 0 \) is an exact sequence in \( \text{mod} R \) where \( P \) is a projective module, and if \( Z \in C \), then \( X \in C \).

Note that for a given \( Z \in C \), the module \( X \) in the above exact sequence is unique up to a projective summand. We denote \( X \) by \( \Omega Z \) as an object in \( C \). Thus, \( C \) is closed under \( \Omega \) if and only if \( \Omega X \in C \) whenever \( X \in C \).

Similarly to (4), the closedness under \( \text{Tr} \) is defined. Actually, we say that \( C \) is closed under \( \text{Tr} \) if \( \text{Tr} X \in C \) whenever \( X \in C \).

Note that the categories \( G \) and \( H \) satisfy any of the above first four conditions and that \( G \) is closed under \( \text{Tr} \). We also note the following lemma.

**Lemma 3.2** Let \( C \) be a subcategory of \( \text{mod} R \) which contains the projectives.

1. If \( C \) is closed under kernels of epimorphisms, then it is closed under \( \Omega \).
2. If \( C \) is extension-closed and closed under \( \Omega \), then it is resolving.

**Proof.** (1) Trivial.

(2) To show that \( C \) is closed under kernels of epimorphisms, let 0 → \( \omega \rightarrow Y \rightarrow Z \rightarrow 0 \) be an exact sequence in \( \text{mod} R \) and assume that \( Y, Z \in C \). Taking a projective
cover $P \to Z$ and taking the pull-back diagram, we have the following commutative
diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\Omega Z & \Omega Z \\
\downarrow & \downarrow \\
0 & \to & X & \to & E & \to & P & \to & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0
\end{array}
\]

Since $C$ is closed under $\Omega$, we have $\Omega Z \in C$. Then, since $C$ is extension-closed, we have $E \in C$. Noting that the middle row is a split exact sequence, we have $X \in C$ since $C$ is additively closed. □

We terminologically say that $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ is an exact sequence in a subcategory $C \subseteq \text{mod} R$ if it is an exact sequence in $\text{mod} R$ and such that $X_i \in C$ for all $i$.

**Proposition 3.3** Let $C$ be a subcategory of $\text{mod} R$ which contains the projectives and which is closed under kernels of epimorphisms.

1. Then $\text{mod} C$ is an abelian category with enough projectives.

2. For any $F \in \text{mod} C$, there is a short exact sequence in $C$

\[
0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0
\]

such that $F$ has a projective resolution of the following type:

\[
\begin{array}{cccc}
\cdots \longrightarrow & \text{Hom}_R(\ , \Omega^2 X_2)|_C & \longrightarrow & \text{Hom}_R(\ , \Omega^2 X_1)|_C & \longrightarrow & \text{Hom}_R(\ , \Omega^2 X_0)|_C \\
\longrightarrow & \text{Hom}_R(\ , \Omega X_2)|_C & \longrightarrow & \text{Hom}_R(\ , \Omega X_1)|_C & \longrightarrow & \text{Hom}_R(\ , \Omega X_0)|_C \\
\longrightarrow & \text{Hom}_R(\ , X_2)|_C & \longrightarrow & \text{Hom}_R(\ , X_1)|_C & \longrightarrow & \text{Hom}_R(\ , X_0)|_C \\
\longrightarrow & F & \longrightarrow & 0
\end{array}
\]

**Proof.** (1) Note that $\text{mod} C$ is naturally embedded into an abelian category $\text{Mod} C$. Let $\varphi : F \to G$ be a morphism in $\text{mod} C$. It is easy to see just from the definition that $\text{Coker}(\varphi) \in \text{mod} C$. If we prove that $\text{Ker}(\varphi) \in \text{mod} C$, then we see that $\text{mod} C$ is an
abelian category, since it is a full subcategory of the abelian category Mod\( \mathcal{C} \) which is closed under kernels and cokernels. (See also [5, §2], in which Auslander and Reiten call this property the existence of pseudo-kernels and pseudo-cokernels.) Now we prove that \( \text{Ker}(\varphi) \) is finitely presented.

(i) For the first case, we prove it when \( F \) and \( G \) are projective. So let \( \varphi : \text{Hom}_R(\cdot, X_1) \to \text{Hom}_R(\cdot, X_0) \). In this case, by Yoneda’s lemma, \( \varphi \) is induced from \( f : X_1 \to X_0 \). If necessary, adding a projective summand to \( X_1 \), we may assume that \( f : X_1 \to X_0 \) is an epimorphism in \( \text{mod} \ R \). Setting \( X_2 \) as the kernel of \( f \), we have an exact sequence

\[
0 \to X_2 \to X_1 \xrightarrow{f} X_0 \to 0.
\]

Since \( \mathcal{C} \) is closed under kernels of epimorphisms, we have \( X_2 \in \mathcal{C} \). Then it follows from Lemma 2.7 that the sequence

\[
\text{Hom}_R(\cdot, X_2) \xrightarrow{\psi} \text{Hom}_R(\cdot, X_1) \xrightarrow{\varphi} \text{Hom}_R(\cdot, X_0)
\]

is exact in \( \text{mod} \mathcal{C} \). Applying the same argument to \( \ker(\psi) \), we see that \( \ker(\varphi) \) is finitely presented as required.

(ii) Now we consider a general case. The morphism \( \varphi : F \to G \) induces the following commutative diagram whose horizontal sequences are finite presentations of \( F \) and \( G \):

\[
\begin{array}{ccc}
\text{Hom}_R(\cdot, X_1) \xrightarrow{a} & \text{Hom}_R(\cdot, X_0) \xrightarrow{b} & F \to 0 \\
\downarrow u & \downarrow v & \downarrow \varphi \\
\text{Hom}_R(\cdot, Y_1) \xrightarrow{c} & \text{Hom}_R(\cdot, Y_0) \xrightarrow{d} & G \to 0
\end{array}
\]

Now we define \( H \) by the following exact sequence:

\[
0 \to H \to \text{Hom}_R(\cdot, X_0) \oplus \text{Hom}_R(\cdot, Y_1) \xrightarrow{(v,c)} \text{Hom}_R(\cdot, Y_0)
\]

It follows from the first step of this proof, we have \( H \in \text{mod} \mathcal{C} \). On the other hand, it is easy to see that there is an exact sequence:

\[
\ker(c) \oplus \text{Hom}_R(\cdot, X_1) \to H \to \ker(\varphi) \to 0
\]

Here, from the first step again, we have \( \ker(c) \in \text{mod} \mathcal{C} \). Now since \( \text{mod} \mathcal{C} \) is closed under cokernels in \( \text{Mod} \mathcal{C} \), we finally have \( \ker(\varphi) \) as required.

(2) Let \( F \) be an arbitrary object in \( \text{mod} \mathcal{C} \) with the finite presentation

\[
\text{Hom}_R(\cdot, X_1) \xrightarrow{\varphi} \text{Hom}_R(\cdot, X_0) \to F \to 0.
\]

Then, as in the first step of the proof of (1), we may assume that there is a short exact sequence in \( \mathcal{C} \)

\[
0 \to X_2 \to X_1 \xrightarrow{f} X_0 \to 0
\]
such that \( \varphi \) is induced by \( f \). Applying Lemma 2.7, we have the following exact sequence

\[
\text{Hom}_R( , X_2) \mid \mathcal{C} \longrightarrow \text{Hom}_R( , X_1) \mid \mathcal{C} \stackrel{\varphi}{\longrightarrow} \text{Hom}_R( , X_0) \mid \mathcal{C} \longrightarrow F \longrightarrow 0.
\]

Similarly to the proof of Lemma 3.2(2), taking a projective cover of \( X_0 \) and taking the pull-back, we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & \\
\downarrow & & \\
\Omega X_0 & \longrightarrow & \Omega X_0 & \\
\downarrow & & \\
0 & \longrightarrow & X_2 & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\
& & \| & & & & \\
0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
\downarrow & & \\
0 & & & & & & 0
\end{array}
\]

Since the second row is a split exact sequence, we get the exact sequence

\[
0 \longrightarrow \Omega X_0 \longrightarrow X_2 \oplus P \longrightarrow X_1 \longrightarrow 0,
\]

where \( P \) is a projective module. Then it follows from Lemma 2.7 that there is an exact sequence

\[
\text{Hom}_R( , \Omega X_0) \mid \mathcal{C} \longrightarrow \text{Hom}_R( , X_2) \mid \mathcal{C} \longrightarrow \text{Hom}_R( , X_1) \mid \mathcal{C}
\]

Continue this procedure, and we shall obtain the desired projective resolution of \( F \) in \( \text{mod}\mathcal{C} \). \( \square \)

Note that the proof of the first part of Theorem 3.3 is completely similar to that of [11, Lemma (4.17)], in which it is proved that \( \text{mod}\mathcal{C} \) is an abelian category when \( R \) is a Cohen-Macaulay local ring and \( \mathcal{C} \) is the category of maximal Cohen-Macaulay modules.

**Definition 3.4** A category \( \mathcal{A} \) is said to be a Frobenius category if it satisfies the following conditions:

1. \( \mathcal{A} \) is an abelian category with enough projectives and with enough injectives.
2. All projective objects in \( \mathcal{A} \) are injective.
3. All injective objects in \( \mathcal{A} \) are projective.
Likewise, a category $\mathcal{A}$ is said to be a quasi-Frobenius category if it satisfies the conditions:

1. $\mathcal{A}$ is an abelian category with enough projectives.
2. All projective objects in $\mathcal{A}$ are injective.

**Theorem 3.5** Let $\mathcal{C}$ be a subcategory of $\text{mod} \mathcal{R}$ which contains the projectives and which is closed under kernels of epimorphisms. If $\mathcal{C} \subseteq \mathcal{H}$ then $\text{mod}\mathcal{C}$ is a quasi-Frobenius category.

To prove this theorem, we prepare the following lemma. Here we recall that the full embedding $\iota: \text{mod}\mathcal{C} \to \text{mod}\mathcal{C}$ is the functor induced by the natural functor $\mathcal{C} \to \mathcal{C}$. 

**Lemma 3.6** Let $\mathcal{C}$ be a subcategory of $\text{mod} \mathcal{R}$ which contains the projectives and which is closed under kernels of epimorphisms. Then the following conditions are equivalent for each $F \in \text{mod}\mathcal{C}$.

1. $F$ is an injective object in $\text{mod}\mathcal{C}$.
2. $\iota F \in \text{mod}\mathcal{C}$ is half-exact as a functor on $\mathcal{C}$.

**Proof.** As we have shown in the previous proposition, the category $\text{mod}\mathcal{C}$ is an abelian category with enough projectives. Therefore an object $F \in \text{mod}\mathcal{C}$ is injective if and only if $\text{Ext}^1_{\text{mod}\mathcal{C}}(G, F) = 0$ for any $G \in \text{mod}\mathcal{C}$. But for a given $G \in \text{mod}\mathcal{C}$, there is a short exact sequence in $\mathcal{C}$

\[
\begin{align*}
\ast & \quad 0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0,
\end{align*}
\]

such that $G$ has a projective resolution

\[
\begin{align*}
\text{Hom}_R(\ , X_2)_{\mathcal{C}} & \longrightarrow \text{Hom}_R(\ , X_1)_{\mathcal{C}} \longrightarrow \text{Hom}_R(\ , X_0)_{\mathcal{C}} \longrightarrow G \longrightarrow 0.
\end{align*}
\]

Conversely, for any short exact sequence in $\mathcal{C}$ such as $(\ast)$, the cokernel functor $G$ of $\text{Hom}_R(\ , X_1)_{\mathcal{C}} \to \text{Hom}_R(\ , X_0)_{\mathcal{C}}$ is an object of $\text{mod}\mathcal{C}$. Therefore $F$ is injective if and only if it satisfies the following condition:

The induced sequence

\[
\begin{align*}
\text{Hom}(\text{Hom}_R(\ , X_0)_{\mathcal{C}}, F) \to \text{Hom}(\text{Hom}_R(\ , X_1)_{\mathcal{C}}, F) \to \text{Hom}(\text{Hom}_R(\ , X_2)_{\mathcal{C}}, F)
\end{align*}
\]

is exact, whenever $0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$.

It follows from Yoneda’s lemma that this is equivalent to saying that $F(X_0) \longrightarrow F(X_1) \longrightarrow F(X_2)$ is exact whenever $0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$. This exactly means that $\iota F$ is half-exact as a functor on $\mathcal{C}$. $\square$
Proof. Now we proceed to the proof of Theorem 3.5. We have already shown that mod$\mathcal{C}$ is an abelian category with enough projectives. It remains to show that any projective module $\text{Hom}_R(\ , X)\mid_{\mathcal{C}}$ $(X \in \mathcal{C})$ is an injective object in mod$\mathcal{C}$. Since $\mathcal{C}$ is a subcategory of $\mathcal{H}$, it follows from Corollary 2.8 that $\text{Hom}_R(\ , X)\mid_{\mathcal{C}} = \iota(\text{Hom}_R(\ , X)\mid_{\mathcal{C}})$ is a half-exact functor, hence it is injective by the previous lemma. $\square$

Before stating the next theorem, we should remark that the syzygy functor $\Omega$ gives an automorphism on $\mathcal{G}$.

Theorem 3.7 Let $\mathcal{C}$ be a subcategory of mod$R$. And suppose the following conditions.

1. $\mathcal{C}$ is a resolving subcategory of mod$R$.
2. $\mathcal{C} \subseteq \mathcal{H}$.
3. The functor $\Omega : \mathcal{C} \to \mathcal{C}$ yields a surjective map on the set of isomorphism classes of the objects in $\mathcal{C}$.

Then mod$\mathcal{C}$ is a Frobenius category. In particular, mod$\mathcal{G}$ is a Frobenius category.

Proof. Since $\mathcal{C}$ is subcategory of $\mathcal{H}$ that is closed under kernels of epimorphisms in mod$R$, mod$\mathcal{C}$ is a quasi-Frobenius category by the previous theorem. It remains to prove that mod$\mathcal{C}$ has enough injectives and all injectives are projective.

(i) For the first step of the proof we show that each $\mathcal{C}$-module $F \in \text{mod}\mathcal{C}$ can be embedded into a projective $\mathcal{C}$-module $\text{Hom}_R(\ , Y)\mid_{\mathcal{C}}$ for some $Y \in \mathcal{C}$.

In fact, as we have shown in Proposition 3.3, for a given $F \in \text{mod}\mathcal{C}$, there is a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

such that $F$ has a projective resolution

$$\text{Hom}_R(\ , X_1)\mid_{\mathcal{C}} \longrightarrow \text{Hom}_R(\ , X_0)\mid_{\mathcal{C}} \longrightarrow F \longrightarrow 0.$$ 

From the assumption (3), there is an exact sequence in $\mathcal{C}$

$$0 \longrightarrow X_2 \longrightarrow P \longrightarrow Y \longrightarrow 0,$$

where $P$ is a projective module. Then, similarly to the argument in the proof of Lemma 3.2, just taking the push-out, we have a commutative diagram with exact rows and
Since $\text{Ext}_R^1(X_0, P) = 0$, we should note that the second row is splittable. Hence we have a short exact sequence of the type
\[ 0 \rightarrow X_1 \rightarrow X_0 \oplus P \rightarrow Y \rightarrow 0. \]
Therefore we have from Lemma 2.7 that there is an exact sequence
\[ \text{Hom}_R( , X_1)|_\mathcal{C} \rightarrow \text{Hom}_R( , X_0)|_\mathcal{C} \rightarrow \text{Hom}_R( , Y)|_\mathcal{C}. \]
Thus $F$ can be embedded into $\text{Hom}_R( , Y)|_\mathcal{C}$ as desired.

(ii) Since all projective modules in $\text{mod}\mathcal{C}$ are injective, it follows from (i) that $\text{mod}\mathcal{C}$ has enough injectives.

(iii) To show that every injective module in $\text{mod}\mathcal{C}$ is projective, let $F$ be an injective $\mathcal{C}$-module in $\text{mod}\mathcal{C}$. By (i), $F$ is a $\mathcal{C}$-submodule of $\text{Hom}_R( , Y)|_\mathcal{C}$ for some $Y \in \mathcal{C}$, hence $F$ is a direct summand of $\text{Hom}_R( , Y)|_\mathcal{C}$. Since it is a summand of a projective module, $F$ is projective as well. $\square$

4 Characterizing subcategories of $\mathcal{H}$

In this section we always assume that $R$ is a henselian local ring with maximal ideal $m$ and with the residue class field $k = R/m$. In the following, what we shall need from this assumption is the fact that $X \in \text{mod} R$ is indecomposable only if $\text{End}_R(X)$ is a (noncommutative) local ring. In fact we can show the following lemma.

Lemma 4.1 Let $(R, m)$ be a henselian local ring. Then $\text{mod}\mathcal{C}$ is a Krull-Schmidt category for any subcategory $\mathcal{C}$ of $\text{mod} R$.

Proof. We have only to prove that $\text{End}_{\text{mod}\mathcal{C}}(F)$ is a local ring for any indecomposable $\mathcal{C}$-module $F \in \text{mod}\mathcal{C}$. 

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First we note that $\text{End}_{\text{mod} C}(F)$ is a module-finite algebra over $R$. In fact, since there is a finite presentation

$$\Hom_R( , X_1)|_C \longrightarrow \Hom_R( , X_0)|_C \longrightarrow F \longrightarrow 0,$$

$F(X_0)$ is a finite $R$-module. On the other hand, taking the dual by $F$ of the above sequence and using Yoneda’s lemma, we can see that there is an exact sequence of $R$-modules

$$0 \longrightarrow \text{End}_{\text{mod} C}(F) \longrightarrow F(X_0) \longrightarrow F(X_1).$$

As a submodule of a finite module, $\text{End}_{\text{mod} C}(F)$ is finite over $R$.

Now suppose that $\text{End}_{\text{mod} C}(F)$ is not a local ring. Then there is an element $e \in \text{End}_{\text{mod} C}(F)$ such that both $e$ and $1 - e$ are nonunits. Let $\bar{R}$ be the image of the natural ring homomorphism $R \to \text{End}_{\text{mod} C}(F)$. We consider the subalgebra $\bar{R}[e]$ of $\text{End}_{\text{mod} C}(F)$. Since $\bar{R}[e]$ is a commutative $R$-algebra which is finite over $R$, it is also henselian. Since $e, 1 - e \in \bar{R}[e]$ are both nonunits, $\bar{R}[e]$ should be decomposed into a direct product of rings, by the henselian property. In particular, $\bar{R}[e]$ contains a nontrivial idempotent. This implies that $\text{End}_{\text{mod} C}(F)$ contains a nontrivial idempotent, hence that $F$ is decomposable in $\text{mod} C$.

We claim the validity of the converse of Theorem 3.5.

**Theorem 4.2** Let $C$ be a resolving subcategory of $\text{mod} R$. Suppose that $\text{mod} C$ is a quasi-Frobenius category. Then $C \subseteq \mathcal{H}$.

In a sense $\mathcal{H}$ is the largest resolving subcategory $C$ of $\text{mod} R$ for which $\text{mod} C$ is a quasi-Frobenius category.

**Proof.** Let $X$ be an indecomposable nonfree module in $C$. It is sufficient to prove that $\text{Ext}^1_R(X, R) = 0$.

In fact, if it is true for any $X \in C$, then $\text{Ext}^1_R(\Omega^i X, R) = 0$ for any $i \geq 0$, since $C$ is closed under $\Omega$. This implies that $\text{Ext}^{i+1}_R(X, R) = 0$ for $i \geq 0$, hence $X \in \mathcal{H}$.

Now we assume that $\text{Ext}^1_R(X, R) \neq 0$ to prove the theorem by contradiction. Let $\sigma$ be a nonzero element of $\text{Ext}^1_R(X, R)$ that corresponds to the nonsplit extension

$$\sigma : 0 \longrightarrow R \longrightarrow Y \overset{p}{\longrightarrow} X \longrightarrow 0.$$

Since $C$ is closed under extension, we should note that $Y \in C$. On the other hand, noting that $R$ is the zero object in $C$, we have from Lemma 2.7 that there is an exact sequence of $C$-modules

$$0 \longrightarrow \Hom_R( , Y)|_C \overset{p^*}{\longrightarrow} \Hom_R( , X)|_C.$$

Since $\text{mod} C$ is a quasi-Frobenius category, this monomorphism is a split one, hence $\Hom_R( , Y)|_C$ is a direct summand of $\Hom_R( , X)|_C$ through $p_*$. Since the embedding
\( \mathcal{C} \to \text{mod} \mathcal{C} \) is full, and since we assumed \( X \) is indecomposable, \( \text{Hom}_R(\ , X)|_\mathcal{C} \) is indecomposable in \( \text{mod} \mathcal{C} \) as well. Hence \( p_* \) is either an isomorphism or \( p_* = 0 \).

We first consider the case that \( p_* \) is an isomorphism. In this case, we can take a morphism \( q \in \text{Hom}_R(X, Y) \) such that \( q_* : \text{Hom}_R(\ , X)|_\mathcal{C} \to \text{Hom}_R(\ , Y)|_\mathcal{C} \) is the inverse of \( p_* \). Then \( p_* q_* = (pq)_* \) is the identity on \( \text{Hom}_R(\ , X)|_\mathcal{C} \). Since \( \text{End}_R(X) \cong \text{End}(\text{Hom}_R(\ , X)|_\mathcal{C}) \), we see that \( pq = 1 \) in \( \text{End}_R(X) \). Since \( \text{End}_R(X) \) is a local ring and since \( \text{End}_R(X) \) is a residue ring of \( \text{End}_R(X) \), we see that \( pq \in \text{End}_R(X) \) is a unit. This shows that the extension \( \sigma \) splits, and this is a contradiction. Hence this case never occurs.

As a result we have that \( p_* = 0 \), which implies \( \text{Hom}_R(\ , Y)|_\mathcal{C} = 0 \). Since the embedding \( \mathcal{C} \to \text{mod} \mathcal{C} \) is full, this is equivalent to saying that \( Y \) is a projective, hence free, module. Thus it follows from the extension \( \sigma \) that \( X \) has projective dimension exactly one. (We have assumed that \( X \) is nonfree.) Thus the extension \( \sigma \) should be a minimal free resolution of \( X \):

\[
0 \longrightarrow R \overset{\alpha}{\longrightarrow} R^e \overset{p}{\longrightarrow} X \longrightarrow 0.
\]

where \( \alpha = (a_1, \ldots, a_r) \) is a matrix with entries in \( \mathfrak{m} \). Now let \( x \in \mathfrak{m} \) be any element and let us consider the extension corresponding to \( x \sigma \in \text{Ext}^1_R(X, R) \). By making push-out, we obtained this extension as the second row in the following commutative diagram with exact rows:

\[
\begin{array}{c}
\sigma : 0 \longrightarrow R \overset{\alpha}{\longrightarrow} R^e \overset{p}{\longrightarrow} X \longrightarrow 0 \\
\downarrow x \quad \downarrow \quad \| \\
x \sigma : 0 \longrightarrow R \longrightarrow Z \overset{p'}{\longrightarrow} X \longrightarrow 0
\end{array}
\]

Note that there is an exact sequence

\[
R \overset{(x, \alpha)}{\longrightarrow} R \oplus R^e \longrightarrow Z \longrightarrow 0,
\]

where all entries of the matrix \( (x, \alpha) \) are in \( \mathfrak{m} \). Thus the \( R \)-module \( Z \) is not free, and thus \( p'_* : \text{Hom}_R(\ , Z)|_\mathcal{C} \to \text{Hom}_R(\ , X)|_\mathcal{C} \) is a nontrivial monomorphism. Then, repeating the argument in the first case to the extension \( x \sigma \), we must have \( x \sigma = 0 \) in \( \text{Ext}^1_R(X, R) \). Since this is true for any \( x \in \mathfrak{m} \) and for any \( \sigma \in \text{Ext}^1_R(X, R) \), we obtain that \( \mathfrak{m} \text{Ext}^1_R(X, R) = 0 \). On the other hand, by computation, we have \( \text{Ext}^1_R(X, R) \cong R/(a_1, \ldots, a_r) \), and hence we must have \( \mathfrak{m} = (a_1, \ldots, a_r)R \). Since the residue field \( k \) has a free resolution of the form

\[
R^e \overset{t(a_1, \ldots, a_r)}{\longrightarrow} R \longrightarrow k \longrightarrow 0,
\]

comparing this with the extension \( \sigma \), we have that \( X \cong \text{Tr}k \). What we have proved so far is the following:
Suppose there is an indecomposable nonfree module $X$ in $\mathcal{C}$ which satisfies $\text{Ext}^1_R(X, R) \neq 0$. Then $X$ is isomorphic to $\text{Tr}k$ as an object in $\mathcal{C}$ and $X$ has projective dimension one.

If $R$ is a field then the theorem is obviously true. So we assume that the local ring $R$ is not a field. Then we can find an indecomposable $R$-module $L$ of length 2 and a nonsplit exact sequence

$$0 \longrightarrow k \longrightarrow L \longrightarrow k \longrightarrow 0.$$  

Note that $L = R/I$ for some $m$-primary ideal $I$. Note also that depth $R \geq 1$, since there is a module of projective dimension one. Therefore there is no nontrivial $R$-homomorphism from $L$ to $R$, and thus we have $\text{End}_R(L) = \text{End}_R(L) \cong R/I$. Similarly we have $\text{End}_R(k) \cong k$.

It also follows from $\text{Hom}_R(k, R) = 0$ that there is an exact sequence of the following type:

$$(*) \quad 0 \longrightarrow \text{Tr}k \longrightarrow \text{Tr}L \oplus P \longrightarrow \text{Tr}k \longrightarrow 0,$$

where $P$ is a suitable free module. Since $X = \text{Tr}k$ is in $\mathcal{C}$, and since $\mathcal{C}$ is extension-closed, we have $\text{Tr}L \in \mathcal{C}$ as well. Note that $\text{Tr}$ is a duality on $\text{mod} R$, hence we see that $\text{End}_R(\text{Tr}L) \cong \text{End}_R(L)$ that is a local ring. As a consequence we see that $\text{Tr}L$ is indecomposable in $\mathcal{C}$.

We claim that $\text{Tr}L$ has projective dimension exactly one, hence in particular $\text{Ext}^1_R(\text{Tr}L, R) \neq 0$. In fact, we have from $(*)$ that $\text{Tr}L$ has projective dimension at most one as well as $X = \text{Tr}k$. If $\text{Tr}L$ were free then $X = \text{Tr}k$ would be its own first syzygy from $(*)$ and hence free because $X$ has projective dimension one. But this is a contradiction.

Thus it follows from the above claim that $\text{Tr}L$ is isomorphic to $\text{Tr}k$ in $\text{mod} R$. Taking the transpose again, we finally have that $L$ is isomorphic to $k$ in $\text{mod} R$. But this is absurd, because $\text{End}_R(L) \cong R/I$ and $\text{End}_R(k) \cong k$. Thus the proof is complete. \(\square\)

**Corollary 4.3** The following conditions are equivalent for a henselian local ring $R$.

1. $\text{mod} (\text{mod} R)$ is a quasi-Frobenius category.
2. $\text{mod} (\text{mod} R)$ is a Frobenius category.
3. $R$ is an artinian Gorenstein ring.

**Proof.**

(3) \(\Rightarrow\) (2): If $R$ is an artinian Gorenstein ring, then we have $\text{mod} R = \mathcal{G}$. Hence this implication follows from Theorem 3.7.

(2) \(\Rightarrow\) (1): Obvious.

(1) \(\Rightarrow\) (3): Suppose that $\text{mod} (\text{mod} R)$ is a quasi-Frobenius category. Then, by Theorem 4.2, any indecomposable $R$-module $X$ is in $\mathcal{H}$. In particular, the residue field $k = R/m$ is in $\mathcal{H}$, hence by definition, $\text{Ext}^i_R(k, R) = 0$ for any $i > 0$. This happens only if $R$ is an artinian Gorenstein ring. \(\square\)
5 Characterizing subcategories of $G$

Lemma 5.1 Let $R$ be a henselian local ring and let $C$ be an extension-closed subcategory of $\text{mod} \ R$. For objects $X, Y \in C$, we assume the following:

(1) There is a monomorphism $\varphi$ in $\text{Mod}C$:

$$\varphi : \text{Hom}_R(\ ,Y)|_C \to \text{Ext}^1(\ ,X)|_C$$

(2) $X$ is indecomposable in $C$.

(3) $Y \not\cong 0$ in $C$.

Then the module $X$ is isomorphic to a direct summand of $\Omega Y$.

Proof. We note from Yoneda’s lemma that there is an element $\sigma \in \text{Ext}^1_R(Y, X)$ which corresponds to the short exact sequence:

$$\sigma : 0 \longrightarrow X \longrightarrow L \longrightarrow Y \longrightarrow 0$$

such that $\varphi$ is induced by $\sigma$ as follows:

For any $W \in C$ and for any $f \in \text{Hom}_R(W, Y)$, consider the pull-back diagram to get the following commutative diagram with exact rows:

$$\begin{array}{cccccc}
0 & \longrightarrow & X & \longrightarrow & L & \longrightarrow & Y & \longrightarrow & 0 \\
| & & | & \uparrow f & & \uparrow & & \uparrow & \\
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

Setting the second exact sequence as $f^*\sigma$, we have $\varphi(f) = f^*\sigma$.

Note that $L \in C$, since $C$ is extension-closed. Also note that $\sigma$ is nonsplit. In fact, if it splits, then $\varphi$ is the zero map, hence $\text{Hom}_R(\ ,Y)|_C = 0$ from the assumption. Since the embedding $C \hookrightarrow \text{mod} \ C$ is full, this implies that $Y = 0$ in $C$, which is a contradiction.

Now let $P \to Y$ be a surjective $R$-module homomorphism where $P$ is a projective module. Then there is a commutative diagram with exact rows:

$$\begin{array}{cccccc}
0 & \longrightarrow & X & \longrightarrow & L & \longrightarrow & Y & \longrightarrow & 0 \\
g \uparrow & & & & & & \uparrow & & & & & & \uparrow & \\
0 & \longrightarrow & \Omega Y & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

Note that the extension $\sigma$ induces the exact sequence of $C$-modules:

$$\begin{array}{ccccccc}
\text{Hom}_R(\ ,L)|_C & \longrightarrow & \text{Hom}_R(\ ,Y)|_C & \longrightarrow & \text{Ext}^1_R(\ ,X)|_C & \\
p^* & & \varphi & & & \\
\text{Hom}_R(\ ,L)|_C & \longrightarrow & \text{Hom}_R(\ ,Y)|_C & \longrightarrow & \text{Ext}^1_R(\ ,X)|_C
\end{array}$$

Since $\varphi$ is a monomorphism, the morphism $\text{Hom}_R(\ ,L)|_C \to \text{Hom}_R(\ ,Y)|_C$ is the zero morphism. In particular, the map $p \in \text{Hom}_R(L, Y)$ is the zero element by Yoneda’s
lemma. (Note that we use the fact \( L \in \mathcal{C} \) here.) This is equivalent to saying that \( p : L \to Y \) factors through a projective module, hence that it factors through the map \( \pi \). As a consequence, there are maps \( k : L \to P \) and \( \ell : X \to \Omega Y \) which make the following diagram commutative:

\[
\begin{array}{c}
0 \longrightarrow X \xrightarrow{a} L \xrightarrow{p} Y \longrightarrow 0 \\
\downarrow \ell \downarrow k \quad \downarrow \phi \\
0 \longrightarrow \Omega Y \xrightarrow{\alpha} P \xrightarrow{\pi} Y \longrightarrow 0
\end{array}
\]

Then, since \( p(1 - hk) = 0 \), there is a map \( b : L \to X \) such that \( 1 - hk = ab \). Likewise, since \( \pi(1 - kh) = 0 \), there is a map \( \beta : P \to \Omega Y \) such that \( 1 - kh = \alpha \beta \). Note that there are equalities:

\[
a(1 - g\ell) = a - ag\ell = a - h\alpha\ell = a - hka = (1 - hk)a = aba
\]

Since \( a \) is a monomorphism, we hence have \( 1 - g\ell = ba \). Thus we finally obtain the equality \( 1 = ba + g\ell \) in the local ring \( \text{End}_R(X) \).

Since \( \sigma \) is a nonsplit sequence, \( ba \in \text{End}_R(X) \) never be a unit, and it follows that \( g\ell \) is a unit in \( \text{End}_R(X) \). This means that the map \( g : \Omega Y \to X \) is a split epimorphism, hence \( X \) is isomorphic to a direct summand of \( \Omega Y \) as desired. \( \square \)

**Theorem 5.2** Let \( R \) be a henselian local ring. Suppose that

1. \( \mathcal{C} \) is a resolving subcategory of \( \mod R \).
2. \( \mod \mathcal{C} \) is a Frobenius category.
3. There is no nonprojective module \( X \in \mathcal{C} \) with \( \text{Ext}^1_R(\phantom{1} , X)_{|\mathcal{C}} = 0 \).

Then \( \mathcal{C} \subseteq \mathcal{G} \).

**Proof.** As the first step of the proof, we prove the following:

(i) For a nontrivial indecomposable object \( X \in \mathcal{C} \), there is an object \( Y \in \mathcal{C} \) such that \( X \) is isomorphic to a direct summand of \( \Omega Y \).

To prove this, let \( X \in \mathcal{C} \) be nontrivial and indecomposable. Consider the \( \mathcal{C} \)-module \( F := \text{Ext}^1_R(\phantom{1} , X)_{|\mathcal{C}} \). The third assumption assures us that \( F \) is a nontrivial \( \mathcal{C} \)-module. Hence there is an indecomposable module \( W \in \mathcal{C} \) such that \( F(W) \neq 0 \). Take a nonzero element \( \sigma \) in \( F(W) = \text{Ext}^1_R(W, X) \) that corresponds to an exact sequence

\[
0 \longrightarrow X \longrightarrow E \longrightarrow W \longrightarrow 0.
\]

Note that \( E \in \mathcal{C} \), since \( \mathcal{C} \) is extension-closed. Then we have an exact sequence of \( \mathcal{C} \)-modules

\[
\text{Hom}_R(\phantom{1} , E)_{|\mathcal{C}} \longrightarrow \text{Hom}_R(\phantom{1} , W)_{|\mathcal{C}} \longrightarrow \text{Ext}^1_R(\phantom{1} , X)_{|\mathcal{C}}
\]
We denote by $F_\sigma$ the image of $\varphi$. Of course, $F_\sigma$ is a nontrivial $C$-submodule of $F$ which is finitely presented. Since we assume that $\text{mod}C$ is a Frobenius category, we can take a minimal injective hull of $F_\sigma$ that is projective as well, i.e. there is a monomorphism $i : F_\sigma \to \text{Hom}_R(\ , Y)|_C$ for some $Y \in C$ that is an essential extension. Since $\text{Ext}^1_R(\ , X)|_C$ is half-exact as a functor on $C$, we can see by a similar method to that in the proof of Lemma 3.6 that $\text{Hom}(\ , \text{Ext}^1_R(\ , X))$ is an exact functor on $\text{mod}C$. It follows from this that the natural embedding $F_\sigma \to F$ can be enlarged to the morphism $g : \text{Hom}_R(\ , Y)|_C \to F$. Hence there is a commutative diagram

$$
\begin{array}{ccc}
F_\sigma & \xrightarrow{c} & \text{Hom}_R(\ , Y)|_C \\
\cap & \downarrow & \cap \\
F & \to & \text{Ext}^1_R(\ , X).
\end{array}
$$

Since $i$ is an essential extension, we see that $\text{Ker} \ g = 0$, hence we have $\text{Hom}_R(\ , Y)|_C$ is a submodule of $F$. Hence by the previous lemma we see that $X$ is isomorphic to a direct summand of $\Omega Y$. Thus the claim (i) is proved.

Now we prove the theorem. Since $\text{mod}C$ is a quasi-Frobenius category, we know from Theorem 4.2 that $C \subseteq H$. To show $C \subseteq G$, let $X$ be a nontrivial indecomposable module in $C$. We want to prove that $\text{Ext}^1_R(\text{Tr}X, R) = 0$ for $i > 0$. It follows from the claim (i) that there is $Y \in C$ such that $X$ is a direct summand of $\Omega Y$. Note that $Y \in H$. From the obvious sequence

$$
0 \to \Omega Y \to P \to Y \to 0
$$

with $P$ being a projective module, it is easy to see that there is an exact sequence of the type

$$
(*) \quad 0 \to \text{Tr}Y \to P' \to \text{Tr}\Omega Y \to 0,
$$

where $P'$ is projective. Since $\Omega Y$ is a torsion-free module, it is obvious that $\text{Ext}^1_R(\text{Tr}\Omega Y, R) = 0$. Since $X$ is a direct summand of $\Omega Y$, we have that $\text{Ext}^1_R(\text{Tr}X, R) = 0$ as well. This is true for any indecomposable module in $C$, hence for each indecomposable summand of $Y$. Therefore we have $\text{Ext}^1_R(\text{Tr}Y, R) = 0$. Then it follows from this together with $(*)$ that $\text{Ext}^2_R(\text{Tr}\Omega Y, R) = 0$, hence we have that $\text{Ext}^2_R(\text{Tr}X, R) = 0$. Continuing this procedure, we can show $\text{Ext}^i_R(\text{Tr}X, R) = 0$ for any $i > 0$. □

**Remark 5.3** We conjecture that $G$ should be the largest resolving subcategory $C$ of $\text{mod}R$ such that $\text{mod}C$ is a Frobenius category.

Theorem 5.2 together with Theorem 3.7 say that this is true modulo Auslander-Reiten conjecture:

**AR** If $\text{Ext}^i_R(X, X \oplus R) = 0$ for any $i > 0$ then $X$ should be projective.

In fact, if the conjecture (AR) is true, then the third condition of the previous theorem is automatically satisfied.
Definition 5.4 Let $\mathcal{A}$ be any additive category. We denote by $\text{Ind}(\mathcal{A})$ the set of nonisomorphic modules which represent all the isomorphism classes of indecomposable objects in $\mathcal{A}$. If $\text{Ind}(\mathcal{A})$ is a finite set, then we say that $\mathcal{A}$ is a category of finite type.

The following theorem is a main theorem of this paper, which claims that any resolving subcategory of finite type in $\mathcal{H}$ are contained in $\mathcal{G}$.

Theorem 5.5 Let $R$ be a henselian local ring and let $\mathcal{C}$ be a subcategory of $\text{mod} R$ which satisfies the following conditions.

(1) $\mathcal{C}$ is a resolving subcategory of $\text{mod} R$.

(2) $\mathcal{C} \subseteq \mathcal{H}$.

(3) $\mathcal{C}$ is of finite type.

Then, $\text{mod}\mathcal{C}$ is a Frobenius category and $\mathcal{C} \subseteq \mathcal{G}$.

Lemma 5.6 Let $\mathcal{C}$ be a subcategory of $\text{mod} R$ which contains the projectives and suppose that $\mathcal{C}$ is of finite representation type. Then the following conditions are equivalent for a contravariant additive functor $F$ from $\mathcal{C}$ to $\text{mod} R$.

(1) $F$ is finitely presented, i.e. $F \in \text{mod}\mathcal{C}$.

(2) $F(W)$ is a finitely generated $R$-module for each $W \in \text{Ind}(\mathcal{C})$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial from the definition. We prove (2) $\Rightarrow$ (1). For this, note that for each $X, W \in \text{Ind}(\mathcal{C})$ and for each $f \in \text{Hom}_R(X, W)$, the induced map $F(f) : F(W) \rightarrow F(X)$ is an $R$-module homomorphism and satisfies that $F(af) = aF(f)$ for $a \in R$. Therefore the $\mathcal{C}$-module homomorphism

$$\varphi_W : \text{Hom}_R(X, W) \otimes_R F(W) \rightarrow F$$

which sends $f \otimes x$ to $F(f)(x)$ is well-defined. Now let $\{W_1, \ldots, W_m\}$ be the complete list of elements in $\text{Ind}(\mathcal{C})$. Then the $\mathcal{C}$-module homomorphism

$$\Phi = \bigoplus_{i=1}^m \varphi_{W_i} : \bigoplus_{i=1}^m \text{Hom}_R(X, W_i) \otimes_R F(W_i) \rightarrow F$$

is defined, and it is clear that $\Phi$ is an epimorphism in $\text{Mod}\mathcal{C}$. Therefore $F$ is finitely generated, and this is true for $\text{Ker}(\Phi)$ as well. Hence $F$ is finitely presented. $\Box$

Lemma 5.7 Let $\mathcal{C}$ be a subcategory of $\text{mod} R$ which contains the projectives and which is of finite type. And let $X$ be a nontrivial indecomposable module in $\mathcal{C}$. Suppose that $\mathcal{C}$ is closed under kernels of epimorphisms. Then $\mathcal{C}$ admits an AR-sequence ending in $X$, that is, there is a nonsplit exact sequence in $\mathcal{C}$

$$0 \longrightarrow \tau X \longrightarrow L \xrightarrow{p} X \longrightarrow 0,$$
such that for any indecomposable \( Y \in \mathcal{C} \) and for any morphism \( f : Y \to X \) which is not a split epimorphism, there is a morphism \( g : Y \to L \) that makes the following diagram commutative.

\[
\begin{array}{ccc}
L & \xrightarrow{p} & X \\
\uparrow g & & \uparrow f \\
Y & \xrightarrow{} & L \\
\end{array}
\]

**Proof.** Let \( \text{rad} \hom_R(\cdot, X)_{\mathcal{C}} \) be the radical functor of \( \hom_R(\cdot, X)_{\mathcal{C}} \); i.e. for each \( W \in \text{Ind}(\mathcal{C}) \), if \( W \not\cong X \) then \( \text{rad} \hom_R(W, X) = \hom_R(W, X) \), on the other hand, if \( W = X \) then \( \text{rad} \hom_R(X, X) \) is the unique maximal ideal of \( \text{End}_R(X) \). Since \( \text{rad} \hom_R(\cdot, X)_{\mathcal{C}} \) is a \( \mathcal{C} \)-submodule of \( \hom_R(\cdot, X)_{\mathcal{C}} \), it follows from the previous lemma that \( \text{rad} \hom_R(\cdot, X)_{\mathcal{C}} \) is finitely presented, hence there is an \( L \in \mathcal{C} \) and a morphism \( p : L \to X \) such that \( p_* : \text{Hom}_R(\cdot, L)_{\mathcal{C}} \to \text{rad} \hom_R(\cdot, X)_{\mathcal{C}} \) is an epimorphism. Adding a projective summand to \( L \) if necessary, we may assume that the \( R \)-module homomorphism \( p : L \to X \) is surjective. Setting \( \tau X = \ker(p) \), we see that \( \tau X \in \mathcal{C} \), since \( \mathcal{C} \) is closed under kernels of epimorphisms. And it is clear that the obtained sequence \( 0 \to \tau X \to L \to X \to 0 \) satisfies the required condition to be an AR-sequence. \( \square \)


**Lemma 5.8** Let \( \mathcal{C} \) be a resolving subcategory of \( \text{mod}R \). Suppose that \( \mathcal{C} \) is of finite type. Then for any \( X, Y \in \mathcal{C} \), the \( R \)-module \( \text{Ext}_R^1(X, Y) \) is of finite length.

**Proof.** It is sufficient to prove the lemma in the case that \( X \) and \( Y \) are indecomposable. For any \( x \in \mathfrak{m} \) and for any \( \sigma \in \text{Ext}_R^1(X, Y) \), it is enough to show that \( x^n\sigma = 0 \) for a large integer \( n \).

Now suppose that \( x^n\sigma \neq 0 \) for any integer \( n \), and we shall show a contradiction. Let us take an AR-sequence ending in \( X \) as in the previous lemma

\[
\alpha : 0 \longrightarrow \tau X \longrightarrow L \xrightarrow{p} X \longrightarrow 0,
\]

and a short exact sequence that corresponds to each \( x^n\sigma \in \text{Ext}_R^1(X, Y) \)

\[
x^n\sigma : 0 \longrightarrow Y \longrightarrow L_n \xrightarrow{p_n} X \longrightarrow 0.
\]

Since \( p_n \) is not a split epimorphism, the following commutative diagram is induced:

\[
\begin{array}{ccc}
0 & \longrightarrow & Y \\
\downarrow h_n & & \downarrow \quad \|
0 & \longrightarrow & \tau X \\
& & \quad \|
& & \downarrow \\
0 & \longrightarrow & 0 \\
& & \quad \|
\end{array}
\]

The morphism \( h_n \) induces an \( R \)-module map

\[
(h_n)_* : \text{Ext}_R^1(X, Y) \longrightarrow \text{Ext}_R^1(X, \tau X)
\]

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which sends $x^n\sigma$ to the AR-sequence $\alpha$. Since $(h_n)_* \in R$-linear, we have $\alpha = x^n(h_n)_*(\sigma) \in x^n\text{Ext}_R^1(X, \tau X)$. Note that this is true for any integer $n$ and that $\bigcap_{i=1}^{\infty} x^n\text{Ext}_R^1(X, \tau X) = (0)$. Therefore we must have $\alpha = 0$. This contradicts to that $\alpha$ is a nonsplit exact sequence. □

**Remark 5.9** Compare the proof of the above lemma with that in [11, Theorem (3.4)]. Also remark that the same lemma is proved by Huneke and Leuschke [8] in the case that $\mathcal{C}$ is the subcategory of maximal Cohen-Macaulay modules.

Now we proceed to the proof of Theorem 5.5. For this, let $\mathcal{C}$ be a subcategory of $\text{mod}R$ that satisfies three conditions as in the theorem. The proof will be done step by step.

For the first step we show that

(Step 1) the category $\text{mod}\mathcal{C}$ is a quasi-Frobenius category.

This has been proved in Theorem 3.5, since $\mathcal{C}$ is a resolving subcategory of $\mathcal{H}$. □

Now we prove the following.

(Step 2) Any $\mathcal{C}$-module $F \in \text{mod}\mathcal{C}$ can be embedded in an injective $\mathcal{C}$-module of the form $\text{Ext}_R^1(\cdot, X)|_{\mathcal{C}}$ for some $X \in \mathcal{C}$. In particular, $\text{mod}\mathcal{C}$ has enough injectives.

**Proof.** As we have shown in Lemma 3.3 that for a given $F \in \text{mod}\mathcal{C}$, there is a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

such that a projective resolution of $F$ in $\text{mod}\mathcal{C}$ is given as in Lemma 3.3(2). It is easy to see from the above exact sequence that there is an exact sequence

$$\cdots \longrightarrow \text{Hom}_R(\cdot, X_1)|_{\mathcal{C}} \longrightarrow \text{Hom}_R(\cdot, X_0)|_{\mathcal{C}} \longrightarrow \text{Ext}_R^1(\cdot, X_2)|_{\mathcal{C}} \longrightarrow \cdots$$

Hence there is a monomorphism $F \rightarrow \text{Ext}_R^1(\cdot, X_2)|_{\mathcal{C}}$. Note that $\text{Ext}_R^1(W, X_2)$ is a finitely generated $R$-module for each $W \in \text{Ind}(\mathcal{C})$. Hence it follows from Lemma 5.6 that $\text{Ext}_R^1(\cdot, X_2)|_{\mathcal{C}} \in \text{mod}\mathcal{C}$. On the other hand, since $\text{Ext}_R^1(\cdot, X_2)$ is a half-exact functor on $\mathcal{C}$, we see from Lemma 3.6 that $\text{Ext}_R^1(\cdot, X_2)|_{\mathcal{C}}$ is an injective object in $\text{mod}\mathcal{C}$. □

(Step 3) For each indecomposable module $X \in \mathcal{C}$, the $\mathcal{C}$-module $\text{Ext}_R^1(\cdot, X)|_{\mathcal{C}}$ is projective in $\text{mod}\mathcal{C}$. In particular, $\text{mod}\mathcal{C}$ is a Frobenius category.
Proof. For the proof, we denote the finite set \( \text{Ind}(\mathcal{C}) \) by \( \{W_1, \ldots, W_m\} \) where \( m = |\text{Ind}(\mathcal{C})| \). Setting \( E := \text{Ext}^1_R(\ , W_i) \) for any one of \( i \) \((1 \leq i \leq m)\), we want to prove that \( E \) is projective in \( \text{mod}\mathcal{C} \).

Firstly, we show that \( E \) is of finite length as an object in the abelian category \( \text{mod}\mathcal{C} \), that is, there is no infinite sequence of strict submodules

\[
E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots.
\]

To show this, set \( W = \oplus_{i=1}^m W_i \) and consider the sequence of \( R \)-submodules

\[
E(W) \supset E_1(W) \supset E_2(W) \supset \cdots \supset E_n(W) \supset \cdots
\]

Since we have shown in Lemma 5.8 that \( E(W) = \text{Ext}^1_R(W, X) \) is an \( R \)-module of finite length, this sequence will terminate, i.e. there is an integer \( n \) such that \( E_n = E_{n+1} = E_{n+2} = \cdots \) as functors on \( \mathcal{C} \). Therefore \( E \) is of finite length.

In particular, \( E \) contains a simple module in \( \text{mod}\mathcal{C} \) as a submodule.

Now note that there are only \( m \) nonisomorphic indecomposable projective modules in \( \text{mod}\mathcal{C} \); in fact they are \( \text{Hom}_R(\ , W_i)\mathcal{C} (i = 1, 2, \ldots, m) \). Corresponding to indecomposable projectives, there are only \( m \) nonisomorphic simple modules in \( \text{mod}\mathcal{C} \) which are

\[
S_i = \text{Hom}_R(\ , W_i)\mathcal{C} / \text{rad} \text{Hom}_R(\ , W_i)\mathcal{C} \quad (i = 1, 2, \ldots, m).
\]

Since we have shown in the steps 1 and 2 that \( \text{mod}\mathcal{C} \) is an abelian category with enough projectives and with enough injectives, each simple module \( S_i \) has the injective hull \( I(S_i) \) for \( i = 1, 2, \ldots, m \).

Since we have proved that \( E \) is an injective module of finite length, we see that \( E \) is a finite direct sum of \( I(S_i) \) \((i = 1, 2, \ldots, m)\). Since any module in \( \text{mod}\mathcal{C} \) can be embedded into a direct sum of injective modules of the form \( E = \text{Ext}^1_R(\ , W_i) \), we conclude that all nonisomorphic indecomposable injective modules in \( \text{mod}\mathcal{C} \) are \( I(S_i) \) \((i = 1, 2, \ldots, m)\).

Note that these are exactly \( m \) in number.

Since \( \text{mod}\mathcal{C} \) is a quasi-Frobenius category, any of indecomposable projective modules in \( \text{mod}\mathcal{C} \) are indecomposable injective. Hence the following two sets coincide:

\[
\{\text{Hom}_R(\ , W_i)\mathcal{C} \mid i = 1, 2, \ldots, m\} = \{I(S_i) \mid i = 1, 2, \ldots, m\}.
\]

As a result, every injective module is projective. And we have shown that \( \text{mod}\mathcal{C} \) is a Frobenius category.

Remark 5.10 We should remark that the proof of the step 3 is the same as the proof of Nakayama’s theorem that states the following:

Let \( A \) be a finite dimensional algebra over a field. Then \( A \) is left selfinjective if and only if \( A \) is right selfinjective. In particular, \( \text{mod}A \) is a quasi-Frobenius category if and only if so is \( \text{mod}A^{op} \). And in this case \( \text{mod}A \) is a Frobenius category.

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See [10] for example.

Now we proceed to the final step of the proof. If we prove the following, then the category $\mathcal{C}$ satisfies all the assumptions in Theorem 5.2, hence we obtain $\mathcal{C} \subseteq \mathcal{G}$. And this will complete the proof.

(Step 4) If $X \in \mathcal{C}$ such that $X \not\cong 0$ in $\mathcal{C}$, then we have $\text{Ext}^1_R(\cdot, X)|_{\mathcal{C}} \neq 0$.

**Proof.** As in the proof of the step 3 we set $\text{Ind}(\mathcal{C}) = \{W_i \mid i = 1, 2, \ldots, m\}$. It is enough to show that $\text{Ext}^1_R(\cdot, W_i)|_{\mathcal{C}} \neq 0$ for each $i$. Now assume that $\text{Ext}^1_R(\cdot, W_1)|_{\mathcal{C}} = 0$ and we shall show a contradiction. In this case, it follows from the step 2 that any module in $\text{mod}\mathcal{C}$ can be embedded into a direct sum of copies of $(m - 1)$ modules $\text{Ext}^1_R(\cdot, W_2)|_{\mathcal{C}}, \ldots, \text{Ext}^1_R(\cdot, W_m)|_{\mathcal{C}}$. In particular, any indecomposable injective modules appear in these $(m - 1)$ modules as direct summands. But we have shown in the proof of the step 3 that there are $m$ indecomposable injective modules $I(S_i)$ ($i = 1, 2, \ldots, m$). Hence at least one of $\text{Ext}^1_R(\cdot, W_2)|_{\mathcal{C}}, \ldots, \text{Ext}^1_R(\cdot, W_m)|_{\mathcal{C}}$ contains two different indecomposable injective modules as direct summands. Since $\text{mod}\mathcal{C}$ is a Frobenius category, we see in particular that it is decomposed nontrivially into a direct sum of projective modules in $\text{mod}\mathcal{C}$. We may assume that $\text{Ext}^1_R(\cdot, W_2)|_{\mathcal{C}}$ is decomposed as

$$\text{Ext}^1_R(\cdot, W_2)|_{\mathcal{C}} \cong \text{Hom}_R(\cdot, Z_1)|_{\mathcal{C}} \oplus \text{Hom}_R(\cdot, Z_2)|_{\mathcal{C}},$$

where $Z_1, Z_2(\not\cong 0) \in \mathcal{C}$. Then it follows from Lemma 5.1 that $W_2$ is isomorphic to a direct summand of $\Omega Z_1 \oplus \Omega Z_2$. But since $W_2$ is indecomposable, we may assume that $W_2$ is isomorphic to a direct summand of $\Omega Z_1$. Then $\text{Ext}^1_R(\cdot, W_2)|_{\mathcal{C}}$ is a direct summand of $\text{Ext}^1_R(\cdot, \Omega Z_1)|_{\mathcal{C}} \cong \text{Hom}_R(\cdot, Z_1)|_{\mathcal{C}}$. This is a contradiction, because $\text{mod}\mathcal{C}$ is a Krull-Schmidt category by Lemma 4.1. $\square$

**References**


