MIXED EXPANSION FORMULA FOR THE RECTANGULAR SCHUR FUNCTIONS AND THE AFFINE LIE ALGEBRA $A_1^{(1)}$

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Abstract. Formulas are obtained that express the Schur S-functions indexed by Young diagrams of rectangular shape as linear combinations of "mixed" products of Schur's S- and Q-functions. The proof is achieved by using representations of the affine Lie algebra of type $A_1^{(1)}$. A realization of the basic representation that is of "$D_2^{(2)}$"-type plays the central role.

1. Introduction

We derive formulas of combinatorial nature that express the Schur S-functions indexed by Young diagrams of rectangular shape, the rectangular S-functions for short, as linear combinations of "mixed" products of S- and Q-functions.

The rectangular S-functions are studied in [4, 7] from a viewpoint of representations of the affine Lie algebra of type $A_1^{(1)}$ and $A_2^{(2)}$. In the work, these functions appear as certain distinguished weight vectors in the so called *homogeneous* realization of the basic representation $L(\Lambda_0)$ of $A_1^{(1)}$ (see [5]). On the other hand, the Schur Q-functions arise naturally in the representation of $D_{2l}^{(2)}$-type Lie algebras ([9]). In the subsequent pursuit of various realizations of $L(\Lambda_0)$, our formula has come out as an application of the isomorphism $D_2^{(2)} \cong A_1^{(1)}$. Roughly speaking, we can realize the space $L(\Lambda_0)$ as a tensor product of the spaces of the Schur S- and Q-functions. We say such a "mixed" realization as homogeneous of type $D_2^{(2)}$.

Let us describe our main result in more detail. Let μ be a partition and $S_\mu(t)$ be the corresponding Schur S-function, where $t = (t_1, t_2, t_3, \ldots)$, each t_j $(j = 1, 2, \ldots)$ is the j-th *power sum* p_j divided by j. Let $Q_\lambda(t)$ denote the Schur Q-function indexed by a *strict* partition λ, where $t = (t_1, t_3, t_5, \ldots)$. Let $\square(m, n)$ denote the Young diagram of the rectangular shape (n^m). Set also $S_\mu(t') = S_\mu(t_2, t_4, t_6, \ldots)$. Note that the set $\{Q_\lambda(t)S_\mu(t'); \lambda, \mu\}$ forms a basis of the space of the symmetric functions. For each strict partition μ, we associate a strict partition $\mu^{b[0]}$ and a partition $\mu^{b[1]}$ in a combinatorial manner (see Section 4).

Let m, n be non-negative integers. Our formula (Theorem 5.3), called "mixed expansion formula", reads:

$$\sum_{\mu} \delta(\mu) Q_{\mu^{b[0]}}(t) S_{\mu^{b[1]}}(t') = S_{\square(m,n)}(t), \tag{1}$$

where the summation runs over a certain finite set of strict partitions determined by m and n. Here $\delta(\mu) = \pm 1$ is a sign given in a combinatorial way. We prove the formula (1) by comparing two realizations of $L(\Lambda_0)$ mentioned above. The left hand side stems from combinatorial descriptions of actions of Chevalley generators in the homogeneous realization of type $D_2^{(2)}$, whereas the right hand side is obtained via "vertex operator calculus" (as employed in [4]) in the homogeneous realization of type $A_1^{(1)}$.

The paper is organized as follows. In Section 2 we recall the spin representation of $A_1^{(1)}$. In Section 3 we describe the action of $A_1^{(1)}$ in terms of Young diagrams. In Section 4 we recall some combinatorics of the strict partitions. In Section 5 we state our main theorem on rectangular Schur functions. In Section 6 we recall the boson-fermion correspondence and obtain weight vectors as a sum of combination of S- and Q- functions. As the conclusion of this section, we obtain LHS of our formula (1). In Section 7 we consider f_i-action ($i = 0, 1$) and obtain rectangular Schur functions appearing in RHS of our formula through the vertex operator calculation. Section 8 is devoted to the proof of the main theorem.

2. The spin representation of $A_1^{(1)}$

We consider the associative $\mathbb{C}$-algebra $\mathbb{B}$ defined by the generators β_n ($n \in \mathbb{Z}$) and the anti-commutation relations

$$[\beta_m, \beta_n]_+ = (-1)^m \delta_{m+n,0}.$$

Consider a left $\mathbb{B}$-module $\mathcal{F} = \mathbb{B}/\sum_{n<0} \mathbb{B}\beta_n$ called *the Fock module*. It is generated by a nonzero vector $|\mathrm{vac}\rangle \in \mathcal{F}$ called *the vacuum* satisfying

$$\beta_n |\mathrm{vac}\rangle = 0 \quad (n < 0).$$

A *partition* is any non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ containing only finitely many non-zero terms. We regard two partitions as the same that differ only by a string of zeros at the end. The non-zero λ_i are called the *parts* of λ. The number of parts is the *length* of λ, denoted by $\ell(\lambda)$. A partition is *strict* if all parts are distinct. Denote by $\mathcal{P}$ (resp. $\mathcal{SP}$) the set of all partitions (resp. strict partitions).

Definition 2.1. *Let λ be a strict partition, which we may write in the form $\lambda = (\lambda_1, \dots, \lambda_{2k})$ where $\lambda_1 > \cdots > \lambda_{2k} \geq 0$. Let $|\lambda\rangle$ denote the vector*

$$|\lambda\rangle = \beta_{\lambda_1} \cdots \beta_{\lambda_{2k}} |\text{vac}\rangle \in \mathcal{F}.$$

Set $f_i^\infty = (-1)^i \beta_{i+1}\beta_{-i}$ $(i \geq 0)$. They have the following nice combinatorial property:

Proposition 2.2. *Let λ be a strict partition. For $i > 0$, if i is a part of λ and $i+1$ is not a part of λ, then we have*

$$f_i^\infty |\lambda\rangle = |\mu\rangle$$

where μ is obtained from λ by replacing its part i by $i+1$, else we have $f_i^\infty|\lambda\rangle = 0$. If 1 is not a part of λ, then we have

$$f_0^\infty |\lambda\rangle = \begin{cases} 2^{-1}|\mu\rangle & (\ell(\lambda)\text{: odd }) \\ |\mu\rangle & (\ell(\lambda)\text{: even }) \end{cases}$$

where $\ell(\lambda)$ is the length of λ, and μ is obtained from λ by adding a part 1, else we have $f_0^\infty|\lambda\rangle = 0$.

We shall use standard notation on the affine Lie algebra $A_1^{(1)}$. Namely $e_i, f_i, h_i (i = 0, 1)$ are the Chevalley generators, α_0, α_1 are the simple roots, $\Lambda_i (i = 0, 1)$ are the fundamental weights. We refer [5]. The affine Lie algebra $A_1^{(1)}$ acts on $\mathcal{F}$ by

$$f_0 = \sqrt{2} \sum_{n \in \mathbb{Z}} \beta_{-4n+1}\beta_{4n}, \quad f_1 = \sqrt{2} \sum_{n \in \mathbb{Z}} \beta_{-4n-1}\beta_{4n+2},$$

$$e_0 = \sqrt{2} \sum_{n \in \mathbb{Z}} \beta_{4n}\beta_{-4n-1}, \quad e_1 = \sqrt{2} \sum_{n \in \mathbb{Z}} \beta_{4n-2}\beta_{-4n+1},$$

$$h_1 = -h_0 + 1 = 2 \sum_{n \in \mathbb{Z}} : \beta_{4n-1}\beta_{-4n+1} : .$$

Let $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) be a submodule of $\mathcal{F}$ generated by $|0\rangle$ (resp. $\beta_0|0\rangle$). Let $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_1$ and $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. Each summand $\mathcal{F}_i$ is isomorphic to the irreducible highest weight module $L(\Lambda_0)$. Note that the following expressions:

$$f_0 = \sqrt{2} \sum_{j \geq 0} f_{4j}^\infty + \sqrt{2} \sum_{j \geq 0} f_{4j+3}^\infty, \quad f_1 = \sqrt{2} \sum_{j \geq 0} f_{4j+1}^\infty + \sqrt{2} \sum_{j \geq 0} f_{4j+2}^\infty.$$

3. Combinatorial description of the action

Let $\lambda \in \mathcal{SP}$. Any box $x \in \lambda$ in the j-th row, we endow a color $c(x)$ with it

$$c(x) = \begin{cases} 0 & (j \equiv 0, 1 \mod 4) \\ 1 & (j \equiv 2, 3 \mod 4) \end{cases} .$$

For example, $\lambda = (5, 4, 2, 1)$ is colored as

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \\ 0 & 1 & & & \\ 0 & & & & \end{matrix} .$$

We say that a node x is i-good to λ, if $\lambda \cup \{x\}$ is a strict partition and $c(x) = i$. The following nodes indicated by dots are the 1-good nodes:

$$\begin{matrix} 0 & 1 & 1 & 0 & 0 & \bullet \\ 0 & 1 & 1 & 0 & & \\ 0 & 1 & \bullet & & & \\ 0 & & & & & \end{matrix} .$$

If we get a strict partition μ by adding a node to λ, we denote this node by μ/λ. Such a node is called an good node. If an good node of λ has 4-bar content i, we call it an i-good node of λ.

Set

$$I_i^\ell(\lambda) = \{\mu \in \mathcal{SP} \mid \mu \supset \lambda,\ |\mu| = |\lambda| + \ell,\ \forall x \in \mu - \lambda,\ c(x) = i\}.$$

It is the set of strict partitions obtained from λ by adding i-nodes ℓ times in succession.

Lemma 3.1. *A weight vector of the weight $\Lambda_0 - m^2\delta + m\alpha_1$ is given by $|c_m\rangle$, where for $m > 0$ we set $c_m = (4m-3, \ldots, 5, 1)$, and for $m < 0$, $c_m = (-4m-1, \ldots, 7, 3)$.*

The strict partitions c_m $(m \in \mathbb{Z})$ appeared in Lemma 3.1 form the set of "4-bar cores", introduced in [9]. Set $I_i^\ell(m) = I_i^\ell(c_m)$.

Example 3.2. *For $m = -2$ and $i = 0$ we have*

$$I_0^1(-2) = \{(8,3), (7,4), (7,3,1,0)\},$$

$$I_0^3(-2) = \{(9,4), (8,5), (9,3,1,0), (8,4,1,0), (7,5,1,0)\}$$

and

$$I_0^2(-1) = \{(5,0), (4,1)\}.$$

We need the following combinatorial lemma:

Lemma 3.3.
$$\frac{f_i^\ell}{\ell!}|c_m\rangle = \sqrt{2}^{-\varepsilon_m} \sum_{\lambda \in I_i^\ell(m)} \sqrt{2}^{a(\lambda)} |\lambda\rangle.$$
where $a(\lambda) = \sharp\{j \mid \lambda_j \equiv 0 \mod 2\}$, *and* $\varepsilon_m = 1$ *if* m *is odd, and* $\varepsilon_m = 0$ *if* m *is even.*

Proof. Firstly we consider the case of $i = 1$. For $\lambda \in I_1^\ell(m)$, put
$$\lambda - c_m = (r_1, r_2, \cdots, r_m).$$
We compute
$$\begin{aligned} r_1! r_2! \cdots r_m! &= 2^{\sharp\{j; r_j = 2\}} \\ &= 2^{(\ell - \sharp\{j; r_j = 1\})/2} \\ &= 2^{(\ell - a(\lambda) + \varepsilon_m)/2}, \end{aligned}$$
where we note that $a(\lambda)$ counts a 0 if $m \equiv 1 \pmod 2$. Then the coefficient of $|\lambda\rangle$ is
$$\frac{\sqrt{2}^\ell}{\ell!} \frac{\ell!}{r_1! r_2! \cdots r_m!} = \sqrt{2}^{a(\lambda) - \varepsilon_m}.$$
Secondary we consider the case of $i = 0$. In this case we have to remark that the action of $\beta_1\beta_0$-part. Let
$$\lambda - c_{-m} = (r_1, r_2, \cdots, r_m, r_{m+1})$$
for $\lambda \in I_0^\ell(m)$. Then the coefficient of $|\lambda\rangle$ is
$$\frac{1}{2^{r_{m+1}\varepsilon_m}} \frac{\sqrt{2}^\ell}{\ell!} \frac{\ell!}{r_1! r_2! \cdots r_m!}. \tag{2}$$
The similar computation above gives
$$r_1! r_2! \cdots r_m! = 2^{\sharp\{j; r_j = 2\}} = 2^{(\ell - \sharp\{j; r_j = 1, j \leq m\} - r_{m+1})/2}.$$
Here we divide our argument into two cases. First we assume that m is even. We have
$$\sharp\{j; r_j = 1, j \leq m\} = \begin{cases} a(\lambda) & (\lambda_{m+1} = 0) \\ a(\lambda) - 1 & (\lambda_{m+1} = 1) \end{cases}$$

Second we assume that m is odd. We have
$$\sharp\{j; r_j = 1, j \leq m\} = \begin{cases} a(\lambda) - 1 & (\lambda_{m+1} = 0) \\ a(\lambda) & (\lambda_{m+1} = 1) \end{cases}$$

By substituting these four results into (2) we obtain

$$\frac{\sqrt{2}^{\ell}}{\ell!}\frac{\ell!}{r_1!r_2!\cdots r_m!} = \sqrt{2}^{a(\lambda)-\varepsilon_m}.$$

□

Example 3.4. $f_0|c_{-2}\rangle = \sqrt{2}\,\beta_8\beta_3|\mathrm{vac}\rangle + \sqrt{2}\,\beta_7\beta_4|\mathrm{vac}\rangle + \sqrt{2}\,\beta_7\beta_3\beta_1\beta_0|\mathrm{vac}\rangle$.

4. 4-bar quotient

Let us introduce the notion of 4-bar quotient. We shall give a bijection

$$\mathcal{SP} \to \mathbb{Z} \times \mathcal{SP} \times \mathcal{P}, \quad \lambda \mapsto (m, \lambda[0], \lambda[1]).$$

For $\lambda \in \mathcal{SP}$, the pair $(\lambda[0], \lambda[1])$ is called the 4-bar quotient of λ. Moreover, the corresponding $c_m \in \mathcal{SP}$ is called the 4-bar core of λ.

Let us identify the strict partition λ with the subset $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_s\}$ of $\mathbb{N}$. For $a = 0, 1, 2, 3$, we set $\boldsymbol{\lambda}^{(a)} = \{\lambda_j \in \boldsymbol{\lambda} \mid \lambda_j \equiv a \mod 4\}$. Namely

$$\boldsymbol{\lambda}^{(a)} = \boldsymbol{\lambda} \cap (4\mathbb{N} + a) \quad (a = 0, 1, 2, 3)$$

and we have $\boldsymbol{\lambda} = \sqcup_{a=0}^{3} \boldsymbol{\lambda}^{(a)}$. The even part $\boldsymbol{\lambda}^{(0)} \cup \boldsymbol{\lambda}^{(2)} \subset 2\mathbb{N}$ of $\boldsymbol{\lambda}$ gives a strict partition $\lambda[0]$ via the inclusion

$$\boldsymbol{\lambda}^{(0)} \cup \boldsymbol{\lambda}^{(2)} \subset 2\mathbb{N} \longrightarrow \mathbb{N}, \quad 2k \mapsto k.$$

By the odd parts $\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(3)}$, we define a partition $\lambda[1]$ in the following way: First consider two bijections

$$\iota : 4\mathbb{N} + 1 \longrightarrow \mathbb{Z}_{\geq 0} \quad (4k + 1 \mapsto k), \quad \iota^* : 4\mathbb{N} + 3 \longrightarrow \mathbb{Z}_{<0} \quad (4k + 3 \mapsto -k - 1).$$

Then define a subset

$$\mathcal{M}(\lambda) = \iota(\boldsymbol{\lambda}^{(1)}) \cup (\mathbb{Z}_{<0} - \iota^*(\boldsymbol{\lambda}^{(3)}))$$

of $\mathbb{Z}$. This is a "Maya diagram" in the sense that, if we express $\mathcal{M}(\lambda)$ as an descending sequence $i_1 > i_2 > i_3 > \cdots$, there exists a unique integer m such that $i_k = -k + m$ for $k \ll 0$. Then we can define

$$\lambda[1] = (i_1 + 1 - m, i_2 + 2 - m, i_3 + 3 - m, \ldots) \in \mathcal{P}.$$

The integer m is called the charge of $\mathcal{M}(\lambda)$.

Lemma 4.1. *(cf. [8]) The map*

$$\mathcal{SP} \to \mathbb{Z} \times \mathcal{SP} \times \mathcal{P}, \quad \lambda \mapsto (m, \lambda[0], \lambda[1])$$

is a bijection.

We can illustrate the above construction. Let us look at a particular example, $\lambda = (11, 9, 6, 2, 1)$. We draw a "4-bar abacus":

$$
\begin{array}{ccc}
0 & \textcircled{1} & 3 \\
\textcircled{2} & & \\
4 & 5 & 7 \\
\textcircled{6} & & \\
8 & \textcircled{9} & \textcircled{11} \\
10 & & \\
12 & 13 & 15
\end{array}
$$

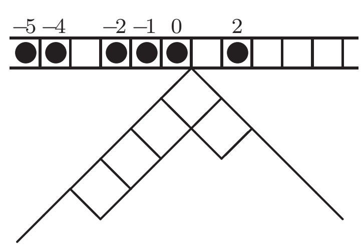
We can read $\boldsymbol{\lambda}^{(0)} \bigcup \boldsymbol{\lambda}^{(1)}$ from first column. Then we have $\lambda[0] = (3, 1)$. From second and third columns, we can read

$$\iota(\boldsymbol{\lambda}^{(1)}) = (2, 0)$$
$$\iota^*(\boldsymbol{\lambda}^{(3)}) = (-3).$$

Then we obtain

$$M = (2, 0, -1, -2, -4, -5, \cdots)$$

and draw a Maya diagram;



.

Finally we have $\lambda[1] = (2, 1, 1, 1)$ and $m = 1$.

5. Main Result

Define $h_n(t)$ by $\exp\left(\sum_{n=1}^{\infty} t_n z^n\right) = \sum_{n=0}^{\infty} h_n(t) z^n$. Let λ be a partition. The Schur S-function with shape λ is defined as

$$S_\lambda(t) = \det(h_{\lambda_i + j - i}(t)).$$

Define $q_n(t)$ by $\exp\left(\sum_{n=1}^{\infty} t_{2n-1} z^{2n-1}\right) = \sum_{n=0}^{\infty} q_n(t) z^n$. For $m > n \geq 0$, we put

$$Q_{m,n}(t) = q_m(t) q_n(t) + 2 \sum_{i=1}^{n} (-1)^i q_{m+i}(t) q_{n-i}(t).$$

If $m \leq n$ we define $Q_{m,n}(t) = -Q_{n,m}(t)$. Let $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ be a strict partition, where $\lambda_1 > \cdots > \lambda_{2n} \geq 0$. Then the $2n \times 2n$ matrix $M_\lambda = (Q_{\lambda_i,\lambda_j})$ is skew-symmetric. The Q-function Q_λ is defined as

$$Q_\lambda(t) = \mathrm{Pf}(M_\lambda).$$

Each strict partition μ in $I_1^\ell(c_m)$ or $I_0^\ell(c_m)$ has its own sign determined by beads configuration.

Definition 5.1. *Draw the 4-bar abacus of $\lambda \in I_i^\ell(c_m)$ $(i = 1, 2)$ as follows.*

(1) *If $m > 0$, then we do not put a bead on 0.*
(2) *If $m < 0$, we have to put a bead on 0 unless $\lambda_{m+1} = 1$.*

Let $g(\lambda)$ be the number of beads on the central runner at the positions bigger than that of each bead on the leftmost runner. For a strict partition $\lambda \in I_i^\ell(c_m)$, we define the sign by

$$\delta(\lambda) = (-1)^{g(\lambda)}.$$

Example 5.2. *We consider the case of $i = 1, m = 3, \ell = 3$ and $\lambda = (11, 5, 2, 0)$.*

0	1	3
②		
4	⑤	7
6		
8	9	⑪

We have $\delta(\lambda) = (-1)^1 = -1$. In the case of $i = 0, m = -4, \ell = 5$ and $\lambda = (15, 13, 8, 5)$, we have to put a bead on 0.

⓪	1	3
2		
4	⑤	7
6		
⑧	9	11
10		
12	⑬	⑮

We have $\delta(\lambda) = (-1)^{2+1} = -1$.

We can now state our formula.

Theorem 5.3. *(Mixed expansion formula) For non-negative integers m and n, we have*

$$\sum_{\mu\in I_1^n(c_m)} \delta(\mu)Q_{\mu[0]}(t)S_{\mu[1]}(t') = S_{\square(2m-n,n)}(t),$$

$$\sum_{\mu\in I_0^n(c_{-m})} \delta(\mu)Q_{\mu[0]}(t)S_{\mu[1]}(t') = S_{\square(n,2m+1-n)}(t),$$

where $t = (t_1, t_2, t_3, \cdots)$ *and* $S_\nu(t') = S_\nu(u)|_{u_j\mapsto t_{2j}}$.

6. Bosonization

We introduce the operators ϕ_n, ψ_n, ψ_n^* $(n \in \mathbb{Z})$ by

$$\beta_{4n} = \phi_{2n}, \quad \beta_{4n+1} = \sqrt{-1}\,\psi_n, \quad \beta_{4n+2} = \sqrt{-1}\,\phi_{2n+1}, \quad \beta_{4n+3} = \sqrt{-1}\,\psi^*_{-n-1},$$

satisfying the anti-commutation relations

$$[\psi_m, \psi_n^*]_+ = \delta_{m,n}, \quad [\psi_m^*, \psi_n^*]_+ = [\psi_m, \psi_n]_+ = 0, \tag{3}$$

$$[\phi_m, \phi_n]_+ = (-1)^m\delta_{m+n,0}, \tag{4}$$

$$[\psi_m^*, \phi_n]_+ = [\psi_m, \phi_n]_+ = 0. \tag{5}$$

Let us introduce the bosonic current operators

$$H_{2m} = \sum_{k\in\mathbb{Z}} :\psi_k\psi^*_{k+m}: , \quad H_{2m+1} = \frac{1}{2}\sum_{k\in\mathbb{Z}} (-1)^{k+1}\phi_k\phi_{-k-(2m+1)}.$$

One has

$$[H_m, H_n] = \frac{m}{2}\,\delta_{m+n,0}\,c.$$

Thus the Lie algebra $\mathfrak{H} = \oplus_{n\neq 0}\mathbb{C}H_n \oplus \mathbb{C}c$ is an infinite-dimensional Heisenberg algebra.

We have a canonical $\mathfrak{H}$-module $S[\mathfrak{H}_-]$ where $\mathfrak{H}_- = \oplus_{n<0}\mathbb{C}H_n$, and S stands for the symmetric algebra. Let $t_n = \frac{2}{n}H_{-n}$ $(n > 0)$. Then we can identify $S[\mathfrak{H}_-]$ with the ring $\mathbb{C}[\,t\,] = \mathbb{C}[\,t_1, t_2, t_3, \dots]$ of polynomials in infinitely many variables t_n. The representation of $\mathfrak{H}$ on $\mathbb{C}[\,t\,]$ is described as follows:

$$H_nP(t) = \frac{\partial}{\partial t_n}P(t), \quad H_{-n}P(t) = \frac{n}{2}\,t_nP(t) \quad (n > 0,\ P(t) \in \mathbb{C}[\,t\,]),$$

so that c acts as an identity.

If we introduce the space of highest weight vectors with respect to $\mathfrak{H}$ by

$$\Omega = \{|v\rangle \in \mathcal{F} \mid H_m|v\rangle = 0\ (\forall m > 0)\}.$$

Ω has a basis $\{|\sigma, m\rangle\ (m \in \mathbb{Z},\ \sigma = 0, 1)\}$, where

$$|0,m\rangle = \begin{cases} \psi_{m-1}\cdots\psi_0|\mathrm{vac}\rangle & (m \geq 0) \\ \psi^*_{-m}\cdots\psi^*_{-1}|\mathrm{vac}\rangle & (m < 0) \end{cases}, \quad |1,m\rangle = \begin{cases} \sqrt{2}\phi_0\psi_{m-1}\cdots\psi_0|\mathrm{vac}\rangle & (m \geq 0) \\ \sqrt{2}\phi_0\psi^*_{-m}\cdots\psi^*_{-1}|\mathrm{vac}\rangle & (m < 0) \end{cases}.$$

Note that

$$\phi_n|\sigma,m\rangle = 0 \quad (n<0), \quad \psi_n|\sigma,m\rangle = 0 \quad (n<m), \quad \psi_n^*|\sigma,m\rangle = 0 \quad (n\geq m).$$

Let the operator H_0 also act on $\mathbb{C}[\,t\,]\otimes\mathbf{\Omega}$ via

$$H_0\left(P(t)\,\theta^\sigma e^{m\alpha}\right) = m\cdot P(t)\,\theta^\sigma e^{m\alpha} \quad (P(t)\in\mathbb{C}[t]).$$

Lemma 6.1.

$$|c_m\rangle = (\sqrt{-1})^{-|m|}\,\sqrt{2}^{-\varepsilon_m}|\,\varepsilon_m, m\rangle.$$

Proof. We can easily obtain the equation by direct calculation. □

We introduce a formal symbol $\theta^\sigma e^{m\alpha}$ corresponding to $|\sigma,m\rangle\in\Omega$, and define

$$\mathbf{\Omega} = \bigoplus_{m\in\mathbb{Z},\,\sigma=0,1}\mathbb{C}\,\theta^\sigma e^{m\alpha}.$$

Proposition 6.2. [1, 2] *There exists a canonical isomorphism of $\mathfrak{H}$-modules*

$$\Phi : \mathcal{F}\longrightarrow \mathbb{C}[\,t\,]\otimes_{\mathbb{C}}\mathbf{\Omega}$$

such that $\Phi(|\sigma,m\rangle) = \theta^\sigma e^{m\alpha}(m\in\mathbb{Z},\sigma=0,1)$.

When we write $\Phi(|v\rangle) = \sum_{m,\sigma}P_{m,\sigma}(t)\,\theta^\sigma e^{m\alpha}$ for $|v\rangle\in\mathcal{F}$, the coefficient $P_{m,\sigma}(t)\in\mathbb{C}[t]$ can be expressed in terms of the vacuum expectation value on $\mathbb{B}$ as follows:

$$P_{m,\sigma}(t) = \langle m,\sigma|e^{H(t)}|v\rangle, \quad H(t)=\sum_{n=1}^{\infty}t_nH_n.$$

Consider the right $\mathbb{B}$-module $\mathcal{F}^\dagger = \mathbb{B}/\sum_{n>0}\beta_n\mathbb{B} = \langle\mathrm{vac}|\mathbb{B}$. We have a bilinear pairing

$$\mathcal{F}^\dagger\otimes_{\mathbb{B}}\mathcal{F}\to\mathbb{C}, \quad \langle u|\otimes_{\mathbb{B}}|v\rangle\mapsto\langle u|v\rangle$$

normalized $\langle\mathrm{vac}|\mathrm{vac}\rangle = 1$. Introduce the vectors $\langle m,\sigma|\in\mathcal{F}^\dagger$ $(m\in\mathbb{Z},\sigma=0,1)$ which are characterized by $\langle m,\sigma|\sigma',n\rangle = \delta_{m,n}\delta_{\sigma,\sigma'}$ $(m,n\in\mathbb{Z},\sigma,\sigma'=0,1)$ and

$$\langle m,\sigma|\phi_n = 0 \quad (n>0), \quad \langle m,\sigma|\psi_n = 0 \quad (n\geq m), \quad \langle m,\sigma|\psi_n^* = 0 \quad (n<m).$$

We denote by $\mathbb{W}_\phi$ the linear subspace of $\mathbb{B}$ spanned by ϕ_j $(j\in\mathbb{Z})$.

Lemma 6.3. *If $\langle u|\in\mathcal{F}^\dagger, |v\rangle\in\mathcal{F}$ be such that $\langle u|\phi_j = 0$ $(j>0)$, $\phi_j|v\rangle = 0$ $(j<0)$, then for $w_i\in\mathbb{W}_\phi$ $(i=1,\ldots,2k)$ we have*

$$\langle u|w_1\cdots w_{2k}|v\rangle = \langle u|v\rangle\,\mathrm{Pf}(\langle w_iw_j\rangle)$$

Proof. A bilinear form on $\mathbb{B}_\phi$ is defined by $(a,b)\mapsto\langle u|ab|v\rangle$ which has all the properties of vacuum expectation value on $\mathbb{B}_\phi$ except for the normalization condition. Obviously, the normalization factor is given by $\langle u|v\rangle$. Hence the lemma follows. □

Lemma 6.4. [1, 2, 10] *We have*

$$\Phi(\psi_{i_1}\cdots\psi_{i_s}|0,m\rangle) = S_{(i_1-m,i_2-m,\ldots,i_s-m)-\delta_s}(t_{even})\, e^{(m+s)\alpha} \quad (i_1>\cdots>i_s>m),$$

$$\Phi(\phi_{j_1}\cdots\phi_{j_a}|\mathrm{vac}\rangle) = \sqrt{2}^{-a} Q_{j_1,\ldots,j_a}(t_{odd})\,\theta^a \quad (j_1>\cdots>j_a\geq 0),$$

where $\delta_s=(s-1,s-2,\ldots,1,0)$ *and* $t_{even}=(t_2,t_4,\ldots)$, $t_{odd}=(t_1,t_3,t_5,\ldots)$.

Lemma 6.3 and 6.4 give us

Lemma 6.5. *Let* $j_1>\cdots>j_a\geq 0$, $i_1>\cdots>i_s>m$. *We have*

$$\Phi(\phi_{j_1}\cdots\phi_{j_a}\psi_{i_1}\cdots\psi_{i_s}|0,m\rangle) = \sqrt{2}^{-a}\, Q_{j_1,\ldots,j_a}(t_{odd})\, S_{(i_1-m,\ldots,i_s-m)-\delta_s}(t_{even})\theta^a e^{(m+s)\alpha}.$$

Consequently we obtain the following theorem.

Theorem 6.6. *Let* $\lambda\in F_i^\ell(m)$. *There exists a 4-th root of unity* $\zeta_{m,\ell,i}$ *such that*

$$\Phi(\sqrt{2}^{a(\lambda)}|\lambda\rangle) = \zeta_{m,\ell,i}\,\delta(\lambda)\,Q_{\lambda[0]}(t_{odd})S_{\lambda[1]}(t_{even})\,\theta^{m+\ell}\,e^{(m+(-1)^i\ell)\alpha}.$$

7. Vertex operators

In this section we realize f_i on $\mathcal{B}$ in terms of vertex operators. We introduce the formal generating functions

$$\phi(z)=\sum_{n\in\mathbb{Z}}\phi_n z^n, \quad \psi(z)=\sum_{n\in\mathbb{Z}}\psi_n z^{2n}, \quad \psi^*(z)=\sum_{n\in\mathbb{Z}}\psi^*_{-n}z^{2n-2}.$$

For $x=(x_1,x_2,x_3,\ldots)$, set

$$\xi(x,z)=\sum_{n=1}^{\infty}x_n z^n, \quad \xi_0(x,z)=\sum_{n=1}^{\infty}x_{2n}z^{2n}, \quad \xi_1(x,z)=\sum_{n=1}^{\infty}x_{2n-1}z^{2n-1}.$$

On the space $\mathbf{\Omega}$, we define the operators $\theta, e^{\pm\alpha}$ by

$$\theta.(\theta\, e^{m\alpha})=e^{m\alpha}, \quad \theta.\, e^{m\alpha}=\theta\, e^{m\alpha}, \quad e^{\pm\alpha}.(\theta\, e^{m\alpha})=-\theta\, e^{(m\pm1)\alpha}, \quad e^{\pm\alpha}.e^{m\alpha}=e^{(m\pm1)\alpha}.$$

Note that $e^{\pm\alpha}\theta=-\theta\, e^{\pm\alpha}$.

Proposition 7.1. [1, 2] *One has*

$$\Phi\ \phi(z)\Phi^{-1}=\sqrt{2}^{-1}e^{\xi_1(t,z)}e^{-2\xi_1(\tilde{\partial}_t,z^{-1})}\theta, \tag{6}$$

$$\Phi\ \psi(z)\Phi^{-1}=e^{\xi_0(t,z)}e^{-2\xi_0(\tilde{\partial}_t,z^{-1})}e^{\alpha}z^{2H_0}, \tag{7}$$

$$\Phi\ \psi^*(z)\Phi^{-1}=e^{-\xi_0(t,z)}e^{2\xi_0(\tilde{\partial}_t,z^{-1})}e^{-\alpha}z^{-2H_0}. \tag{8}$$

Lemma 7.2. [1, 2] *Let* $V_1(z)=\sqrt{2}\ \Phi\,\phi(-z)\psi^*(z)\,\Phi^{-1}$, $V_0(z)=\sqrt{2}\ \Phi\,\phi(z)\psi(z)\,\Phi^{-1}$. *Then we have*

$$V_1(z)=e^{-\xi(t,z)}e^{2\,\xi(\tilde{\partial}_t,z^{-1})}\theta e^{-\alpha}z^{-2H_0}, \quad V_0(z)=e^{\xi(t,z)}e^{-2\,\xi(\tilde{\partial}_t,z^{-1})}\theta e^{\alpha}z^{2H_0}. \tag{9}$$

Due to Lemma 7.2, we can write the actions of f_i on $\mathbb{C}[t]\otimes_{\mathbb{C}}\mathbf{\Omega}$ in terms of a formal contour integrals

$$f_0 = \sqrt{-1}^{-1}\oint z^{-1}V_0(z)dz, \quad f_1 = -\oint V_1(z)dz, \tag{10}$$

where for $A(z)=\sum_n A_n z^n$, we set $\oint A(z)dz = A_{-1}$.

Lemma 7.3.

$$V_i(z_\ell)\cdots V_i(z_2)V_i(z_1) = (-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2 e^{(-1)^i\sum_j \xi(t,z_j)}e^{2(-1)^{i+1}\sum_j \xi(\widetilde{\partial}_t,z_j^{-1})}\theta^\ell e^{(-1)^i\ell\alpha}(z_1\cdots z_\ell)^{2(-1)^i H_0}$$

Proof. By $V_i^0(z)$, we denote "the zero mode" $\theta e^{(-1)^i\alpha}z^{2(-1)^iH_0}$ of $V_i(z)$. Then by using the relations $\theta e^{\pm\alpha} = -e^{\pm\alpha}\theta$, and $z_j^{\pm 2H_0}e^{\pm\alpha} = z_j^2\cdot e^{\pm\alpha}z_j^{\pm 2H_0}$, we have

$$V_i^0(z_\ell)\cdots V_i^0(z_2)V_i^0(z_1) = (-1)^{\frac{\ell(\ell-1)}{2}}\left(\prod_{j=1}^{\ell} z_j^{2j-2}\right)\theta^\ell e^{(-1)^i\ell\alpha}(z_1\cdots z_\ell)^{2(-1)^iH_0}.$$

On the other hand, by the standard calculus of vertex operators, we have

$$V_i^+(z_2)V_i^-(z_1) = \left(1-\frac{z_1}{z_2}\right)^2 V_i^-(z_1)V_i^+(z_2),$$

where we set $V_i^-(z) = e^{(-1)^i\xi(t,z)}$, $V_i^+(z) = e^{(-1)^{i+1}2\xi(\widetilde{\partial}_t,z^{-1})}$. Then the lemma follows immediately. □

For $\lambda\in\mathcal{P}$, we denote by $\mathrm{S}_\lambda(z)$ the Schur function $\det(z_i^{\lambda_j+j-1})/\det(z_i^{j-1})$ with respect to $z=(z_1,\dots,z_\ell)$. We use the well-known orthogonality relation

$$\frac{1}{(2\pi\sqrt{-1})^\ell}\int_{T^\ell}\mathrm{S}_\lambda(z)\overline{\mathrm{S}_\mu(z)}\,|\Delta(z)|^2\frac{dz_1}{z_1}\cdots\frac{dz_\ell}{z_\ell} = \ell!\,\delta_{\lambda,\mu},$$

where we denote by T^ℓ the ℓ-dimensional torus $\{z=(z_j)\in\mathbb{C}^\ell \mid |z_j|=1\}$. Since $\overline{\mathrm{S}_\mu(z)} = \mathrm{S}_\mu(z^{-1}) = \mathrm{S}_\mu(z_1^{-1},\dots,z_\ell^{-1})$ for $z\in T^\ell$, we can rewrite this relation as

$$\oint\cdots\oint \mathrm{S}_\lambda(z)\mathrm{S}_\mu(z^{-1})(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2(z_1\cdots z_\ell)^{-\ell}\,dz_1\cdots dz_\ell = \ell!\,\delta_{\lambda,\mu}. \tag{11}$$

We also utilize the following form of the Cauchy identity:

$$e^{\sum_{j=1}^{\ell}\xi(t,z_j)} = \sum_{\lambda\in\mathcal{P}}\mathrm{S}_\lambda(z)S_\lambda(t). \tag{12}$$

Lemma 7.4. *For $m>0$, we have*

$$f_0^{(\ell)}\theta^m e^{-m\alpha} = \zeta'_{m,\ell,1}S_{\square(\ell,2m+1-\ell)}(t)\,\theta^{m+\ell}e^{(m-\ell)\alpha},$$

where

$$\zeta'_{m,\ell,0} = \sqrt{-1}^{(2\ell-1)m}\ (1\le\ell\le 2m-1),$$

and

$$f_1^{(\ell)}\theta^m e^{m\alpha} = \zeta'_{m,\ell,0} S_{\square(\ell,2m-\ell)}(t)\,\theta^{m+\ell} e^{(-m+\ell)\alpha},$$

where

$$\zeta'_{-m,\ell,1} = \sqrt{-1}^{-\ell+(2\ell-1)m}\ (1 \le \ell \le 2m).$$

Proof. In view of the relation $e^{\ell\alpha}\cdot\theta^m = (-1)^{\ell m}\theta^m\cdot e^{\ell\alpha}$, we have by Lemma 7.3

$$V_0(z_\ell)\cdots V_0(z_1)\,\theta^m e^{-m\alpha} = (-1)^{\ell m}(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2 e^{\sum_j \xi(t,z_j)}\theta^{m+\ell}e^{(-m+\ell)\alpha}(z_1\cdots z_\ell)^{-2m}.$$

Using this, we have

$$\begin{aligned}
&f_0^{(\ell)}\theta^m e^{-m\alpha}\\
&= \frac{\sqrt{-1}^{-\ell}}{\ell!}\oint\cdots\oint(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(z)^2(z_1\cdots z_\ell)^{-2m-1}e^{\sum_j\xi(t,z_j)}dz_1\cdots dz_\ell\cdot(-1)^{\ell m}\theta^{m+\ell}e^{(-m+\ell)\alpha}\\
&= \sqrt{-1}^{-\ell}(-1)^{\ell m}S_{\square(\ell,2m+1-\ell)}(t)\,\theta^{m+\ell}e^{(-m+\ell)\alpha}
\end{aligned}$$

where we carried out the contour integral by using (11) and (12). Then combined with Lemma 6.1, we have $\zeta'_{m,\ell,0} = \sqrt{-1}^{-\ell+(2\ell-1)m}$.

In a similar way, we have $\zeta'_{m,\ell,1} = \sqrt{-1}^{(2\ell-1)m}$. We just note that $S_{\square(2m-\ell,\ell)}(-t) = (-1)^{\ell(2m-\ell)}S_{\square(\ell,2m-\ell)}(t)$, and detail of the calculation is left for the reader. □

8. Proof of the main theorem

Here we show an outline of a proof of the main theorem. First we calculate

$$\begin{aligned}
\Phi\left(f_i^{(\ell)}|c_m\rangle\right) &= \sqrt{2}^{-\varepsilon_m}\sum_{\lambda\in I_i^\ell(m)}\Phi\left(\sqrt{2}^{a(\lambda)}|\lambda\rangle\right) \quad \text{by Lemma 3.3}\\
&= \zeta_{m,\ell,i}\sqrt{2}^{-\varepsilon_m}\sum_{\lambda\in I_i^\ell(m)}\delta(\lambda)\,Q_{\lambda[0]}(t_{odd})S_{\lambda[1]}(t_{even})\,\theta^{m+\ell}\,e^{(m+(-1)^i\ell)\alpha},
\end{aligned} \tag{13}$$

where $\zeta_{m,\ell,i}$, is a certain 4-th root of unity.

Secondly we show

$$f_i^{(\ell)}\Phi\left(|c_m\rangle\right) = \begin{cases}\zeta'_{m,\ell,1}\,\sqrt{2}^{-\varepsilon_m}S_{\square(2m-\ell,\ell)}(t)\,\theta^{m+\ell}\,e^{(m-\ell)\alpha}\\ \zeta'_{m,\ell,0}\,\sqrt{2}^{-\varepsilon_m}S_{\square(\ell,2m+1-\ell)}(t)\,\theta^{m+\ell}\,e^{(m+\ell)\alpha}.\end{cases} \tag{14}$$

We have already shown this in Lemma 7.4. Therefore all we have to show is the following:

Lemma 8.1.

$$\zeta_{m,\ell,i} = \zeta'_{m,\ell,i}.$$

We give an example before proving this lemma.

Example 8.2. *Let $m = 4, \ell = 4$ and $\lambda = (13, 10, 7, 2)$. Then we have $\delta(\lambda) = 1$ from 4-bar abacus of λ and $\zeta'_{4,4,1} = 1$.*

$$\begin{aligned}\beta_{13}\beta_{10}\beta_7\beta_3|vac\rangle &= \sqrt{-1}^4\psi_3\phi_5\psi^*_{-2}\phi_1|vac\rangle \\ &= \psi_3\phi_5\phi_1\psi_{-1}|0,2\rangle \\ &= (-1)^2\phi_5\phi_1\psi_3\psi_{-1}|0,2\rangle.\end{aligned}$$

In the second equation above we have to count carefully the number of ϕ's and ψ's jumped by ψ^*'s. The sign appearing in the last equation is $\delta(\lambda)$. Therefore we have

$$\beta_{13}\beta_{10}\beta_7\beta_3|vac\rangle = \delta(\lambda)\zeta_{4,4,1}\phi_5\phi_1\psi_3\psi_{-1}|0,2\rangle$$

and $\zeta_{4,4,1} = \zeta'_{4,4,1}$.

Proof. We write $|\lambda\rangle = \beta_{\lambda_1}\cdots\beta_{\lambda_{2s}}|\mathrm{vac}\rangle$, where $\lambda_1 > \cdots > \lambda_{2s} \geq 0$. If we neglect the factor $\sqrt{-1}$ in (3), the set $\{\beta_{\lambda_1}, \ldots, \beta_{\lambda_{2s}}\}$ is decomposed into the three parts

$$\{\psi_{i_1}, \ldots, \psi_{i_N}\}, \quad \{\psi^*_{j_1}, \ldots, \psi^*_{j_{N^*}}\}, \quad \{\phi_{k_1}, \ldots, \phi_{k_a}\},$$

where $i_1 > \cdots > i_N \geq 0 > j_1 > \cdots > j_{N^*}$ and $k_1 > \cdots > k_a \geq 0$. Actually, $I = \{i_1, \ldots, i_N\}$ (resp. $J = \{j_1, \ldots, j_{N^*}\}$) is nothing but $\iota(\boldsymbol{\lambda}^{(1)})$ (resp. $\iota^*(\boldsymbol{\lambda}^{(3)})$), and $K = \{\phi_{k_1}, \ldots, \phi_{k_a}\}$ corresponds to $\boldsymbol{\lambda}^{(0)} \cup \boldsymbol{\lambda}^{(2)}$. We shall rearrange these generators so that we can compare $|\lambda\rangle$ with its "normal form" such as in Lemma 6.5. We denote by a, N and N^* the number of ϕ's, ψ's and ψ^*'s respectively.

Case 1: $i = 1, m = 2n > 0$. We start with $\beta_{\lambda_1}\cdots\beta_{\lambda_{2s}}|\mathrm{vac}\rangle$. Note that, in this case, for any $\phi_{k_j} \in K$, k_j is odd. Therefore any β_{λ_j} has the factor $\sqrt{-1}$ when we identify it, according to (3), with one of ψ, ψ^*, ϕ. We neglect the factor for the moment.

By the anti-commutation relation, we first move $\psi^*_{j_1}$ to the right end of β's,

$$(-1)^{-j_1-1}\beta_{\lambda_1}\cdots\widehat{\beta_{\lambda_j}}\cdots\beta_{\lambda_{2s}}\psi^*_{j_1}|\mathrm{vac}\rangle$$

where β_{λ_j} corresponds to $\psi^*_{j_1}$, and the hat symbol indicates that we omit β_{λ_j} from the sequence of β's. Note that β_{λ_j} is located on the $(-j_1)$-th term of the sequence $\beta_{\lambda_1}, \ldots, \beta_{\lambda_{2s}}$ from the right. Since the vacuum $|\mathrm{vac}\rangle$ can be written as $|\mathrm{vac}\rangle = \psi_{-1}\psi_{-2}\cdots\psi_{j_{N^*}}|0, j_{N^*}\rangle$, we have $\psi^*_{j_1}|\mathrm{vac}\rangle = (-1)^{-j_1-1}\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\psi_{j_{N^*}}|0, j_{N^*}\rangle$, and therefore we have

$$|\lambda\rangle = \beta_{\lambda_1}\cdots\widehat{\beta_{\lambda_j}}\cdots\beta_{\lambda_{2s}}\cdot\psi_{-1}\psi_{-2}\cdots\widehat{\psi_{j_1}}\cdots\psi_{j_{N^*}}|0, j_{N^*}\rangle.$$

Here one may think a pair annihilation procedure is occur. By repeating this procedure, we can erase the ψ^*'s from the sequence $\beta_{\lambda_1}, \ldots, \beta_{\lambda_{2s}}$. Hence by the definition of the sign $\delta(\lambda)$, if we move ϕ's to the left of ψ's, we have

$$|\lambda\rangle = \delta(\lambda)\sqrt{-1}^{m}\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}|J^c\rangle, \tag{15}$$

where $|J^c\rangle = (-1)^{j_1+\cdots+j_{N^*}-N^*}\psi^*_{j_1}\cdots\psi^*_{j_{N^*}}|0,0\rangle$.

Here we recover the neglected factor $\sqrt{-1}^m$. Thus we have $\zeta_{m,\ell,1} = \sqrt{-1}^m$, which is equal to $\zeta'_{m,\ell,1}$ since m is even.

Case 2: $i = 1, m = 2n+1 > 0$. In this case, we always have $\phi_0 = \beta_0$ in the right end of β's. We have

$$|\lambda\rangle = \delta(\lambda)\sqrt{-1}^{m}(-1)^{m-\ell}\phi_{k_1}\cdots\phi_{k_a}\psi_{i_1}\cdots\psi_{i_N}|J^c\rangle.$$

The only difference from (15) is the appearance of $(-1)^{m-\ell}$. This factor is explained as follows: Firstly, when we erase ψ^*'s as in (15), the additional sign $(-1)^{N^*}$ is in need, because of the existence of ϕ_0. Secondly, after we move $\phi_j (j > 0)$ to the left (with the sign change $\delta(\lambda)$), we have to move ϕ_0 to the left also. This causes the sign $(-1)^N$. Now $(-1)^{N+N^*}$ is equal to $(-1)^{m-\ell}$, because $N + N^* = (m-\ell) - 2N^*$. Note that $m - \ell = \mathrm{ch}(\lambda)$. Thus we have $\zeta_{m,\ell,1} = \sqrt{-1}^m(-1)^{m-\ell}$, which is equal to $\zeta'_{m,\ell,1}$ as m is odd.

Case 3: $i = 0, m = 2n+1 > 0$. Consider $\lambda \in I^\ell_0(-m)$. Remark that $|c_{-m}\rangle$ has the end term $\beta_0 = \phi_0$. We divide this case into subcases: (a) λ has the end term ϕ_0, (b) λ does not have the end term ϕ_0. Let us first consider subcase (a). We have

$$\zeta_{-m,\ell,0} = \sqrt{-1}^{m+1-a}(-1)^{N^*}.$$

The sign $(-1)^{N^*}$ comes from the exchanges of ϕ_0 and ψ^*'s. Let us rewrite this expression. Now we have the obvious relations $N - N^* = -m+\ell,\ a+N+N^* = m+1$. We eliminate N from these equations to get $a + 2N^* = 2m - \ell + 1$. Then we have $\zeta_{-m,\ell,0} = \sqrt{-1}^{-m+\ell}$, which is equal to $\zeta'_{-m,\ell,0}$. For subcase (b), we have ψ_0 instead of the absence of ϕ_0. So the same factor $(-1)^{N^*}$ occurs when we exchange ψ_0 and ψ^*'s. Then the formula is the same as (a).

Case 4: $i = 0, m = 2n > 0$. Consider $\lambda \in I^\ell_0(-m)$. We divide this case into the subcases: (c) λ ends with $\psi_0\phi_0$ and (d) λ does not contain ϕ_0 nor ψ_0. In the case (c), remark that ϕ_0 should be exchange with ψ's to cause $(-1)^N$, which is included into $\zeta_{-m,\ell,0}$. By the similar argument of the case 1, ψ^* can be erased without causing any sign change. So we have $\zeta_{-m,\ell,0} = \sqrt{-1}^{m+2-a}(-1)^N$. Now since we have $a+2N = \ell+2$, and m is odd, we can see $\zeta_{-m,\ell,0} = \zeta'_{-m,\ell,0}$. The subcase (d) is the most cumbersome one. By the definition, we include in $\delta(\lambda)$ the sign $(-1)^N$ arising from the exchanges of the dummy "ϕ_0" and ψ's. So have to compensate the same factor to get $\zeta_{-m,\ell,0} = \sqrt{-1}^{m-a}(-1)^N$. Now using $a+2N = \ell$, we have $\zeta_{-m,\ell,0} = \sqrt{-1}^{m-\ell}$, which is $\sqrt{-1}^{-m-\ell}$ as m is even. □

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