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INSTABILITY OF PLANAR TRAVELING WAVES IN BISTABLE REACTION-DIFFUSION SYSTEMS

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Abstract. This paper is concerned with the stability of a planar traveling wave in a cylindrical domain. The equation describes activator-inhibitor systems in chemistry or biology. The wave has a thin transition layer and is constructed by singular perturbation methods. Let ε be the width of the layer. We show that, if the cross section of the domain is narrow enough, the traveling wave is asymptotically stable, while it is unstable if the cross section is wide enough by studying the linearized eigenvalue problem. For the latter case, we study the wavelength associated with an eigenvalue with the largest real part, which is called the fastest growing wavelength. We prove that this wavelength is $O(\varepsilon^{1/3})$ as ε goes to zero mathematically rigorously. This fact shows that, if unstable planar waves are perturbed randomly, this fastest growing wavelength is selectively amplified with as time goes on. For this analysis, we use a new uniform convergence theorem for some inverse operator and carry out the Lyapunov-Schmidt reduction.

1. **Introduction.** This paper is concerned with the stability of planar traveling waves in a cylindrical domain. The equation is expressed as

$$\begin{aligned} \varepsilon\tau u_t &= \varepsilon^2\Delta u + f(u,v) \\ v_t &= \Delta v + g(u,v) \end{aligned} \qquad t>0,\ (z_1,\dots,z_N)\in \boldsymbol{R}\times\Omega. \tag{1.1}$$

Here f and g are, for instance, given by

$$f(u,v) = u^2(1-u) - \beta_0 uv, \quad g(u,v) = uv + \beta_1 v - \beta_2 v^2, \tag{1.2}$$

where β_0, β_1 and β_2 are positive constants with $2\beta_0(1+2\beta_1) < \beta_2$. For the assumption of f and g, see (A1)–(A4) in this section.

Equation (1.1) has two stable constant solutions. We denote them by $(\overline{u}_+, \overline{v}_+)$ and $(\overline{u}_-, \overline{v}_-)$ with $\overline{u}_- < \overline{u}_+$ and $\overline{v}_- < \overline{v}_+$. The domain $\boldsymbol{R}\times\Omega$ is a cylinder in $\boldsymbol{R}^N$ with an integer $N>1$. The cross section Ω is a bounded domain in $\boldsymbol{R}^{N-1}$ with piecewise smooth boundary. The letter Δ is Laplacian $\sum_{j=1}^N (\partial/\partial z_j)^2$. We impose the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \boldsymbol{n}} = 0, \ \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on } \boldsymbol{R}\times\partial\Omega.$$

Here τ is a positive constant, and $\partial/\partial\boldsymbol{n}$ is the outward conormal derivative on $\partial\Omega$.

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Equation (1.1) describes activator-inhibitor systems in chemistry or biology as in Fife [3]. Examples are as follows. Reaction terms (1.2) describe the dynamics of prey and predator system in mathematical ecology. See Mimura, Nishiura and Yamaguti [9]. This system consists of two diffusing populations interacting each other, and has two stable constant states. Traveling waves that connect these two stable states are observed in these systems ([3], [6]). For chemical models, Ortoleva and Ross [12] and Collis and Ross [2] studied several chemical systems including Belousov-Zhabotinski reaction with two stable resting states, and studied propagating waves in these systems. Equation (1.1) includes the FitzHugh-Nagumo equation of bistable type. In an appropriate ionic environment, electronic transmission in nerve or muscle fibers have two resting states and have propagating waves connecting them. See Rinzel and Terman [13] for instance. For population models of excitatory and inhibitory model neurons in nervous systems, see Wilson and Cowan [19].

In several activator-inhibitor systems, an activator diffuses much slowly and reacts fast, that is, $\varepsilon > 0$ is very small. For prey-predator systems, population models of botanical planktons and planktonic animals agree with this situation if the increase of botanical planktons and the mutual interaction are very fast. We always assume $\varepsilon > 0$ is small in this paper.

A one-dimensional problem has a traveling wave solution. Let c be the velocity. Putting $x_1 = z_1 + ct$, we see that a one-dimensional traveling wave solution $(u_0(x_1), v_0(x_1))$ satisfies

$$\begin{aligned} \varepsilon^2 \tfrac{d^2 u_0}{dx_1^2} - \varepsilon\tau c \tfrac{du_0}{dx_1} + f(u_0(x_1), v_0(x_1)) = 0 \\ \tfrac{d^2 v_0}{dx_1^2} - c\tfrac{dv_0}{dx_1} + g(u_0(x_1), v_0(x_1)) = 0 \end{aligned} \qquad -\infty < x_1 < \infty, \tag{1.3}$$

subject to

$$u_0(\pm\infty) = \overline{u}_\pm, \quad v_0(\pm\infty) = \overline{v}_\pm. \tag{1.4}$$

Equation (1.3)–(1.4) has at least one traveling wave solution, and the stability condition is explicitly given by [11] and [7]. See [6] for existence and construction. Let $(u_0(x_1;\varepsilon), v_0(x_1;\varepsilon))$ be one of stable traveling wave solutions of (1.3)–(1.4), and let c^ε denote its velocity.

For (1.1), we use a coordinate as

$$x \stackrel{\text{def}}{=} {}^t(x_1, x_2, \ldots, x_N) = {}^t(z_1 + ct, z_2, \ldots, z_N),$$

where the superscript ^{t} stands for the transpose. We put

$$\overline{u}_0(x;\varepsilon) = u_0(x_1;\varepsilon), \quad \overline{v}_0(x;\varepsilon) = v_0(x_1;\varepsilon) \qquad x \in \boldsymbol{R} \times \Omega. \tag{1.5}$$

Then $(\overline{u}_0(x;\varepsilon), \overline{v}_0(x;\varepsilon))$ has a flat thin transition layer, which we call a planar traveling wave. From now on, we denote $(\overline{u}_0(x), \overline{v}_0(x))$ simply by $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ since no confusion may occur. This solution satisfies

$$\begin{aligned} \varepsilon^2 \Delta_x u_0 - \varepsilon\tau c^\varepsilon \tfrac{\partial u_0}{\partial x_1} + f(u_0(x;\varepsilon), v_0(x;\varepsilon)) = 0 \\ \Delta_x v_0 - c^\varepsilon \tfrac{\partial v_0}{\partial x_1} + g(u_0(x;\varepsilon), v_0(x;\varepsilon)) = 0 \end{aligned} \qquad x \in \boldsymbol{R} \times \Omega.$$

Here Δ_x is the Laplacian with respect to the variable x, that is, $\sum_{j=1}^N (\partial/\partial x_j)^2$.

We study the stability of the planar traveling wave solution (u_0, v_0). We show that it is unstable in some conditions, and show that a characteristic wavelength is selectively amplified as time goes on if an unstable planar traveling wave is perturbed randomly. Our purpose is to show that such selective amplification of random external perturbation occurs in the traveling waves in chemical or biological models stated above, and determine the characteristic wavelength precisely. One may refer to [16] for stationary planar fronts in a bounded domain for this problem. This selective amplification reminds us of pattern selection mechanism of dendritic solidification ([8]).

For this purpose, we set $u(z,t) = u_0(x;\varepsilon) + \widehat{u}(x,t)$, $v(z,t) = v_0(x;\varepsilon) + \widehat{v}(x,t)$. By a general theory for semilinear parabolic equations, $(\widehat{u}(x,t), \widehat{v}(x,t))$ is well approximated, if it is small enough, by

$$\begin{aligned} \varepsilon\tau\widehat{u}_t &= \varepsilon^2\Delta_x\widehat{u} - \varepsilon\tau c^\varepsilon \tfrac{\partial\widehat{u}}{\partial x_1} + f_u(u_0(x;\varepsilon), v_0(x;\varepsilon))\widehat{u} + f_v(u_0(x;\varepsilon), v_0(x;\varepsilon))\widehat{v} \\ \widehat{v}_t &= \Delta_x\widehat{v} - c^\varepsilon \tfrac{\partial\widehat{v}}{\partial x_1} + g_u(u_0(x;\varepsilon), v_0(x;\varepsilon))\widehat{u} + g_v(u_0(x;\varepsilon), v_0(x;\varepsilon))\widehat{v} \end{aligned} \tag{1.6}$$

in $\boldsymbol{R} \times \Omega$, and

$$\frac{\partial \widehat{u}}{\partial \boldsymbol{n}} = 0, \quad \frac{\partial \widehat{v}}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \boldsymbol{R} \times \partial\Omega.$$

See [5] for instance. Putting $\widehat{u}(x,t) = e^{\lambda t}w(x)$ and $\widehat{v}(x,t) = e^{\lambda t}z(x)$, we obtain the linearized eigenvalue problem

$$\lambda\begin{pmatrix} w \\ z \end{pmatrix} = \mathcal{L}^\varepsilon \begin{pmatrix} w \\ z \end{pmatrix} \qquad \text{in } \boldsymbol{R} \times \Omega \tag{1.7}$$

with the Neumann boundary conditions on $\boldsymbol{R} \times \partial\Omega$. Here $\mathcal{L}^\varepsilon$ is given by

$$\begin{pmatrix} \frac{1}{\varepsilon\tau}(\varepsilon^2\Delta_x + f_u(u_0(x;\varepsilon), v_0(x;\varepsilon))) - c^\varepsilon \frac{\partial}{\partial x_1} & \frac{1}{\varepsilon\tau} f_v(u_0(x;\varepsilon), v_0(x;\varepsilon)) \\ g_u(u_0(x;\varepsilon), v_0(x;\varepsilon)) & \Delta_x + g_v(u_0(x;\varepsilon), v_0(x;\varepsilon)) - c^\varepsilon \frac{\partial}{\partial x_1} \end{pmatrix}.$$

The location of the spectrum set of $\mathcal{L}^\varepsilon$ determines the stability. We consider the stability in the space $\boldsymbol{X}^{\gamma_1}$, where $\boldsymbol{X} = W^{k,2}(\boldsymbol{R}^N) \times \cdots \times W^{k,2}(\boldsymbol{R}^N)$. We take $k > 0$ and $\gamma_1 \in (0,1)$ so that $k/2 < (k+2)\gamma_1 - N/2$ is satisfied. Then the continuous embedding $\boldsymbol{X}^{\gamma_1} \subset C^{\nu_1}(\boldsymbol{R}^N) \times \cdots \times C^{\nu_1}(\boldsymbol{R}^N)$ is valid for some $\nu_1 > k/2$. From this embedding the nonlinear stability in $\boldsymbol{X}^{\gamma_1}$ follows from the location of the eigenvalues. From now on, we abbreviate $f_u(u_0(x_1;\varepsilon), v_0(x_1;\varepsilon))$ to $f_u^0(x_1;\varepsilon)$. The following is the result for the essential spectrum of $\mathcal{L}^\varepsilon$.

LEMMA 1.1. *Assume $\varepsilon > 0$ is small enough. The operator $\mathcal{L}^\varepsilon$ has only isolated eigenvalues in the domain $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda > -\lambda_0/2\}$. The essential spectrum of $\mathcal{L}^\varepsilon$ lies in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \le -\lambda_0/2\}$. Here λ_0 is a positive number independent of ε.*

Thus it suffices to consider only eigenvalues of $\mathcal{L}^\varepsilon$ in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \ge 0\}$. Define $x' = {}^t(x_2, \dots, x_N)$ and $\Delta' \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_N}\right)^2$. Let $\{(\kappa_n, \varphi_n(x')) | n = 1, 2, \dots\}$ be the pairs of eigenvalues and the associated eigenfunction of $-\Delta'$ with

$$-\Delta'\varphi_n = \kappa_n\varphi_n \quad \text{in } \Omega, \qquad \frac{\partial\varphi_n}{\partial\boldsymbol{n}} = 0 \quad \text{on } \partial\Omega.$$

Without loss of generality we assume $0 = \kappa_0 < \kappa_1 \le \kappa_2 \le \cdots \le \kappa_n \le \kappa_{n+1} \le \cdots$. We set

$$w(x) = \sum_{n=0}^{\infty} w_n(x_1)\varphi_n(x'), \quad z(x) = \sum_{n=0}^{\infty} z_n(x_1)\varphi_n(x').$$

If

$$\begin{pmatrix} w(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} w_n(x_1)\varphi_n(x') \\ z_n(x_1)\varphi_n(x') \end{pmatrix} \tag{1.8}$$

is an eigenfunction of $\mathcal{L}^\varepsilon$ with the associated eigenvalue λ, then we have

$$\mathcal{L}_n^\varepsilon \begin{pmatrix} w_n(x_1) \\ z_n(x_1) \end{pmatrix} = \lambda \begin{pmatrix} w_n(x_1) \\ z_n(x_1) \end{pmatrix} \qquad \text{in } \boldsymbol{R}. \tag{1.9}$$

Here we put

$$\mathcal{L}_n^\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon\tau}\left(\varepsilon^2 \frac{d^2}{dx_1^2} - \varepsilon^2\kappa_n + f_u^0(x_1;\varepsilon)\right) - c^\varepsilon \frac{d}{dx_1} & \frac{1}{\varepsilon\tau} f_v^0(x_1;\varepsilon) \\ g_u^0(x_1;\varepsilon) & \frac{d^2}{dx_1^2} - \kappa_n + g_v^0(x_1;\varepsilon) - c^\varepsilon \frac{d}{dx_1} \end{pmatrix}$$

with $f_u^0(x_1;\varepsilon) = f_u(u_0(x_1;\varepsilon), v_0(x_1;\varepsilon))$, and so on. On the contrary, if (w_n, z_n) satisfies (1.9), then (1.8) satisfies (1.7). For every $(w(x), z(x))$ that are not identically zero, there exists n such that $(w_n(x_1), z_n(x_1)) \not\equiv (0,0)$ with (1.9). Thus it suffices to study (1.9) for all n.

In general, the most prevalent way to study a linearized eigenvalue problem is to construct the Evans function and seek for eigenvalues as the zero points of this function. Another way is the Lyapunov-Schmidt method. Nishiura, Mimura, Ikeda and Fujii [11] studied (1.9) when $n = 0$ by this method using the spectral gap condition of an Allen-Cahn operator $L(\varepsilon, c^\varepsilon)$ given by (2.11), and derived a scalar equation for eigenvalues called the SLEP equation. Ikeda, Nishiura and Suzuki [7] proved the equivalence of these two methods. The Lyapunov-Schmidt method needs less calculation compared with that of the Evans function. This might be because $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ has a thin transition layer and is constructed by singularly perturbation methods.

For Lyapunov-Schmidt method for (1.9), the uniform convergence of $L(\varepsilon, c^\varepsilon)^{-1}Q^\varepsilon$ as ε goes to zero is essentially important. Here Q^ε is the projection associated with a spectral set of $L(\varepsilon, c^\varepsilon)$. See §2 for the precise definition. The convergence should be uniform in some function space because the eigenfunctions always depend on ε. A strong convergence theorem of $L(\varepsilon, c^\varepsilon)^{-1}Q^\varepsilon$ and an exponentially weighted normed space have been used so far. However, the author cannot follow the Lyapunov-Schmidt reduction by a strong convergence theorem when the given interval is unbounded. Recently the author proved in [15] a uniform convergence theorem of $L(\varepsilon, c^\varepsilon)^{-1}Q^\varepsilon$ in $\mathcal{L}(L^2(\boldsymbol{R}), H^{-s}(\boldsymbol{R}))$ for any $s \in (0, \frac{1}{2})$. Using this convergence result, we derive the SLEP equation mathematically rigorously for every n. Then using the method of [16], we can study the distribution of the eigenvalues precisely. Because $L(\varepsilon, c^\varepsilon)$ is not self-adjoint and the given interval is unbounded, we need careful analysis.

The following is the main assertion in this paper.

Theorem 1.1. *Assume $\varepsilon > 0$ is small enough. The set of eigenvalues of (1.7) in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \geq 0\}$ is expressed by $\{\Lambda(\kappa_n) \mid 0 \leq \kappa_n \leq \sigma(\varepsilon)\varepsilon^{-1},\ n = 0, 1, \cdots\}$. Here $\Lambda(\,\cdot\,)$ is a real-valued function that has a unique maximizer $\kappa(\varepsilon) \in (0, \sigma(\varepsilon)\varepsilon^{-1})$ with $\Lambda''(\kappa(\varepsilon)) < 0$. It holds that $\Lambda'(\kappa) > 0$ for $\kappa \in [0, \kappa(\varepsilon))$ and $\Lambda'(\kappa) < 0$ for $\kappa \in (\kappa(\varepsilon), \sigma(\varepsilon)\varepsilon^{-1}]$. The following asymptotic estimates*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{2}{3}}\kappa(\varepsilon) = \left(4^{-1}k_3\right)^{\frac{2}{3}} > 0, \quad \lim_{\varepsilon\to 0} \Lambda(\kappa(\varepsilon)) = \tau^{-1}\widehat{\zeta}(0)$$

hold true. The value $\sigma(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = \widehat{\zeta}(0) > 0$. Here $\widehat{\zeta}(0)$ is as in Lemma 2.3, and k_3 are positive constants as in (2.14).

Using this theorem, we obtain the stability criterion.

Theorem 1.2. *Assume $\varepsilon > 0$ is small enough. If $\sigma(\varepsilon)\varepsilon^{-1} < \kappa_1$, a planar traveling wave $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ is asymptotically stable. If $\kappa_1 < \sigma(\varepsilon)\varepsilon^{-1}$, it is unstable.*

If $\kappa_1 = \sigma(\varepsilon)\varepsilon^{-1}$, the linearized eigenvalue problem (1.7) has double zero eigenvalue. The stability of $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ is yet to be studied in this exceptional case.

Theorem 1.3. *Let $\lambda_{\max}(\varepsilon)$ be the largest eigenvalue for the linearized problem (1.7). Then $\lim_{\varepsilon\to 0}\lambda_{\max}(\varepsilon) = \widehat{\zeta}(0)/\tau > 0$ holds true. The associated eigenspace is a linear hull of one or two eigenfunctions, each of which can be expressed as $(w(x), z(x)) = (w_n(x_1)\varphi_n(x'), z_n(x_1)\varphi_n(x'))$, where n is a positive integer with $\kappa_{n-1} \le \kappa(\varepsilon) \le \kappa_n$ or $\kappa_n \le \kappa(\varepsilon) \le \kappa_{n+1}$. Here functions w_n, z_n are given by (3.9).*

Theorem 1.3 implies that selective amplification of random external perturbation can occur for unstable planar traveling waves in general activator-inhibitor systems.

Example 1. Suppose $\Omega = (0,\ell)$ with $\ell > 0$ and $N = 2$. If $0 < \ell < \ell_C(\varepsilon)$, a planar traveling wave is asymptotically stable. If $\ell_C(\varepsilon) < \ell$, it is unstable. Here $\ell_C(\varepsilon)$ is a function of ε with $\ell_C(\varepsilon) = 2\pi\widehat{\zeta}(0)^{-1/2}\varepsilon^{1/2} + o(\varepsilon^{1/2})$ as $\varepsilon \to 0$. Let ε go to zero, then the eigenfunctions associated with $\lambda_{\max}(\varepsilon)$ is given by $\big(w_n(x_1)\cos(\ell^{-1}n\pi x_2), z_n(x_1)\cos(\ell^{-1}n\pi x_2)\big)$ with n that satisfies $\left|\ell^{-1}n\pi - \kappa(\varepsilon)^{1/2}\right| \le \ell^{-1}\pi$. Thus there exists wavelength associated with $\lambda_{\max}(\varepsilon)$, which is called the fastest growing wavelength. This wavelength is $2\pi\left(4k_3^{-1}\right)^{1/3}\varepsilon^{1/3} + o(\varepsilon^{1/3})$ as ε tends to zero.

Example 2. Assume Ω is a rectangle $\Omega = (0,p)\times(0,q)$ in $\boldsymbol{R}^2$, and hence $\boldsymbol{R}\times\Omega$ is a prismatic domain. From Theorem 1.2, a planar traveling wave is stable if $\sigma(\varepsilon)\varepsilon^{-1} < \pi^2\min\{p^{-2}, q^{-2}\}$, and is unstable if $\pi^2\min\{p^{-2}, q^{-2}\} < \sigma(\varepsilon)\varepsilon^{-1}$. It holds that $\lim_{n\to\infty} n^{-1}\kappa_n = 4\pi(pq)^{-1}$. Let $\varepsilon \to 0$, then $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ becomes unstable. From Theorem 1.3, it follows that the eigenspace associated with $\lambda_{\max}(\varepsilon)$ is a linear hull of $(w_n(x_1)\varphi_n(x'), z_n(x_1)\varphi_n(x'))$, where

$$\varphi_n(x') = 2(pq)^{-\frac{1}{2}}\cos\left(p^{-1}\ell\pi x_2\right)\cos\left(q^{-1}m\pi x_3\right)$$

and ℓ, m are some positive integers with

$$\left|\pi^2\left(p^{-2}\ell^2 + q^{-2}m^2\right) - \kappa(\varepsilon)\right| \le 4p^{-1}q^{-1}\pi + \theta_1(\varepsilon).$$

Here $\theta_1(\varepsilon)$ is a positive number with $\lim_{\varepsilon\to 0}\theta_1(\varepsilon) = 0$.

Now we state the standing assumptions on f and g throughout this paper.

(A1) There exist constants $v_{\min}$, $v_{\max}$ with $v_{\min} < v_{\max}$ and three functions $h_-(v)$, $h_0(v)$, $h_+(v)$ with

$$h_-(v) < h_0(v) < h_+(v) \qquad \text{for } v_{\min} < v < v_{\max},$$
$$h_-(v_{\min}) = h_0(v_{\min}), \quad h_0(v_{\max}) = h_+(v_{\max})$$

such that $\{(u,v)\,|\,f(u,v)=0\}$ equals

$$\{(h_-(v),v)\,|\,v_{\min}\le v\} \cup \{(h_0(v),v)\,|\,v_{\min}\le v\le v_{\max}\} \cup \{(h_+(v),v)\,|\,v\le v_{\max}\}.$$

(A2) Functions f, g are smooth functions of u, v in $\mathcal{S} = [h_-(v_{\max}), h_+(v_{\min})] \times [v_{\min}, v_{\max}]$.

(A3) There exist constants $\overline{v}_-$, $\overline{v}_0$, $\overline{v}_+$ with $v_{\min} < \overline{v}_- < \overline{v}_0 < \overline{v}_+ < v_{\max}$ such that $\{(u,v)\,|\,f=0, g=0\}$ consists of $(u,v) = (\overline{u}_-, \overline{v}_-)$, $(\overline{u}_0, \overline{v}_0)$, $(\overline{u}_+, \overline{v}_+)$, where $\overline{u}_- = h_-(\overline{v}_-)$, $\overline{u}_0 = h_0(\overline{v}_0)$, $\overline{u}_+ = h_+(\overline{v}_+)$.

(A4) The following inequalities

$$f_u(h_\pm(v),v)<0,\ \ g_u(h_\pm(v),v)>0,\ \ g_v(h_\pm(v),v)<0,$$
$$g(h_-(v),v)\le 0<g(h_+(v),v)$$

hold true for any $v\in[\overline{v}_-,\overline{v}_+]$. Moreover $f_v(u,v)<0$ is valid for every (u,v) with $\overline{v}_-\le v\le\overline{v}_+$, $h_-(v)\le u\le h_+(v)$.

2. **One-dimensional traveling waves.** In this section, we state on one-dimensional traveling waves. We denote x_1, z_1 simply by x, z in this section. The equation is written as

$$\begin{aligned}\varepsilon\tau u_t&=\varepsilon^2u_{zz}+f(u(z,t),v(z,t))\\ v_t&=v_{zz}+g(u(z,t),v(z,t))\end{aligned}\qquad z\in\boldsymbol{R},\ t>0. \tag{2.1}$$

We seek for a traveling wave solution $(u_0(z+ct),v_0(z+ct))$ that satisfies (1.3)–(1.4). For simplicity, we denote $u_0(x)$, $v_0(x)$ by $u(x)$, $v(x)$. Separating $\boldsymbol{R}$ into two subintervals $\boldsymbol{R}_-=(-\infty,0)$ and $\boldsymbol{R}_+=(0,+\infty)$, we consider (1.3)-(1.4) on each subintervals:

$$\begin{aligned}\varepsilon^2(u^\pm)_{xx}-\varepsilon\tau c(u^\pm)_x+f(u^\pm(x),v^\pm(x))&=0\\ (v^\pm)_{xx}-c(v^\pm)_x+g(u^\pm(x),v^\pm(x))&=0\end{aligned}\qquad x\in\boldsymbol{R}_\pm, \tag{2.2}$$

subject to the following conditions: $u^\pm(\pm\infty)=\overline{u}_\pm$, $u^\pm(0)=\alpha$, $v^\pm(\pm\infty)=\overline{v}_\pm$. Here $\alpha\in(\overline{u}_-,\overline{u}_+)$ is any fixed constant. Moreover we impose a condition $v^\pm(0)=\beta$ in addition with a constant β that will be fixed later. Putting formally $\varepsilon=0$, we obtain

$$\begin{aligned}f(u^\pm(x),v^\pm(x))&=0\\ (v^\pm)_{xx}-c(v^\pm)_x+g(u^\pm(x),v^\pm(x))&=0\end{aligned}\qquad x\in\boldsymbol{R}_\pm. \tag{2.3}$$

We solve the first relation as $u^\pm(x)=h_\pm(v^\pm(x))$. Using this, we introduce the following equations

$$\begin{aligned}(V^\pm)_{xx}-c(V^\pm)_x+G_\pm(V^\pm(x))=0\quad x\in\boldsymbol{R}_\pm\\ V^\pm(\pm\infty)=\overline{v}_\pm,\quad V^\pm(0)=\beta.\end{aligned} \tag{2.4}$$

Here we put $G_\pm(v)\stackrel{\text{def}}{=}g(h_\pm(v),v)$. We have $G'_\pm(v)<0$ for $v\in(\overline{v}_-,\overline{v}_+)$, respectively. In the following we use a functional space

$$X^n_{\gamma,\sigma}(I)=\left\{u(x)\ \middle|\ \|u\|_{X^n_{\gamma,\sigma}(I)}<+\infty\right\}, \tag{2.5}$$

with

$$\|u\|_{X^n_{\gamma,\sigma}(I)}=\sum_{k=0}^n\sup_{x\in I}\left|e^{\gamma|x|}\left(\sigma D_x\right)^k u(x)\right|$$

for $\sigma>0$, $\gamma>0$, a non-negative integer n and a subinterval $I\subset\boldsymbol{R}$. Here we put $D_x=d/dx$.

LEMMA 2.1 ([6]). *For any fixed $c\in\boldsymbol{R}$ and $\beta\in(\overline{v}_-,\overline{v}_+)$, there exist unique strictly monotone increasing solutions $V_0^\pm(x;c,\beta)$ of (2.4) with*

$$|V_0^\pm(x;c,\beta)-\overline{v}_\pm|\in X^2_{\gamma_\pm,1}(\boldsymbol{R}_\pm).$$

There exists a unique $\widehat{\beta}(c)$ with $\left(V_0^-\right)_x(0;c,\widehat{\beta}(c))=\left(V_0^+\right)_x(0;c,\widehat{\beta}(c))$ for each c. This value $\widehat{\beta}(c)$ is a strictly monotone decreasing function in c with $\widehat{\beta}(-\infty)=\overline{v}_+$ and $\widehat{\beta}(+\infty)=\overline{v}_-$. Here γ_+ and γ_- are positive constants.

We put $U_0^{\pm}(x;c,\beta) = h_{\pm}(V_0^{\pm}(x;c,\beta))$, and introduce a new variable $y = x/\varepsilon$. For any fixed $\beta \in [\overline{v}_-, \overline{v}_+]$, we consider the following problem

$$\begin{aligned} (\phi_0)_{yy} - c(\phi_0)_y + f(\phi_0(y), \beta) = 0 \qquad y \in \boldsymbol{R}, \\ \phi_0(\pm\infty) = h_{\pm}(\beta), \quad \phi_0(0) = \alpha. \end{aligned} \tag{2.6}$$

LEMMA 2.2 ([4], [11]). *There exists $c = \widehat{c}(\beta)$ such that (2.6) has a unique strictly monotone increasing solution $\phi_0(y;\beta)$. This solution satisfies*

$$|\phi_0(y;\beta) - h_{\pm}(\beta)| \in X^2_{\sigma_{\pm},1}(\boldsymbol{R}_{\pm}),$$

where σ_+ and σ_- are positive constants. For any fixed $\beta \in [\overline{v}_-, \overline{v}_+]$, $\widehat{c}(\beta)$ is a monotone decreasing continuous function with $\widehat{c}_\beta(\beta) \stackrel{\text{def}}{=} d\widehat{c}/d\beta < 0$ for every $\beta \in (\overline{v}_-, \overline{v}_+)$.

Now we solve $\beta = \widehat{\beta}(c)$ and $c\tau = \widehat{c}(\beta)$ simultaneously. We find $\beta \in (\overline{v}_-, \overline{v}_+)$ with $(\widehat{\beta})^{-1}(\beta) = \tau^{-1}\widehat{c}(\beta)$, that is, $\beta = \widehat{\beta}(\tau^{-1}\widehat{c}(\beta))$. Note that functions $\widehat{\beta}(c)$ and $\widehat{c}(\beta)$ are independent of τ. Because $(\widehat{\beta})^{-1} : (\overline{v}_-, \overline{v}_+) \to (-\infty, \infty)$ continuous, and $\tau^{-1}\widehat{c} : [\overline{v}_-, \overline{v}_+] \to [\tau\widehat{c}(\overline{v}_+), \tau^{-1}\widehat{c}(\overline{v}_-)]$ is continuous and bounded, there exists at least one $\beta^* \in (\overline{v}_-, \overline{v}_+)$ with $(\widehat{\beta})^{-1}(\beta) = \tau^{-1}\widehat{c}(\beta)$. Putting $c^* = \tau^{-1}\widehat{c}(\beta^*)$, we define

$$\begin{aligned} U_{\text{app}}(x;\varepsilon) &= \begin{cases} U_0^-(x;c^*,\beta^*) + \phi_0(x/\varepsilon;\beta^*) - h_-(\beta^*) & x \in \boldsymbol{R}_- \\ U_0^+(x;c^*,\beta^*) + \phi_0(x/\varepsilon;\beta^*) - h_+(\beta^*) & x \in \boldsymbol{R}_+, \end{cases} \\ V_{\text{app}}(x) &= \begin{cases} V_0^-(x;c^*,\beta^*) & x \in \boldsymbol{R}_- \\ V_0^+(x;c^*,\beta^*) & x \in \boldsymbol{R}_+. \end{cases} \end{aligned} \tag{2.7}$$

Assume β^* satisfies $\beta^* = \widehat{\beta}(\tau^{-1}\widehat{c}(\beta^*))$ and

$$\tau > \widehat{\beta}_c(\tau^{-1}\widehat{c}(\beta^*))\widehat{c}_\beta(\beta^*). \tag{2.8}$$

Then (β^*, c^*) is a transversal intersection point of $\beta = \widehat{\beta}(c)$ and $c\tau = \widehat{c}(\beta)$.

THEOREM 2.1 ([6]). *Suppose (A1)-(A4) and (2.8). For sufficiently small $\varepsilon > 0$, there exists a solution $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ of (1.3), (1.4) with*

$$\|u_0(x;\varepsilon) - U_{\text{app}}(x;\varepsilon)\|_{X^1_{\gamma,\varepsilon}(\boldsymbol{R})} + \|v_0(x;\varepsilon) - V_{\text{app}}(x)\|_{X^1_{\gamma,1}(\boldsymbol{R})} \to 0$$

as $\varepsilon \to 0$, where $U_{\text{app}}(x;\varepsilon)$ and $V_{\text{app}}(x)$ are given by (2.7). Furthermore c^ε converges to c^* as $\varepsilon \to 0$. Here γ is a positive constant.

We state the properties of the traveling wave solution $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ given by Theorem 2.1. We define

$$V_0(x) = \begin{cases} V_0^-(x;c^*,\beta^*) & x \in \boldsymbol{R}_- \\ V_0^+(x;c^*,\beta^*) & x \in \boldsymbol{R}_+, \end{cases} \qquad U_0(x) = \begin{cases} U_0^-(x;c^*,\beta^*) & x \in \boldsymbol{R}_- \\ U_0^+(x;c^*,\beta^*) & x \in \boldsymbol{R}_+. \end{cases}$$

We introduce two Sturm-Liouville problems

$$-L(\varepsilon, c^\varepsilon)\phi = \zeta\phi \qquad \text{in } \boldsymbol{R}, \tag{2.9}$$

$$-L(\varepsilon, -c^\varepsilon)\psi = \zeta\psi \qquad \text{in } \boldsymbol{R} \tag{2.10}$$

where

$$L(\varepsilon, c) \stackrel{\text{def}}{=} -\varepsilon^2 D_{xx} + \varepsilon\tau c D_x - f_u^0(x;\varepsilon). \tag{2.11}$$

Here $D_x = d/dx$, $D_{xx} = d^2/dx^2$ and $f_u^0(x;\varepsilon) = f_u(u_0(x;\varepsilon), v_0(x;\varepsilon))$.

LEMMA 2.3 ([11]). *The operators* $-L(\varepsilon, c^\varepsilon)$ *and* $-L(\varepsilon, -c^\varepsilon)$ *have a common algebraically simple eigenvalue* $\zeta(\varepsilon) = \widehat{\zeta}(\varepsilon)\varepsilon$*, where* $\widehat{\zeta}(\varepsilon)$ *is a continuous function up to* $\varepsilon = 0$ *with* $\widehat{\zeta}(0) = -\widehat{c}_\beta(\beta^*)(V_0)_x(0) > 0$*. All other spectrum sets of (2.9) or (2.10) lie in* $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda < -\mu_0\}$ *with a positive constant* μ_0 *that is independent of* ε*.*

Let $\phi(x, \varepsilon)$ be the eigenfunction of $-L(\varepsilon, c^\varepsilon)$ associated with $\zeta(\varepsilon)$, and let $\psi(x, \varepsilon)$ be that of $-L(\varepsilon, -c^\varepsilon)$. We assume $\phi(x, \varepsilon)$ and $\psi(x, \varepsilon)$ are normalized in $L^2(\boldsymbol{R})$. We put $Y = L^2(\boldsymbol{R})$ with usual norm and usual inner product, which we denote by $\|\cdot\|_Y$ and $(\,\cdot\,,\,\cdot\,)$, respectively. Let P^ε and Q^ε be the projections in $Y = L^2(\boldsymbol{R})$ associated with spectral sets $\{\zeta(\varepsilon)\}$ and $\sigma_{\mathrm{e}}(-L(\varepsilon, c^\varepsilon)) \setminus \{\zeta(\varepsilon)\}$, respectively. Here σ_{e} stands for the extended spectrum. Let $Y_1 = P^\varepsilon Y$, $Y_2 = Q^\varepsilon Y$. Then $Y = Y_1 \oplus Y_2$, and each Y_j is invariant under $-L(\varepsilon, c^\varepsilon)$. The spectrum set of $-L(\varepsilon, c^\varepsilon)|_{Y_2}$ lies in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda < -\mu_0\}$. See [17] for instance. From ${}^\perp\mathcal{N}(L(\varepsilon, -c^\varepsilon) + \zeta(\varepsilon)) = \mathcal{R}(L(\varepsilon, c^\varepsilon) + \zeta(\varepsilon))$, $Y_2 = \mathcal{R}(L(\varepsilon, c^\varepsilon) + \zeta(\varepsilon))$ is orthogonal to $\psi(x, \varepsilon) \in \mathcal{N}(L(\varepsilon, -c^\varepsilon) + \zeta(\varepsilon))$. Here $\mathcal{N}$ and $\mathcal{R}$ represent the kernel and the range, respectively. The projections P^ε and Q^ε are expressed by

$$P^\varepsilon = (\phi(x, \varepsilon), \psi(x, \varepsilon))^{-1} \; (\,\cdot\,, \psi(x, \varepsilon)) \; \phi(x, \varepsilon), \quad Q^\varepsilon = I - P^\varepsilon. \tag{2.12}$$

Define

$$\begin{aligned} \widehat{\phi}(y, \varepsilon) &\overset{\text{def}}{=} \varepsilon^{1/2}\phi(\varepsilon y, \varepsilon), \quad \widehat{\psi}(y, \varepsilon) \overset{\text{def}}{=} \varepsilon^{1/2}\psi(\varepsilon y, \varepsilon), \qquad (2.13)\\ h_1(x, \varepsilon) &\overset{\text{def}}{=} \varepsilon^{-1/2}(-f_v^0(x; \varepsilon))\psi(x, \varepsilon), \quad h_2(x, \varepsilon) \overset{\text{def}}{=} \varepsilon^{-1/2} g_u^0(x; \varepsilon)\phi(x, \varepsilon),\\ f_1(x, \varepsilon) &\overset{\text{def}}{=} \exp(2^{-1}c^\varepsilon x) h_1(x; \varepsilon), \quad f_2(x, \varepsilon) \overset{\text{def}}{=} \exp(-2^{-1}c^\varepsilon x) h_2(x; \varepsilon). \end{aligned}$$

Here $f_v^0(x; \varepsilon) = f_v(u_0(x; \varepsilon), v_0(x; \varepsilon))$ and $g_u^0(x; \varepsilon) = g_u(u_0(x; \varepsilon), v_0(x; \varepsilon))$. Let $X = H^1(\boldsymbol{R})$ and let $\|\cdot\|_X$ denote the usual norm of $H^1(\boldsymbol{R})$. Let X' denote the dual space of X with norm $\|\cdot\|_{X'}$. Let $\delta = \delta(x)$ be the Dirac delta function concentrated on $x = 0$. From the continuous embedding $H^1(\boldsymbol{R}) \subset C^{\frac{1}{2}}(\boldsymbol{R})$, $\delta(x)$ belongs to $X' = H^{-1}(\boldsymbol{R})$. We define $\psi_0(y) = (\phi_0)_y(y)\exp(-\tau c^* y)$ and

$$p_1 = \|(\phi_0)_y(y)\|_Y^{-1} > 0, \quad p_2 = \|\psi_0(y)\|_Y^{-1} > 0. \quad p_3 = ((\phi_0)_y, \psi_0)\,,$$

where $\phi_0(y)$ is given by (2.6). We put $p_0 = p_1 p_2 p_3 > 0$ and

$$\begin{aligned} k_1 &= -p_2 p_3 \widehat{c}_\beta(\beta^*) > 0, \quad k_2 = p_1 \left[g(h_+(\beta^*), \beta^*) - g(h_-(\beta^*), \beta^*)\right] > 0,\\ k_3 &= k_1 k_2 / (p_1 p_2 p_3) = -\widehat{c}_\beta(\beta^*) \left[g(h_+(\beta^*), \beta^*) - g(h_-(\beta^*), \beta^*)\right] > 0. \end{aligned} \tag{2.14}$$

The functions stated above have the following properties.

LEMMA 2.4 ([11]). *There exist positive constants* b_1*,* α_1 *that are independent of* ε *with*

$$\left|\widehat{\phi}(y, \varepsilon)\right| < b_1 \exp(-\alpha_1 |y|), \quad \left|\widehat{\psi}(y, \varepsilon)\right| < b_1 \exp(-\alpha_1 |y|) \qquad y \in \boldsymbol{R}.$$

For any fixed compact interval I *in* $\boldsymbol{R}$*,*

$$\lim_{\varepsilon \to 0} \widehat{\phi}(y, \varepsilon) = p_1 (\phi_0)_y(y), \quad \lim_{\varepsilon \to 0} \widehat{\psi}(y, \varepsilon) = p_2 \psi_0(y)$$

hold uniformly in $y \in I$*. Moreover* $\lim_{\varepsilon \to 0} h_1(x, \varepsilon) = k_1 \delta(x)$ *and* $\lim_{\varepsilon \to 0} h_2(x, \varepsilon) = k_2 \delta(x)$ *hold true in* X'*.*

From this lemma and the definitions of $f_1(x, \varepsilon)$ and $f_2(x, \varepsilon)$, we immediately obtain

LEMMA 2.5. *There exist constants* $b_2 > 0$, $\alpha_2 \in (0, \alpha_1)$ *such that*

$$\max\{|f_1(x)|, |f_2(x)|\} < b_2\varepsilon^{-1}\exp(-\alpha_2|x|/\varepsilon) \quad x \in \boldsymbol{R} \tag{2.15}$$

is valid for every $\varepsilon \in (0, \varepsilon_0)$. *Functions* f_1 *and* f_2 *are bounded in* X' *uniformly in* $\varepsilon \in (0, \varepsilon_1)$. *The following relations* $\lim_{\varepsilon\to 0} f_1(x, \varepsilon) = k_1\delta(x)$, $\lim_{\varepsilon\to 0} f_2(x, \varepsilon) = k_2\delta(x)$ *hold true in* X'.

Proof of Lemma 2.5. This lemma follows from Lemma 2.4, the fact that c^ε remains bounded as $\varepsilon \to 0$, and the definitions of $f_1(x; \varepsilon)$, $f_2(x; \varepsilon)$. □

Define $\boldsymbol{C}_+ = \{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \geq 0\}$. We study the inverse operator of $\widehat{T}(\lambda, c) \stackrel{\text{def}}{=} -D_{xx} + cD_x + a(x) + \lambda$ for any $c \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C}_+$. We put $D_x = \frac{d}{dx}$ and $D_{xx} = \frac{d^2}{dx^2}$. Here $a(x)$ is given by

$$a(x) = \left(f_u(U_0, V_0)g_v(U_0, V_0) - f_v(U_0, V_0)g_u(U_0, V_0)\right) / \left(-f_u(U_0, V_0)\right).$$

From (A4) and the definitions of V_0 and U_0, $a(x)$ satisfies $\min_{x\in\boldsymbol{R}} a(x) > 0$. For z^1, $z^2 \in X$, consider a sesquilinear form

$$B_c^\lambda(z^1, z^2) = \left(z_x^1, z_x^2\right) + \left(cz_x^1 + (a(x) + \lambda)z^1, z^2\right)$$

for any $c \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C}_+$. Since

$$\operatorname{Re} B_c^\lambda(z, z) \geq \|z_x\|_Y^2 + \left(\min_{x\in\boldsymbol{R}} a(x)\right)\|z\|_Y^2,$$

the Lax-Milgram theorem implies that the following inverse operator

$$\widehat{K}(\lambda, c) \stackrel{\text{def}}{=} (-D_{xx} + cD_x + a(x) + \lambda)^{-1} \tag{2.16}$$

belongs to $\mathcal{L}(X', X)$ for any $c \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C}_+$. $\widehat{K}(\lambda, c)$ is uniformly bounded in $\mathcal{L}(X', X)$ for all $c \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C}_+$. We put

$$\widehat{z}(x; \lambda, c) = \widehat{K}(\lambda, c)\delta, \tag{2.17}$$

that is, $\widehat{z}(x; \lambda, c) \in X = H^1(\boldsymbol{R})$ is uniquely determined by

$$\begin{aligned} -\widehat{z}_{xx}(x; \lambda, c) + c\widehat{z}_x(x; \lambda, c) + (a(x) + \lambda)\widehat{z}(x; \lambda, c) &= 0 \quad \text{in } \boldsymbol{R} \setminus \{0\}, \\ -\left[\widehat{z}_x(x; \lambda, c)\right]_{x=-0}^{x=+0} &= 1. \end{aligned}$$

Define $p^\varepsilon \stackrel{\text{def}}{=} (\phi(x, \varepsilon), \psi(x, \varepsilon))$. From Lemma 2.4, $(\phi(x, \varepsilon), \psi(x, \varepsilon)) = \int_{-\infty}^{\infty} \widehat{\phi}(y, \varepsilon)\widehat{\psi}(y, \varepsilon)\, dy$ goes to $p_0 = p_1p_2p_3 > 0$ as $\varepsilon \to 0$. Thus $\lim_{\varepsilon\to 0} p^\varepsilon = p_0$ holds true.

The stability of $(u_0(z + ct; \varepsilon), v_0(z + ct; \varepsilon))$ as a solution of (2.1) is as follows.

THEOREM 2.2 ([11],[7]). *Assume (A1)–(A4) and (2.8). For sufficiently small* $\varepsilon > 0$, $(u(z, t), v(z, t)) = (u_0(z + ct; \varepsilon), v_0(z + ct; \varepsilon))$ *of (2.1) is asymptotically stable. The condition (2.8) is equivalent to*

$$p_0\tau > k_1k_2\left\|\widehat{z}(x; 4^{-1}(c^*)^2, 0)\right\|_Y^2. \tag{2.18}$$

In this paper, the stability condition (2.18) is our standing assumption.

3. **Equations for the eigenvalues.** We study the stability of a planar traveling wave solution (1.5) by the linearized eigenvalue problem (1.9). Eigenvalues that concern with stability are those in $\boldsymbol{C}_+ = \{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \geq 0\}$. Thus it suffices to assume $\lambda \in \boldsymbol{C}_+$ without loss of generality. Note that $\varepsilon^2\kappa_n + \varepsilon\tau\lambda \in \boldsymbol{C}_+$ is valid. We obtain a scalar equation for the eigenvalues in this section. Denoting x_1 simply by x and (w_n, z_n) by (w, z), we write (1.9) as

$$(L(\varepsilon, c^\varepsilon) + \varepsilon^2\kappa_n + \varepsilon\tau\lambda)w(x) = f_v^0(x;\varepsilon)z(x) \tag{3.1}$$

$$-z_{xx}(x) + c^\varepsilon z_x(x) + \left(-g_v^0(x;\varepsilon) + \kappa_n + \lambda\right) z(x) - g_u^0(x;\varepsilon)w(x) = 0 \tag{3.2}$$

for $x \in \boldsymbol{R}$. We put

$$(L(\varepsilon, c^\varepsilon) + \lambda)^{-1} Q^\varepsilon = (-f_u(U_0, V_0) + \lambda)^{-1} + r^\varepsilon(\lambda).$$

As in §2, $L(\varepsilon, c^\varepsilon)^{-1}Q^\varepsilon$ satisfies necessary conditions of [15]. Then the following uniform convergence is valid.

LEMMA 3.1 ([15]). *Fix a constant $s \in (0, \frac{1}{2})$ arbitrarily. For sufficiently small $\varepsilon > 0$ and all $\lambda \in \boldsymbol{C}_+$, $(L(\varepsilon, c^\varepsilon) + \lambda)^{-1}Q^\varepsilon$ is uniformly bounded in $\mathcal{L}(Y)$. The following convergence*

$$\lim_{\varepsilon\to 0}(1 + |\lambda|)\|r^\varepsilon(\lambda)\|_{\mathcal{L}(Y, H^{-s}(\boldsymbol{R}))} = 0$$

holds true, where the convergence is uniform in $\lambda \in \boldsymbol{C}_+$.

Define bounded linear operators in Y as

$$S_n(\varepsilon, \kappa, \lambda) \stackrel{\text{def}}{=} -\, g_u^0(x;\varepsilon)\left[(L(\varepsilon, c^\varepsilon) + \varepsilon^2\kappa + \varepsilon\tau\lambda)^{-1}Q^\varepsilon\right]^n \left(f_v^0(x;\varepsilon)\,\cdot\,\right)$$

for $n \in \boldsymbol{N}$, and define $R(\varepsilon, \kappa, \lambda) \stackrel{\text{def}}{=} -g_v^0(x;\varepsilon) + S_1(\varepsilon, \kappa, \lambda)$. Immediately from Lemma 3.1, there exists a constant $B_0 > 0$ with

$$\|R(\varepsilon, \kappa, \lambda)\|_{\mathcal{L}(Y)} < B_0 \tag{3.3}$$

for every $\varepsilon \in (0, \varepsilon_0)$, $\kappa \geq 0$ and $\lambda \in \boldsymbol{C}_+$. From Lemma 3.1, the following convergence

$$\lim_{\varepsilon\to 0} g_u^0(x;\varepsilon)r^\varepsilon(\varepsilon^2\kappa + \varepsilon\tau\lambda)\left(f_v^0(x;\varepsilon)\,\cdot\,\right) = 0 \qquad \text{in } \mathcal{L}(X, X') \tag{3.4}$$

holds true, where the convergence is uniform in $\kappa \in [0, \infty)$ and $\lambda \in \boldsymbol{C}_+$. From (3.4) and the definition of $R(\varepsilon, \kappa, \lambda)$, we have

$$\begin{aligned} &R(\varepsilon, \kappa, \lambda) = -g_v(U_0, V_0) \\ &- g_u(U_0, V_0)(-f_u(U_0, V_0) + \varepsilon^2\kappa + \varepsilon\tau\lambda)^{-1}\left(f_v(U_0, V_0)\,\cdot\,\right) + m^\varepsilon(\varepsilon^2\kappa + \varepsilon\tau\lambda). \end{aligned} \tag{3.5}$$

Here the residual term $m^\varepsilon(\lambda)$ satisfies $\lim_{\varepsilon\to 0}\|m^\varepsilon(\lambda)\|_{\mathcal{L}(X,X')} = 0$, where the convergence is uniform in $\lambda \in \boldsymbol{C}_+$.

Define a quasilinear form on $X = H^1(\boldsymbol{R})$ by

$$B^{\varepsilon,\kappa,\lambda}(z^1, z^2) = (z_x^1, z_x^2) + c^\varepsilon(z_x^1, z^2) + \left((R(\varepsilon, \kappa, \lambda) + \kappa + \lambda)z^1, z^2\right) \tag{3.6}$$

for z^1, $z^2 \in X$.

Define

$$K(\varepsilon, \kappa, \lambda) \stackrel{\text{def}}{=} \left(-D_{xx} + c^\varepsilon D_x + R(\varepsilon, \kappa, \lambda) + \kappa + \lambda\right)^{-1},$$

if it exists. Using (3.5) and the assumption (A4), we obtain

$$\operatorname{Re} B^{\varepsilon,\kappa,\lambda}(z, z) \geq (\kappa + \lambda_R + \alpha_0)\|z\|_X^2$$

for $z \in X$, $\varepsilon \in (0, \varepsilon_0)$, $0 \leq \kappa < \infty$ and $\lambda = \lambda_R + i\lambda_I \in \boldsymbol{C}_+$. Here $\alpha_0 > 0$ is a constant that is independent of ε, κ and λ. Thus we get

$$\|K(\varepsilon, \kappa, \lambda)\|_{\mathcal{L}(X', X)} \leq (\kappa + \lambda_R + \alpha_0)^{-1} \tag{3.7}$$

from the Lax-Milgram theorem.

LEMMA 3.2 ([15]). *For every $\varepsilon \in (0, \varepsilon_0)$ and $\kappa \in [0, \infty)$, $\lambda \in \boldsymbol{C}_+$ is an eigenvalue of (3.1)–(3.2) if and only if*

$$F(\varepsilon, \kappa_n, \lambda) = 0, \tag{3.8}$$

where $F(\varepsilon, \kappa, \lambda) = p^\varepsilon(\widehat{\zeta}(\varepsilon) - \varepsilon\kappa - \tau\lambda) - (K(\varepsilon, \kappa, \lambda)h_2(x, \varepsilon), h_1(x, \varepsilon))$. In this case, the eigenfunctions are given by

$$\begin{aligned} z(x) &= K(\varepsilon, \kappa_n, \lambda)h_2(x, \varepsilon), \\ w(x) &= \varepsilon^{-\frac{1}{2}}\phi(x, \varepsilon) + \left(L(\varepsilon, c^\varepsilon) - \varepsilon^2\kappa_n - \varepsilon\tau\lambda\right)^{-1} Q^\varepsilon(-f_v^0(x; \varepsilon)z). \end{aligned}$$

REMARK 3.1. *From this lemma, $(w_n(x_1), z_n(x_1))$ in Theorem 1.3 is given by*

$$\begin{aligned} &z_n(x_1) = K(\varepsilon, \kappa_n, \lambda_{\max}(\varepsilon))h_2(x_1, \varepsilon), \\ &w_n(x_1) = \varepsilon^{-\frac{1}{2}}\phi(x_1, \varepsilon) + \left(L(\varepsilon, c^\varepsilon) - \varepsilon^2\kappa_n - \varepsilon\tau\lambda_{\max}(\varepsilon)\right)^{-1} Q^\varepsilon(-f_v^0(x_1; \varepsilon)z_n). \end{aligned} \tag{3.9}$$

The following is a priori bound for $\lambda \in \boldsymbol{C}_+$.

LEMMA 3.3. *If $\lambda \in \boldsymbol{C}_+$ is an eigenvalue of (3.1)–(3.2), then $|\lambda|$ is bounded uniformly in ε and κ.*

Proof. From (3.8) we have

$$\lambda < \tau^{-1}\widehat{\zeta}(\varepsilon) - (p^\varepsilon\tau)^{-1}\left(K(\varepsilon, \kappa, \lambda)h_2(x, \varepsilon), h_1(x, \varepsilon)\right)$$

and thus $|\lambda| < \tau^{-1}\widehat{\zeta}(\varepsilon) + (p^\varepsilon\tau)^{-1}\|K(\varepsilon, \kappa, \lambda)h_2(x, \varepsilon)\|_X\|h_1(x, \varepsilon)\|_{X'}$. Note that the right-hand side remains bounded uniformly in ε, κ and λ from (3.7) and Lemma 2.4. This completes the proof. □

From Lemma 3.3, there exists $B_1 > 0$ so that $|\lambda| < B_1$ holds true if $\lambda \in \boldsymbol{C}_+$ satisfies (3.1)–(3.2) with any $\varepsilon \in (0, \varepsilon_0)$, $\kappa \in [0, \infty)$. Using Lemma 3.3 and (3.5), we have

$$\lim_{\varepsilon \to 0} K(\varepsilon, \kappa, \lambda) = \widehat{K}(\kappa + \lambda, c^*) \qquad \text{in } \mathcal{L}(X', X), \tag{3.10}$$

where the convergence is uniform in $\kappa \in [0, \infty)$. Here $\widehat{K}(\kappa + \lambda, c^*)$ is as in (2.16).

Define

$$P(\kappa, c) \stackrel{\text{def}}{=} (-D_{xx} + cD_x + \kappa)^{-1} \quad \text{in } \mathcal{L}(X', X)$$

for any $c \in \boldsymbol{R}$. We put $\mu = \kappa^{\frac{1}{2}}$.

LEMMA 3.4. *For every $\mu > 1$ and $c \in \boldsymbol{R}$, inequalities*

$$\left\|P(\mu^2, c)\right\|_{\mathcal{L}(X', Y)} < \mu^{-1}, \quad \left\|P(\mu^2, c)\right\|_{\mathcal{L}(Y)} \leq \mu^{-2}$$

hold true.

Proof. Assume $-z_{xx} + cz_x + \mu^2 z = h$ for $z \in X$ and $h \in X'$. Multiplying both hands by $\overline{z}$ and integrating the real parts, we have

$$\|z_x\|_Y^2 + \mu^2 \|z\|_Y^2 = \int_{-\infty}^{\infty} h\overline{z}\, dx.$$

Using this relation and

$$\left| \int_{-\infty}^{+\infty} h\overline{z}\, dx \right| \le \|h\|_{X'} \|z\|_X \le \frac{1}{2}\|h\|_{X'}^2 + \frac{1}{2}\|z\|_X^2,$$

we obtain $\left(\mu^2 - \frac{1}{2}\right) \|z\|_Y^2 \le \frac{1}{2}\|h\|_{X'}^2$. Using this and $\mu > 1$, we obtain the first inequality. If $h \in Y$, then we have the Schwarz inequality $|(h, z)| \le \|h\|_Y \|z\|_Y$, and $\mu^2 \|z\|_Y^2 \le \|h\|_Y \|z\|_Y$. Thus we obtain the second inequality. □

Lemma 3.5. *Put $\mu = \kappa^{\frac{1}{2}}$. There exists $B_2 > 0$ such that, if $\mu > B_2$, $K(\varepsilon, \kappa, \lambda)$ can be written as*

$$K(\varepsilon, \kappa, \lambda) = P(\mu^2, c^\varepsilon) + P(\mu^2, c^\varepsilon) Q(\varepsilon, \kappa, \lambda) P(\mu^2, c^\varepsilon)$$

for all $\mu > B_2$ and all $\lambda \in \boldsymbol{C}_+$ with $|\lambda| < B_1$. Here

$$Q(\varepsilon, \kappa, \lambda) \stackrel{\text{def}}{=} -\left(R(\varepsilon, \kappa, \lambda) + \lambda\right) \sum_{k=1}^{\infty} \left[-P(\mu^2, c^\varepsilon)(R(\varepsilon, \kappa, \lambda) + \lambda)\right]^{k-1} \in \mathcal{L}(Y),$$

satisfies $\|Q(\varepsilon, \kappa, \lambda)\|_{\mathcal{L}(Y)} < B_3$, where B_3 a constant independent of ε, κ, λ and μ.

Proof. We denote $R(\varepsilon, \kappa, \lambda)$, $P(\mu, c^\varepsilon)$ simply by R, P. If $\|P(R + \lambda)\|_{\mathcal{L}(Y)} < 1$, then we have $K = [I + P(R + \lambda)]^{-1} P = \sum_{k=0}^{\infty} [-P(R + \lambda)]^k P$. By virtue of Lemma 3.4, we have $\|P(R + \lambda)\|_{\mathcal{L}(Y)} < 1$, if μ is large enough, say, $\mu > B_2$. This completes the proof. □

Combining Lemmas 3.4 and 3.5 we obtain

$$\|K(\varepsilon, \kappa, \lambda)\|_{\mathcal{L}(X', Y)} < \mu^{-1} + B_3\, \mu^{-3}, \tag{3.11}$$

$$\|K(\varepsilon, \kappa, \lambda)\|_{\mathcal{L}(Y)} < \mu^{-2} + B_3\, \mu^{-4}, \tag{3.12}$$

if $\mu > B_2$. We write $K(\varepsilon, \kappa, \lambda)$, $S_n(\varepsilon, \kappa, \lambda)$ simply as K, S_n, respectively.

Lemma 3.6. *$K(\varepsilon, \kappa, \lambda)$ is real-analytic for $\kappa > 0$ and is analytic for $\lambda = \lambda_R + i\lambda_I$ with $\lambda_R > 0$ in the space $\mathcal{L}(X', X)$ with*

$$\begin{aligned}
K_\lambda &= -K(I + \varepsilon\tau S_2)K, \\
K_\kappa &= -K(I + \varepsilon^2 S_2)K, \\
K_{\lambda\lambda} &= 2K(I + \varepsilon\tau S_2)K(I + \varepsilon\tau S_2)K - 2(\varepsilon\tau)^2 K S_3 K, \\
K_{\kappa\lambda} &= 2K(I + \varepsilon^2 S_2)K(I + \varepsilon\tau S_2)K - 2\varepsilon^3 \tau K S_3 K, \\
K_{\kappa\kappa} &= 2K(I + \varepsilon^2 S_2)K(I + \varepsilon^2 S_2)K - 2\varepsilon^4 K S_3 K.
\end{aligned}$$

Note the right-hand sides belong to $\mathcal{L}(X', X)$. Moreover $\widehat{K}(\lambda, c)$ is analytic for λ with $\lambda_R > 0$ in $\mathcal{L}(X', X)$ and satisfies $\left(\widehat{K}(\lambda, c)\right)_\lambda = -\widehat{K}(\lambda, c)^2$, $\left(\widehat{K}(\lambda, c)\right)_{\lambda\lambda} = 2\widehat{K}(\lambda, c)^3$ in $\mathcal{L}(X', X)$. Here $c \in \boldsymbol{R}$ is any fixed number.

Proof. The proof can be done by the same argument as in Lemma 3.1 in [10]. We omit it. □

Using this lemma, we can differentiate $F(\varepsilon, \kappa, \lambda)$ and obtain

$$\begin{aligned}
F_\lambda(\varepsilon, \kappa, \lambda) &= -p^\varepsilon \tau + (K(I + \varepsilon\tau S_2)Kh_2, h_1), && (3.13)\\
F_\kappa(\varepsilon, \kappa, \lambda) &= -\varepsilon p^\varepsilon + \big(K(I + \varepsilon^2 S_2)Kh_2, h_1\big). && (3.14)\\
F_{\kappa\kappa}(\varepsilon, \kappa, \lambda) &= -2\big(K(I + \varepsilon^2 S_2)K(I + \varepsilon^2 S_2)Kh_2, h_1\big) \\
&\quad +2\varepsilon^4 (KS_3Kh_2, h_1) && (3.15)\\
F_{\kappa\lambda}(\varepsilon, \kappa, \lambda) &= -2\big(K(I + \varepsilon^2 S_2)K(I + \varepsilon\tau S_2)Kh_2, h_1\big) \\
&\quad +2\varepsilon^3\tau (KS_3Kh_2, h_1) && (3.16)
\end{aligned}$$

for $\kappa > 0$ and $\lambda = \lambda_R + i\lambda_I$ with $\lambda_R > 0$. Here we denote $h_1(x, \varepsilon)$, $h_2(x, \varepsilon)$ simply by h_1, h_2. From Lemma 3.5, we have

$$\begin{aligned}
& |(K(I + \nu_0 S_2)Kh_2, h_1)| \\
= \;& |((I + QP(\mu, c^\varepsilon))(I + \nu_0 S_2)Kh_2, P(\mu, -c^\varepsilon)h_1)| \\
\le \;& \|I + QP(\mu, c^\varepsilon)\|_{\mathcal{L}(Y)} \|I + \nu_0 S_2\|_{\mathcal{L}(Y)} \|K\|_{\mathcal{L}(X',Y)} \|h_2\|_{X'} \\
& \times \|P(\mu, -c^\varepsilon)\|_{\mathcal{L}(X',Y)} \|h_1\|_{X'}, \qquad (3.17)
\end{aligned}$$

where ν_0 represents ε^2 or $\varepsilon\tau$. The right-hand side of (3.17) is of order $O(\mu^{-2})$ as $\mu \to +\infty$ uniformly in (ε, κ). From (3.7), we obtain

$$|(K(\varepsilon, \kappa, 0)h_2, h_1)| < (\kappa + \alpha_0)^{-1} \|h_1\|_{X'} \|h_2\|_{X'}.$$

We fix $\sigma_0 \in (\frac{1}{2}\widehat{\zeta}(0), \widehat{\zeta}(0))$ arbitrarily. Then for every $\lambda \in \boldsymbol{C}_+$ with $|\lambda| < B_1$ we have

$$\operatorname{Re} F_\lambda(\varepsilon, \kappa, \lambda) = -p^\varepsilon \tau + \operatorname{Re}\, (K(I + \varepsilon\tau S_2)Kh_2, h_1) < 0, \qquad (3.18)$$

$$p^\varepsilon \widehat{\zeta}(\varepsilon) - (K(\varepsilon, \kappa, 0)h_2, h_1) > p^\varepsilon \sigma_0 > 0, \qquad (3.19)$$

if $\kappa = \mu^2$ is large enough, say, if $\kappa > M_1$.

The Green function of $-(d/dx)^2 + \nu^2$ is given by $H(x, \xi; \nu) = (2\nu)^{-1} \exp(-\nu|x - \xi|)$ for every $\nu > 0$.

LEMMA 3.7. *For sufficiently small $\varepsilon > 0$, all $\kappa \in [0, \infty)$ and $\lambda \in \boldsymbol{C}_+$ with $|\lambda| < B_1$,*

$$\operatorname{Re} F_\lambda(\varepsilon, \kappa, \lambda) < -b < 0 \qquad (3.20)$$

holds true. Here b is a positive constant independent of ε, κ and λ.

Proof. It suffices to prove the lemma assuming $0 \le \kappa \le M_1$. From (3.13) and (3.10), the following convergence

$$\lim_{\varepsilon \to 0} F_\lambda(\varepsilon, \kappa, \lambda) = -p_0\tau + k_1k_2 \langle \widehat{K}(\kappa + \lambda, c^*)\delta, \delta\rangle$$

is valid uniformly in $\kappa \in [0, M_1]$ and $\lambda \in \boldsymbol{C}_+$ with $|\lambda| < B_1$. The bracket $\langle \cdot\,, \cdot \rangle$ denotes the scalar product between X and X'. Note that $\langle \widehat{K}(\kappa + \lambda, c^*)\delta, \delta\rangle$ is nothing but $\widehat{z}(0; \kappa + \lambda, c^*)$. It suffices to prove the real part of the right-hand side is negative. Define

$$G(\lambda_R, \lambda_I) = \operatorname{Re} \langle \widehat{K}(\lambda, c^*)^2 \delta, \delta\rangle.$$

Then, by virtue of (2.18), it suffices to prove

$$G(\lambda_R, \lambda_I) \le G(0, 0) \qquad \text{for all } \lambda_R + i\lambda_I \in \boldsymbol{C}_+. \qquad (3.21)$$

Using

$$\widehat{K}(\lambda, c^*) = \exp(2^{-1}c^*x)\widehat{K}(\lambda + 4^{-1}(c^*)^2, 0)\left(\,\cdot\,\exp(2^{-1}c^*x)\right) \quad \text{for } \lambda \in \boldsymbol{C}_+.$$

we obtain

$$\langle \widehat{K}(\lambda, c^*)^2\delta, \delta\rangle = \langle \widehat{K}(\lambda + 4^{-1}(c^*)^2, 0)^2\delta, \delta\rangle.$$

Define

$$A(\lambda_R, \lambda_I) =$$
$$\left(-D_{xx} + a(x) + 4^{-1}(c^*)^2 + \lambda_R - i\lambda_I\right)^{-1}\left(-D_{xx} + a(x) + 4^{-1}(c^*)^2 + \lambda_R + i\lambda_I\right)^{-1}.$$

Then $A(\lambda_R, \lambda_I)$ belongs to $\mathcal{L}(X', X)$ and satisfies

$$\begin{aligned} \frac{\partial}{\partial \lambda_R}A(\lambda_R, \lambda_I) &= -2A(\lambda_R, \lambda_I)\widehat{T}(4^{-1}(c^*)^2 + \lambda_R, 0)A(\lambda_R, \lambda_I) \\ \frac{\partial}{\partial \lambda_I}A(\lambda_R, \lambda_I) &= -2\lambda_I A(\lambda_R, \lambda_I)^2 \end{aligned}$$

in $\mathcal{L}(X', X)$. Using

$$\begin{aligned} \widehat{K}(\lambda, 0) &= \widehat{T}(\lambda, 0)A(\lambda_R, \lambda_I) - 2i\lambda_I A(\lambda_R, \lambda_I) \\ \widehat{K}(\lambda, 0)^2 &= A(\lambda_R, \lambda_I) - 2i\lambda_I \widehat{K}(\lambda_R, 0)A(\lambda_R, \lambda_I)^2 \end{aligned}$$

we obtain

$$G(\lambda_R, \lambda_I) = \langle A(\lambda_R + 4^{-1}(c^*)^2, \lambda_I)\delta, \delta\rangle$$

and thus

$$\begin{aligned} \frac{\partial}{\partial \lambda_R}G(\lambda_R, \lambda_I) &= -2B_0^{\lambda}(A(\kappa + \lambda_R, \lambda_I)\delta, A(\kappa + \lambda_R, \lambda_I)\delta,)\big|_{\lambda = 4^{-1}(c^*)^2 + \kappa + \lambda_R} \le 0 \\ \lambda_I \frac{\partial}{\partial \lambda_I}G(\lambda_R, \lambda_I) &= -2(\lambda_I)^2 \left\| A(\kappa + \lambda_R), \lambda_I \right\|_Y^2 \le 0. \end{aligned}$$

Now we get (3.21) and complete the proof. □

Lemma 3.8. *For every* $\nu > 1$, $\|P(\nu^2, 0)\|_{\mathcal{L}(Y, L^\infty(\boldsymbol{R}))} < \frac{1}{2}\nu^{-\frac{3}{2}}$ *holds true.*

Proof. Let $z = P(\nu^2, 0)h$ for $h \in Y$. Then we obtain $z(x) = \int_{-\infty}^{+\infty} H(x, \xi; \nu)h(\xi)\, d\xi$, and thus $|z(x)| \le \|H(x, \cdot\,; \nu)\|_Y \|h\|_Y$. Direct verification yields

$$\|H(x, \cdot\,; \nu)\|_Y = \left(\int_{-\infty}^{+\infty} H(x, \xi; \nu)^2\, d\xi\right)^{\frac{1}{2}} = \frac{1}{2}\nu^{-\frac{3}{2}}.$$

We obtain $\|z\|_{L^\infty(\boldsymbol{R})} \le \frac{1}{2}\nu^{-\frac{3}{2}}\|h\|_Y$. This completes the proof. □

Lemma 3.9. *For sufficiently small* $\varepsilon > 0$, *there exists a real-valued function* $\Lambda(\kappa)$ *defined for* $\kappa \in [0, \sigma(\varepsilon)\varepsilon^{-1}]$ *with* $\Lambda(0) = 0$, $\Lambda(\sigma(\varepsilon)\varepsilon^{-1}) = 0$ *and* $\Lambda(\kappa) > 0$ *for every* $\kappa \in (0, \sigma(\varepsilon)\varepsilon^{-1})$. *If* $\kappa_n \in [0, \sigma(\varepsilon)\varepsilon^{-1}]$, *(1.9) has a unique eigenvalue* $\Lambda(\kappa_n)$ *in* $\boldsymbol{C}_+$. *If* $\kappa_n \notin [0, \sigma(\varepsilon)\varepsilon^{-1}]$, *(1.9) has no eigenvalues in* $\boldsymbol{C}_+$.

Proof. First we show $F(\varepsilon, \kappa, 0) \ge 0$ for every $0 \le \kappa < M_1$. Differentiating (1.3) by x_1, we see that (1.9) has zero eigenvalue associated with $(\frac{\partial u_0}{\partial x_1}(x_1), \frac{\partial v_0}{\partial x_1}(x_1))$. Combining this fact and Lemma 3.2, we have $F(\varepsilon, 0, 0) = 0$. Using (3.14), we obtain

$$\lim_{\varepsilon \to 0} F_\kappa(\varepsilon, \kappa, 0) = k_1 k_2 \langle \widehat{K}(\kappa, c^*)\delta, \delta\rangle > 0.$$

Therefore $F(\varepsilon,\kappa,0)>0$ if $\kappa\in(0,M_1)$. By the definition of $F(\varepsilon,\kappa,\lambda)$, we have $F(\varepsilon,\kappa,+\infty)=-\infty$. Since $F_\lambda(\varepsilon,\kappa,\lambda)<0$ holds, there exists a unique λ, which we denote by $\Lambda(\kappa)$, that satisfies $F(\varepsilon,\kappa,\lambda)=0$ for $\lambda\in[0,M_1)$.

Next we search for $\kappa\in[M_1,+\infty)$ with

$$F(\varepsilon,\kappa,0)=p^\varepsilon\left(\widehat{\zeta}(\varepsilon)-\varepsilon\kappa\right)-(K(\varepsilon,\kappa,0)h_2,h_1)=0. \tag{3.22}$$

Then from (3.19) it is necessary that $\varepsilon\kappa>\sigma_0$ holds true. For any $\sigma>0$, we have

$$F(\varepsilon,\sigma\varepsilon^{-1},0)=p^\varepsilon(\widehat{\zeta}(\varepsilon)-\sigma)-\left(K(\varepsilon,\sigma\varepsilon^{-1},0)h_2,h_1\right).$$

For $\varepsilon>0$ that is small enough, we have

$$F(\varepsilon,\sigma\varepsilon^{-1},0)\big|_{\sigma=2\widehat{\zeta}(0)}<0<F(\varepsilon,\sigma\varepsilon^{-1},0)\big|_{\sigma=\frac12\widehat{\zeta}(0)}.$$

Hence there exists at least one $\sigma(\varepsilon)\in[\frac12\widehat{\zeta}(0),2\widehat{\zeta}(0)]$ with $F(\varepsilon,\sigma(\varepsilon)\varepsilon^{-1},0)=0$. We will show that $\sigma(\varepsilon)$ is unique. For this purpose, we show

$$\operatorname{Re}F_\kappa(\varepsilon,\kappa,\lambda)=-\varepsilon p^\varepsilon+\operatorname{Re}\left(K(I+\varepsilon^2S_2)Kh_2,h_1\right)<0 \tag{3.23}$$

with every $\kappa>\frac12\widehat{\zeta}(0)\varepsilon^{-1}$ and $\lambda\in\boldsymbol{C}_+$ with $|\lambda|<B_1$ if ε is small enough. Indeed, we have

$$\begin{aligned}\left(K(I+\varepsilon^2S_2)Kh_2,h_1\right)&=\left(P(\kappa,c)(I+QP(\kappa,c))(I+\varepsilon^2S_2)Kh_2,h_1\right)\\&=(P(\kappa,c)(I+QP(\kappa,c))Kh_2,h_1)\\&\quad+\varepsilon^2\left(P(\kappa,c)(I+QP(\kappa,c))S_2Kh_2,h_1\right).\end{aligned}$$

The second term of the right-hand side is of order $O(\varepsilon^2)$, and the first term equals

$$\begin{aligned}&(P(\kappa,c)(I+QP(\kappa,c))Kh_2,h_1)=((I+QP(\kappa,c))Kh_2,P(\kappa,-c)h_1)\\=\;&(P(\kappa,c)h_2,P(\kappa,-c)h_1)\\&+\left(QP(\kappa,c)^2(I+QP(\kappa,c))h_2+P(\kappa,c)QP(\kappa,c)h_2,P(\kappa,-c)h_1\right).\end{aligned}$$

The second term is of order $O(\kappa^{-2})$ as $\kappa\to+\infty$, and is of order $O(\varepsilon^2)$ as $\varepsilon\to0$ when $\kappa>\frac12\widehat{\zeta}(0)\varepsilon^{-1}$. As for the first term, we have

$$(P(\kappa,c)h_2,P(\kappa,-c)h_1)=\left(P(\nu^2,0)f_2,P(\nu^2,0)f_1\right)=\left(P(\nu^2,0)^2f_2,f_1\right). \tag{3.24}$$

with $\nu=\left(\kappa+(c^\varepsilon)^2/4\right)^{\frac12}$. From Lemmas 3.4 and 3.8, we obtain

$$\begin{aligned}\left\|P(\nu^2,0)^2f_2\right\|_{L^\infty(\boldsymbol{R})}&\le\left\|P(\nu^2,0)\right\|_{\mathcal{L}(Y,L^\infty(\boldsymbol{R}))}\left\|P(\nu^2,0)\right\|_{\mathcal{L}(X',Y)}\|f_2\|_{X'}\\&\le\frac12\nu^{-\frac52}\|f_2\|_{X'}.\end{aligned}$$

Combining this inequality and (2.15), we have

$$\left|(P(\nu^2,0)^2f_2,f_1)\right|\le\frac{b_2}{\alpha_2}\nu^{-\frac52}\|f_2\|_{X'}.$$

Using $\nu=\left(\kappa+(c^\varepsilon)^2/4\right)^{\frac12}$ and $\kappa>\frac12\widehat{\zeta}(0)\varepsilon^{-1}$, we obtain $|(P(\nu^2,0)^2f_2,f_1)|$ is at most of order $O(\varepsilon^{\frac54})$ as $\varepsilon\to0$. Thus we get (3.23). Therefore $\sigma(\varepsilon)$ is uniquely determined and it satisfies $\lim_{\varepsilon\to0}\sigma(\varepsilon)=\widehat{\zeta}(0)$.

From (3.23), we have $F(\varepsilon,\kappa,0)>0$ if $\kappa\in(0,\sigma(\varepsilon)\varepsilon^{-1})$. If λ is real, we obtain $\lim_{\lambda\to+\infty}F(\varepsilon,\kappa,\lambda)=-\infty$ for every $\kappa\ge0$ by virtue of (3.7) and (3.8). Thus there exists a real value $\Lambda(\kappa)>0$ such that $F(\varepsilon,\kappa,\Lambda(\kappa))=0$ holds for every $\kappa\in(0,\sigma(\varepsilon)\varepsilon^{-1})$. We put $\Lambda(0)=0$, $\Lambda(\sigma(\varepsilon)\varepsilon^{-1})=0$, and then $\Lambda(\kappa)$ becomes a continuous function for $\kappa \in [0, \sigma(\varepsilon)\varepsilon^{-1}]$. If $\lambda \in \boldsymbol{C}_+$ satisfies (3.8), then $\lambda = \Lambda(\kappa)$. Indeed, using

$$\begin{aligned} 0 &= F(\varepsilon,\kappa,\lambda) = F(\varepsilon,\kappa,\lambda) - F(\varepsilon,\kappa,\Lambda(\kappa)) \\ &= F_\lambda(\varepsilon,\kappa,\theta_0\lambda + (1-\theta_0)\Lambda(\kappa))(\lambda - \Lambda(\kappa)) \qquad (0 < \theta_0 < 1) \end{aligned}$$

and Lemma 3.7, we obtain $\lambda = \Lambda(\kappa)$.

Finally we show that, if $\kappa > \sigma(\varepsilon)\varepsilon^{-1}$, then there exists no $\lambda \in \boldsymbol{C}_+$ with (3.8). Otherwise there exists $\lambda \in \boldsymbol{C}_+$ with (3.8). Then from (3.23), we have $\operatorname{Re} F(\varepsilon, \sigma(\varepsilon)\varepsilon^{-1}, \lambda) > 0$, which contradicts $F(\varepsilon, \sigma(\varepsilon)\varepsilon^{-1}, 0) = 0$ and (3.18). We complete the proof of Lemma 3.9. □

4. **The distribution of the eigenvalues and the proof of main theorems.** In this section, we will show $\Lambda(\,\cdot\,)$ has a unique maximizer $\kappa(\varepsilon)$ in $(0, \sigma(\varepsilon)\varepsilon^{-1})$, and we study the asymptotic behavior of $\kappa(\varepsilon)$ as $\varepsilon \to 0$. From Lemma 3.9, there exists at least one relative maximizer of $\Lambda(\,\cdot\,)$ in $(0, \sigma(\varepsilon)\varepsilon^{-1})$ for each fixed small $\varepsilon > 0$. Let $\kappa(\varepsilon)$ be any one of them. Later we will show that $\kappa(\varepsilon)$ is uniquely determined.

Fix $\theta \in (\frac{1}{2}, 1)$ arbitrarily. We put $\boldsymbol{R} = I(\varepsilon) \cup J(\varepsilon)$ with $I(\varepsilon) = (-\varepsilon^\theta, \varepsilon^\theta)$ and $J(\varepsilon) = \boldsymbol{R} \setminus I(\varepsilon)$. We put $k_j(\varepsilon) = \int_{I(\varepsilon)} f_j(x,\varepsilon)\,dx$ for $j = 1, 2$. From Lemmas 2.4 and 2.5, we have

$$\lim_{\varepsilon\to 0} k_j(\varepsilon) = k_j \quad \text{for } j = 1,2. \tag{4.1}$$

We show that $k_j H(x, 0; \nu)$ is the principal term of $P(\nu^2, 0) f_j$.

Lemma 4.1. *Fix $\sigma_1 > 0$ and $\sigma_2 > 0$ arbitrarily. There exists $M_2 > 0$ such that the following equalities*

$$P(\nu^2, 0) f_j = k_j(\varepsilon) H(x, 0; \nu) + r_j(x, \varepsilon, \nu) \qquad (j = 1, 2)$$

hold true for sufficiently small $\varepsilon > 0$ and $\nu \in (\sigma_1, \sigma_2\varepsilon^{-\frac{1}{2}})$ with

$$|r_j(x, \varepsilon, \kappa)| < M_2 \varepsilon^\theta \exp(-\nu|x|).$$

Thus $r_j(x,\varepsilon,\nu)$ satisfies $\|r_j(\,\cdot\,;\varepsilon,\nu)\|_Y \le M_2\varepsilon^\theta\nu^{-\frac{1}{2}}$. Here M_2 is independent of ε and ν.

Proof. We have

$$P(\nu^2, 0) f_j = \int_{I(\varepsilon)\cup J(\varepsilon)} H(x, \xi; \nu) f_j(\xi)\, d\xi.$$

For $x \in \boldsymbol{R}$, $\xi \in I(\varepsilon)$, we have $||x - \xi| - |x|| \le |\xi| < \varepsilon^\theta$ and

$$\begin{aligned} \exp(\nu|x| - \nu|x-\xi|) - 1 &= \nu(|x| - |x-\xi|)\exp(\omega\nu(|x| - |x-\xi|)) \\ &< \nu\varepsilon^\theta \exp(\omega\nu\varepsilon^\theta) \end{aligned}$$

with $0 < \omega < 1$. Thus we have

$$\left|\nu^{-1}\left(\exp(\nu|x| - \nu|x-\xi|) - 1\right)\right| < \varepsilon^\theta \exp(\omega\nu\varepsilon^\theta).$$

Using $0 \le \limsup_{\varepsilon\to 0} \nu\varepsilon^\theta \le \lim_{\varepsilon\to 0} \sigma_2\varepsilon^{\theta - 1/2} = 0$, we get

$$|H(x,\xi;\nu) - H(x,0;\nu)| < M_3\varepsilon^\theta \exp(-\nu|x|)$$

for $x \in \boldsymbol{R}$, $\xi \in I(\varepsilon)$. Here $M_3 > 0$ is a positive constant independent of ε, ν, x and ξ. Integrating

$$H(x,0;\nu) - M_3\varepsilon^\theta e^{-\nu|x|} < H(x,\xi;\nu) < H(x,0;\nu) + M_3\varepsilon^\theta e^{-\nu|x|}$$

by ξ over $I(\varepsilon)$, we obtain

$$k_j(\varepsilon)(H(x,0;\nu) - M_3\varepsilon^\theta e^{-\nu|x|}) < \int_{I(\varepsilon)} H(x,\xi;\nu)f_j(\xi)\,d\xi < k_j(\varepsilon)(H(x,0;\nu) + M_3\varepsilon^\theta e^{-\nu|x|}). \quad (4.2)$$

For $x \in \boldsymbol{R}$, $\xi \in J(\varepsilon)$, we have

$$\begin{aligned}\int_{J(\varepsilon)} H(x,\xi;\nu)f_j(\xi)\,d\xi &= \int_{J(\varepsilon)} (2\nu)^{-1}\exp(-\nu|x-\xi|)f_j(\xi)\,d\xi \\ &= \int_{T(\varepsilon)} (2\nu)^{-1}\exp(-\nu|x-\varepsilon y|)f_j(\varepsilon y)\varepsilon\,dy,\end{aligned}$$

where $T(\varepsilon) = (-\infty, -\varepsilon^{\theta-1}) \cup (\varepsilon^{\theta-1}, +\infty)$. Using Lemma 2.5, we have

$$\left|\int_{J(\varepsilon)} H(x,\xi;\nu)f_j(\xi)\,d\xi\right| \le \int_{T(\varepsilon)} (2\nu)^{-1}\exp(-\nu|x-\varepsilon y|)b_2\exp(-\alpha_2|y|)\,dy. \quad (4.3)$$

Using $|x-\varepsilon y| \ge |x| - \varepsilon|y|$ and $\lim_{\varepsilon\to 0}(\varepsilon\nu) = 0$, we get

$$\begin{aligned}&\exp(-\nu|x-\varepsilon y|)\exp(-\alpha_2|y|) \\ &\quad\le \exp(-\nu|x|)\exp(-(\alpha_2-\varepsilon\nu)|y|) \le \exp(-\nu|x|)\exp\left(-2^{-1}\alpha_2|y|\right)\end{aligned}$$

for small $\varepsilon > 0$. Thus the right-hand side of (4.3) is less than

$$\int_{T(\varepsilon)} (2\nu)^{-1}\exp(-\nu|x|)\exp\left(-2^{-1}\alpha_2|y|\right)b_2\,dy = \frac{2b_2}{\alpha_2\nu}\exp(-\nu|x|)\exp\left(-2^{-1}\alpha_2\varepsilon^{\theta-1}\right).$$

Combining this fact, (4.2) and (4.3), we complete the proof. □

Lemma 4.2. *Under the same assumption of Lemma 4.1, the following inequality*

$$\left|(P(\nu^2,0)f_2, P(\nu^2,0)f_1) - 4^{-1}k_1(\varepsilon)k_2(\varepsilon)\nu^{-3}\right| < M_2^2\varepsilon^{2\theta}\nu^{-1} + 2^{-1}M_2\varepsilon^\theta\nu^{-2}$$

holds true for sufficiently small $\varepsilon > 0$ and $\nu \in (\sigma_1, \sigma_2\varepsilon^{-\frac{1}{2}})$.

Proof. We begin with

$$\begin{aligned}&(P(\nu^2,0)f_2)(x)(P(\nu^2,0)f_1)(x) \\ =\ &k_1(\varepsilon)k_2(\varepsilon)(H(x,0,\nu)+r_1(x,\varepsilon,\nu))(H(x,0,\nu)+r_2(x,\varepsilon,\nu)),\end{aligned}$$

and thus

$$(P(\nu^2,0)f_2, P(\nu^2,0)f_1) = k_1(\varepsilon)k_2(\varepsilon)\left(g_1(\varepsilon,\nu)+g_2(\varepsilon,\nu)+g_3(\varepsilon,\nu)\right),$$

where

$$\begin{aligned}g_1(\varepsilon,\nu) &= \|H(x,0;\nu)\|_Y^2 = 4^{-1}\nu^{-3}, \\ g_2(\varepsilon,\nu) &= \int_{-\infty}^{\infty} H(x,0;\nu)(r_1(x,\varepsilon,\nu)+r_2(x,\varepsilon,\nu))\,dx, \\ g_3(\varepsilon,\nu) &= \int_{-\infty}^{\infty} r_1(x,\varepsilon,\nu)r_2(x,\varepsilon,\nu)\,dx.\end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned}|g_2(\varepsilon,\nu)| &\le \int_{-\infty}^{\infty} \frac{1}{2\nu}\exp(-\nu|x|)M_2\varepsilon^\theta\exp(-\nu|x|)\,dx = 2^{-1}M_2\varepsilon^\theta\nu^{-2}, \\ |g_3(\varepsilon,\nu)| &\le \int_{-\infty}^{\infty} M_2^2\varepsilon^{2\theta}\exp(-2\nu|x|)\,dx \le M_2^2\varepsilon^{2\theta}\nu^{-1}.\end{aligned}$$

This completes the proof. ☐

Because $F(\varepsilon,\kappa,\lambda)$ is real-analytic in $\kappa>0$ and is analytic in λ, Lemma 3.7 and the implicit function theorem imply that $\Lambda(\kappa)$ is real-analytic in κ with

$$\Lambda'(\kappa) = -F_\lambda(\varepsilon,\kappa,\Lambda(\kappa))^{-1}F_\kappa(\varepsilon,\kappa,\Lambda(\kappa))$$

for $\kappa\in(M_2,\sigma(\varepsilon)\varepsilon^{-1})$. A relative maximizer $\kappa(\varepsilon)$ satisfies $\Lambda'(\kappa(\varepsilon))=0$, which is equivalent to $F_\kappa(\varepsilon,\kappa(\varepsilon),\Lambda(\kappa(\varepsilon)))=0$.

Lemma 4.3. *For $\kappa>M_2$ and $\nu=\left(\kappa+(c^\varepsilon)^2/4\right)^{\frac{1}{2}}$, the following inequality*

$$\left|F_\kappa(\varepsilon,\kappa,\lambda)+p^\varepsilon\varepsilon-(P(\nu^2,0)f_2,P(\nu^2,0)f_1)\right| < M_4(\kappa^{-2}+\varepsilon^2)$$

holds true.

Proof of Lemma 4.3. From the proof of Lemma 3.9, we have

$$F_\kappa(\varepsilon,\kappa,\lambda)+\varepsilon p^\varepsilon-\left(P(\nu^2,0)f_2,P(\nu^2,0)f_1\right) = \left(QP(\kappa,c)^2(I+QP(\kappa,c))h_2,P(\kappa,-c)h_1\right) + \left(P(\kappa,c)QP(\kappa,c)h_2,P(\kappa,-c)h_1\right)+\varepsilon^2\left(P(\kappa,c)(I+QP(\kappa,c))S_2Kh_2,h_1\right).$$

From Lemmas 3.4 and 3.5, the first and the second terms of the right-hand side are of order $O(\kappa^{-2})$ as $\kappa\to+\infty$, while the third terms is of order $O(\varepsilon^2)$ as $\varepsilon\to 0$. This completes the proof. ☐

Now we prove theorems in Section 1.

Proof of Theorem 1.1. From $F_\kappa(\varepsilon,\kappa(\varepsilon),\Lambda(\kappa(\varepsilon)))=0$ and Lemmas 4.2, 4.3, $\kappa(\varepsilon)$ should satisfy

$$\begin{aligned}\left|p^\varepsilon\varepsilon\kappa(\varepsilon)^{\frac{3}{2}}-4^{-1}k_1(\varepsilon)k_2(\varepsilon)\right| &< M_4\left(\kappa(\varepsilon)^{-\frac{1}{2}}+\varepsilon^2\kappa(\varepsilon)\right)\\ &\quad +M_2^2\varepsilon^{2\theta}\kappa(\varepsilon)^{\frac{3}{2}}\nu(\varepsilon)^{-1}+2^{-1}M_2\varepsilon^\theta\kappa(\varepsilon)^{\frac{3}{2}}\nu(\varepsilon)^{-2},\end{aligned}$$

where $\nu(\varepsilon)=\left(\kappa(\varepsilon)+(c^\varepsilon)^2/4\right)^{\frac{1}{2}}$. We see that the right-hand side goes to 0 as $\varepsilon\to 0$, because $\kappa(\varepsilon)<\sigma(\varepsilon)\varepsilon^{-1}$. Therefore we obtain $\lim_{\varepsilon\to 0}\varepsilon\kappa(\varepsilon)^{\frac{3}{2}}=k_1k_2/(4p_0)$, that is,

$$\kappa(\varepsilon)=\left((4p_0)^{-1}k_1k_2\right)^{\frac{2}{3}}\varepsilon^{-\frac{2}{3}}+o(\varepsilon^{-\frac{2}{3}}) \quad \text{as } \varepsilon\to 0. \tag{4.4}$$

We will prove that there exists only one relative maximizer for $\Lambda(\kappa)$. Recall that $\kappa(\varepsilon)$ is any one of the relative maximizers. We will show $\Lambda''(\kappa(\varepsilon))>0$, which will prove that there exist no other relative maximizers except the unique maximizer. Because $\Lambda'(\kappa(\varepsilon))=0$, we have

$$\Lambda''(\kappa(\varepsilon))=-F_\lambda(\varepsilon,\kappa(\varepsilon),\lambda(\varepsilon))^{-1}F_{\kappa\kappa}(\varepsilon,\kappa(\varepsilon),\lambda(\varepsilon)),$$

where $\lambda(\varepsilon)=\Lambda(\kappa(\varepsilon))$. First we calculate

$$\left(P(\mu^2,c^\varepsilon)^3h_2,h_1\right)=\left(P(\mu^2,c^\varepsilon)^2h_2,P(\mu^2,-c^\varepsilon)h_1\right).$$

Using

$$\begin{aligned}P(\mu^2,c^\varepsilon) &= \exp\left(2^{-1}c^\varepsilon x\right)P(\nu^2,0)\left(\exp\left(-2^{-1}c^\varepsilon x\right)\,\cdot\right)\\ P(\mu^2,c^\varepsilon)^2h_2 &= \exp\left(2^{-1}c^\varepsilon x\right)P(\nu^2,0)^2f_2\\ P(\mu^2,-c^\varepsilon)h_1 &= \exp\left(-2^{-1}c^\varepsilon x\right)P(\nu^2,0)f_1\end{aligned}$$

we obtain

$$\left(P(\mu^2,c^\varepsilon)^3h_2,h_1\right)=\left(P(\nu^2,0)^2f_2,P(\nu^2,0)f_1\right).$$

From Lemma 4.1, we have

$$\begin{aligned} P(\nu^2,0)f_1 &= k_1(\varepsilon)H(x,0;\nu)+r_1, \\ P(\nu^2,0)^2f_2 &= k_2(\varepsilon)P(\nu^2,0)H(x,0;\nu)+P(\nu^2,0)r_2. \end{aligned}$$

Hence

$$\begin{aligned} & \left(P(\nu^2,0)^2f_2, P(\nu^2,0)f_1\right) \\ = \ & k_1(\varepsilon)k_2(\varepsilon)\left(P(\nu^2,0)H(x,0;\nu), H(x,0;\nu)\right) \\ & +k_2(\varepsilon)\left(P(\nu^2,0)H(x,0;\nu), r_1\right) \\ & +k_1(\varepsilon)\left(P(\nu^2,0)r_2, H(x,0;\nu)\right)+\left(P(\nu^2,0)r_2, r_1\right). \end{aligned}$$

Direct calculation yields

$$P(\nu^2,0)H(x,0;\nu)=4^{-1}\nu^{-2}\left(\nu^{-1}+|x|\right)\exp(-\nu|x|),$$

and

$$\left(P(\nu^2,0)H(x,0;\nu), H(x,0;\nu)\right)=\frac{3}{32}\nu^{-5}.$$

We put $\nu(\varepsilon)=\left(\kappa(\varepsilon)+(c^\varepsilon)^2/4\right)^{\frac{1}{2}}$, then we have

$$\nu(\varepsilon)=\left((4p_0)^{-1}k_1k_2\right)^{\frac{1}{3}}\varepsilon^{-\frac{1}{3}}+o(\varepsilon^{-\frac{1}{3}}) \quad \text{as } \varepsilon\to 0. \tag{4.5}$$

We have

$$\begin{aligned} \left|\left(P(\nu^2,0)H(x,0;\nu), r_j\right)\right| &\le \|P(\nu^2,0)\|_{\mathcal{L}(Y)}\|H(x,0,\nu)\|_Y\|r_j\|_Y \\ &\le \nu^{-2}\frac{1}{2}\nu^{-\frac{3}{2}}M_2\varepsilon^\theta\nu^{-1}=\frac{1}{2}M_2\varepsilon^\theta\nu^{-\frac{9}{2}} \end{aligned}$$

and

$$\begin{aligned} \left|\left(P(\nu^2,0)r_2, r_1\right)\right| &\le \|P(\nu^2,0)\|_{\mathcal{L}(Y)}\|r_2\|_Y\|r_1\|_Y \\ &\le \nu^{-2}M_2^2\varepsilon^{2\theta}\nu^{-2}=M_2^2\varepsilon^{2\theta}\nu^{-4}. \end{aligned}$$

Recalling $\frac{1}{2}<\theta<1$ and (4.5), we obtain

$$\left(P(\nu(\varepsilon)^2,0)^2f_2, P(\nu(\varepsilon)^2,0)f_1\right)=\left(\frac{3}{32}+o(1)\right)\nu(\varepsilon)^{-5} \quad \text{as } \varepsilon\to 0. \tag{4.6}$$

We study the right-hand side of (3.15). We have

$$F_{\kappa\kappa}=-2(KKKh_2,h_1)+O(\varepsilon^2).$$

We show that $-2(PPf_2,Pf_1)$ is the principal term of the right-hand side. We get

$$\begin{aligned} & (KKKh_2,h_1)=\left(\left(P(\mu^2,c^\varepsilon)+P(\mu^2,c^\varepsilon)QP(\mu^2,c^\varepsilon)\right)^3h_2,h_1\right) \\ = \ & \left((I+QP(\mu^2,c^\varepsilon))(P(\mu^2,c^\varepsilon)+P(\mu^2,c^\varepsilon)QP(\mu^2,c^\varepsilon))^2h_2, P(\mu^2,-c^\varepsilon)h_1\right) \end{aligned}$$

Thus

$$\begin{aligned} & (KKKh_2,h_1)-\left(P(\mu^2,c^\varepsilon)^2h_2, P(\mu^2,-c^\varepsilon)h_1\right) \\ = \ & \left(P(\mu^2,c^\varepsilon)^2h_2, P(\mu^2,-c^\varepsilon)h_1\right) \\ & +\left(\left(P(\nu^2,c^\varepsilon)^2QP(\nu^2,c^\varepsilon)+P(\nu^2,c^\varepsilon)QP(\nu^2,c^\varepsilon)^2\right)h_2, P(\nu^2,-c^\varepsilon)h_1\right) \\ & +\left(P(\nu^2,c^\varepsilon)QP(\nu^2,c^\varepsilon)^2QP(\nu^2,c^\varepsilon)h_2, P(\nu^2,-c^\varepsilon)h_1\right) \\ & +\left(QP(\nu^2,c^\varepsilon)(P(\nu^2,c^\varepsilon)+P(\nu^2,c^\varepsilon)QP(\nu^2,c^\varepsilon)^2h_2, P(\nu^2,-c^\varepsilon)h_1\right) \end{aligned}$$

The right-hand side is of order $O(\nu^{-6})$ as $\nu \to +\infty$. Hence we obtain

$$F_{\kappa\kappa}(\varepsilon, \kappa(\varepsilon), \lambda(\varepsilon)) = -\left(\frac{3}{16} + o(1)\right)\nu(\varepsilon)^{-5} < 0$$

as $\varepsilon \to 0$. Therefore $\Lambda''(\kappa(\varepsilon)) < 0$ for arbitrarily $\kappa(\varepsilon)$ with $\Lambda'(\kappa(\varepsilon)) = 0$. From (4.4), (3.7) and (3.8), we obtain $\lim_{\varepsilon\to 0} \Lambda(\kappa(\varepsilon)) = \tau^{-1}\widehat{\zeta}(0)$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Note that there exists zero eigenvalue $0 = \Lambda(0)$ for the linearized eigenvalue problem (1.7), which is associated with the phase shift. If $\sigma(\varepsilon)\varepsilon^{-1} < \kappa_1$, then $\kappa_1 < \kappa_n$ holds true for every $n = 1, 2, \cdots$. Thus there exist no eigenvalues in $\{\lambda \,|\, \operatorname{Re}\lambda \geq 0\}$ except the zero eigenvalue. Thus (u_0, v_0) is asymptotically stable. If $\kappa_1 < \sigma(\varepsilon)\varepsilon^{-1}$, (1.7) has a positive eigenvalue $\Lambda(\kappa_1) > 0$, and thus (u_0, v_0) is unstable. This completes the proof. □

Proof of Theorem 1.3. The maximizer of $\Lambda(\kappa)$ is given by $\kappa = \kappa(\varepsilon)$. Thus $\lambda_{\max}(\varepsilon) = \max\{\Lambda(\kappa_n) \,|\, n = 0, 1, \dots\}$ is given by κ_n with $\kappa_{n-1} \leq \kappa(\varepsilon) \leq \kappa_n$ or $\kappa_n \leq \kappa(\varepsilon) \leq \kappa_{n+1}$. Using (3.8), (3.7) and (4.4), we obtain $\lim_{\varepsilon\to 0} \lambda_{\max}(\varepsilon) = \tau^{-1}\widehat{\zeta}(0)$. □

5. **Proof of Lemma 1.1.** In this section we will prove Lemma 1.1. For the location of the essential spectrum, one can refer to [18]. See also [5] with exponential dichotomies given by [14].

We set

$$\begin{pmatrix} f_u(\overline{u}_\pm, \overline{v}_\pm) & f_v(\overline{u}_\pm, \overline{v}_\pm) \\ g_u(\overline{u}_\pm, \overline{v}_\pm) & g_v(\overline{u}_\pm, \overline{v}_\pm) \end{pmatrix} = \begin{pmatrix} -a_\pm & -b_\pm \\ c_\pm & -d_\pm \end{pmatrix},$$

respectively. Then $a_\pm$, $b_\pm$, $c_\pm$, $d_\pm$ are positive constants. Let ε be so small that $\varepsilon < \tau$ holds, and let λ_0 be an arbitrary number with

$$0 < 2\lambda_0 < \min\left\{a_+, a_-, d_+, d_-, (\varepsilon\tau)^{-1}a_+, (\varepsilon\tau)^{-1}a_-, (1 - \varepsilon\tau^{-1})d_+, (1 - \varepsilon\tau^{-1})d_-\right\}.$$

First we substitute

$$\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \exp(\lambda t + i\eta x_1) \begin{pmatrix} w(x'; \eta) \\ z(x'; \eta) \end{pmatrix}$$

into (1.6). We write $(w(x'; \eta), z(x'; \eta))$ simply as (w, z) in this section. Then we have

$$\begin{aligned} \varepsilon\tau\lambda w &= -\varepsilon^2\eta^2 w + \varepsilon^2\Delta' w + f_u^0(x_1; \varepsilon)w - \varepsilon\tau c i\eta w + f_v^0(x_1; \varepsilon)z \\ \lambda z &= g_u^0(x_1; \varepsilon)w - \eta^2 z + \Delta' z + g_v^0(x_1; \varepsilon)z - ci\eta z \end{aligned} \tag{5.1}$$

in Ω, and

$$\frac{\partial w}{\partial \boldsymbol{n}} = 0, \quad \frac{\partial z}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \partial\Omega.$$

We put

$$\rho = \lambda + ic\eta.$$

Let $x_1 \to \pm\infty$ respectively in (5.1). Then we have

$$\begin{aligned} \varepsilon\tau\rho w &= (\varepsilon^2\Delta' - \varepsilon^2\eta^2 - a_\pm)w - b_\pm z \\ \rho z &= c_\pm w + (\Delta' - \eta^2 - d_\pm)z \end{aligned} \qquad \text{in } \Omega \tag{5.2}$$

with $\frac{\partial w}{\partial \boldsymbol{n}} = 0$, $\frac{\partial z}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. Consider the union of the spectrum sets of (5.2) with all $\eta \in \boldsymbol{R}$, then it consists of only eigenvalues because Ω is bounded. Let ρ_0 be an eigenvalues with the largest real part in this union.

By [18, chap 4], there exist no more than a finite number of eigenvalues of $\mathcal{L}$ in $\{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda > \operatorname{Re}\rho_0 + \rho_1\}$. Here ρ_1 is any positive number. We will prove

$$\operatorname{Re}\rho_0 \leq -\lambda_0. \tag{5.3}$$

Then Lemma 1.1 follows with $\rho_1 = \lambda_0/2$. Thus it suffices to prove (5.3). We use Δ instead of Δ' for simplicity in what follows. Consider (5.2) with any $\eta \in \boldsymbol{R}$, that is,

$$\begin{array}{c} (\varepsilon^2\Delta - \varepsilon^2\eta^2 - \varepsilon\tau\rho - a_0)w = b_0 z \\ c_0 w + (\Delta - \eta^2 - \rho - d_0)z = 0 \end{array} \quad \text{in } \Omega. \tag{5.4}$$

Here $\eta \in \boldsymbol{R}$, and (a_0, b_0, c_0, d_0) represents either (a_+, b_+, c_+, d_+) or (a_-, b_-, c_-, d_-). It suffices to prove that $\operatorname{Re}\rho > -\lambda_0$ implies $(w, z) \equiv (0, 0)$. From (5.4), we have

$$(\varepsilon^2\Delta - \varepsilon^2\eta^2 - \varepsilon\tau\rho - a_0)(\Delta - \eta - \rho - d_0)z + b_0 c_0 z = 0.$$

We put

$$(\varepsilon^2\Delta - \varepsilon^2\eta^2 - \varepsilon\tau\rho - a_0)(\Delta - \eta^2 - \rho - d_0) + b_0 c_0 = \varepsilon^2(\Delta - t_+)(\Delta - t_-).$$

The eigenvalue of Δ in Ω with the Neumann boundary condition lies in $(-\infty, 0]$ in the complex plane. It suffices to prove the following assertion to obtain Lemma 1.1.

Lemma 5.1. *If* $\operatorname{Re}\rho > -\lambda_0$ *in (5.4), then* $t_\pm \in \boldsymbol{C}\backslash(-\infty, 0]$.

From this lemma there exist no eigenvalues in (5.2) with $\operatorname{Re}\rho > -\lambda_0$. This implies (5.3), and thus gives the proof of Lemma 1.1.

The numbers $t_\pm$ are the solutions of $t^2 - At + B = 0$ with

$$\begin{aligned} A &= \varepsilon^{-2}(a_0 + \varepsilon^2 d_0 + 2\varepsilon^2\eta^2 + \varepsilon\tau\rho + \varepsilon^2\rho) \\ B &= \varepsilon^{-2}\{(\varepsilon^2\eta^2 + \varepsilon\tau\rho + a_0)(\eta^2 + \rho + d_0) + b_0 c_0\} \end{aligned}$$

Thus $t_\pm = \left(A \pm D^{\frac{1}{2}}\right)/2$ is valid respectively. Here $D = A^2 - 4B$. After simple calculations, we obtain

$$\begin{aligned} \varepsilon^4 D &= (a_0 + \varepsilon^2 d_0 + \varepsilon\tau\rho)^2 - 4\varepsilon^2(a_0 d_0 + b_0 c_0) \\ &\quad -2\varepsilon^2(a_0 + 2\varepsilon d_0\tau - \varepsilon^2 d_0)\rho - \varepsilon^3(2\tau - \varepsilon)\rho^2 \end{aligned}$$

Hence we have

$$2\varepsilon^2 t_\pm = m_0(\varepsilon, \rho) + 2\varepsilon^2\eta^2 + \varepsilon^2\rho \pm m_0(\varepsilon, \rho)(1 - \varepsilon M_0(\varepsilon, \rho))^{1/2}, \tag{5.5}$$

where

$$\begin{aligned} m_0(\varepsilon, \rho) &\stackrel{\text{def}}{=} a_0 + \varepsilon^2 d_0 + \varepsilon\tau\rho, \\ M_0(\varepsilon, \rho) &\stackrel{\text{def}}{=} m_0(\varepsilon, \rho)^{-2}\left(4\varepsilon(a_0 d_0 + b_0 c_0) + 2(a_0 + 2\varepsilon d_0\tau - \varepsilon^2 d_0)(\varepsilon\rho) + (2\tau - \varepsilon)(\varepsilon\rho)^2\right). \end{aligned}$$

Lemma 5.2. $M_0(\varepsilon, \rho)$ *is independent of* η*, and is bounded uniformly in* $\varepsilon \in (0, \varepsilon_1)$ *and* ρ *with* $\operatorname{Re}\rho > -\lambda_0$.

Proof of Lemma 5.2. We have

$$\operatorname{Re} m_0(\varepsilon, \rho) = \operatorname{Re}(a_0 + \varepsilon^2 d_0 + \tau\varepsilon\rho) \geq a_0 - \tau\varepsilon\lambda_0 > a_0/2,$$

and hence $|m_0(\varepsilon, \rho)| > a_0/2$. By the triangle inequality we have

$$|a_0 + \varepsilon^2 d_0 + \tau\varepsilon\rho| \geq \left|a_0 + \varepsilon^2 d_0 - \tau\varepsilon|\rho|\right|.$$

Using these inequalities, we obtain

$$|M_0(\varepsilon,\rho)| \le \frac{4\varepsilon(a_0d_0+b_0c_0)+2(a_0+2\varepsilon d_0\tau-\varepsilon^2 d_0)|\varepsilon\rho|+(2\tau-\varepsilon)|\varepsilon\rho|^2}{\max\{a_0^2/4,(a_0+\varepsilon^2 d_0-\tau|\varepsilon\rho|)^2\}},$$

and thus

$$|M_0(\varepsilon,\rho)| \le \sup_{y\ge 0} \frac{4\varepsilon(a_0d_0+b_0c_0)+2(a_0+2\varepsilon d_0\tau-\varepsilon^2 d_0)y+(2\tau-\varepsilon)y^2}{\max\{a_0^2/4,(a_0+\varepsilon^2 d_0-\tau y)^2\}}. \tag{5.6}$$

The right-hand side of (5.6) is bounded uniformly in $\varepsilon \in (0,\varepsilon_1)$ and ρ. This completes the proof of Lemma 5.2. □

Proof of Lemma 5.1 From (5.5) and Lemma 5.2, we have

$$2\varepsilon^2 t_+ = 2m_0(\varepsilon,\rho)\{1+\varepsilon(m_R+im_R)\}+2\varepsilon^2\eta^2+\varepsilon^2\rho \tag{5.7}$$

with real numbers $m_R=m_R(\varepsilon,\rho,\eta)$, $m_I=m_I(\varepsilon,\rho,\eta)$. From Lemma 5.2, $|m_R(\varepsilon,\rho,\eta)|$ and $|m_I(\varepsilon,\rho,\eta)|$ are bounded uniformly in (ε,ρ,η). Taking the real part and the imaginary part, we have

$$\begin{aligned}
2\varepsilon^2\mathrm{Re}\, t_+ &= 2(a_0+\varepsilon^2 d_0+\varepsilon\tau\rho_R)(1+\varepsilon m_R) \\
&\quad -2\varepsilon^2\tau m_I\rho_I+2\varepsilon^2\eta^2+\varepsilon^2\rho_R, &(5.8)\\
2\varepsilon^2\mathrm{Im}\, t_- &= \varepsilon\left\{(2\tau+2\varepsilon\tau m_R+\varepsilon)\rho_I+2(a_0+\varepsilon^2 d_0+\varepsilon\tau\rho_R)m_I\right\}. &(5.9)
\end{aligned}$$

First we will show that $t_+\in \boldsymbol{C}\backslash(-\infty,0]$. Indeed, if $\mathrm{Im}\, t_+=0$, then

$$\rho_I = 2(a_0+\varepsilon^2 d_0+\varepsilon\tau\rho_R)\left(2\tau+2\varepsilon\tau m_R+\varepsilon\right)^{-1}.$$

Substituting this into (5.8), we have

$$\begin{aligned}
&2\varepsilon^2\mathrm{Re}\, t_+ \\
=\ & 2\varepsilon^2\eta^2+\varepsilon^2\rho_R+2(a_0+\varepsilon^2 d_0+\varepsilon\tau\rho_R)\left(1+\varepsilon m_R-\frac{2\varepsilon^2\tau m_I}{2\tau+2\varepsilon\tau m_R+\varepsilon}\right)\\
\ge\ & -\varepsilon^2\lambda_0+2(a_0+\varepsilon^2 d_0-\varepsilon\tau\lambda_0)\left(1+\varepsilon m_R-\frac{2\varepsilon^2\tau m_I}{2\tau+\varepsilon\tau m_R+\varepsilon}\right)
\end{aligned}$$

because $\eta^2\ge 0$ and $\rho_R>-\lambda_0$ are valid. Hence $2\varepsilon^2\mathrm{Re}\, t_+>a_0>0$ for sufficiently small $\varepsilon>0$. This argument implies $t_+\in\boldsymbol{C}\backslash(-\infty,0]$.

Using Lemma 5.2, we have

$$(1-\varepsilon M_0(\varepsilon,\rho))^{1/2} = 1-\frac{1}{2}\varepsilon M_0(\varepsilon,\rho)-\frac{1}{8}\varepsilon^2 M_0(\varepsilon,\rho)^2+O(\varepsilon^3).$$

Here the convergence of $O(\varepsilon^3)$ are uniform in (ρ,η). We will use M_0 instead of $M_0(\varepsilon,\rho)$ in what follows. From this relation and (5.5), we obtain

$$2\varepsilon^2 t_- = 2\varepsilon^2\eta^2+\varepsilon^2\rho+m_0(\varepsilon,\rho)\left(\frac{1}{2}\varepsilon M_0+\frac{1}{8}\varepsilon^2 M_0^2+O(\varepsilon^3)\right). \tag{5.10}$$

From (5.10) and after simple calculation, we have

$$t_- = \eta^2+\rho+m_0(\varepsilon,\rho)\left(\frac{a_0d_0+b_0c_0+d_0(\tau-\varepsilon)\varepsilon\rho}{m_0(\varepsilon,\rho)^2}+\varepsilon(n_R+in_I)\right). \tag{5.11}$$

Here $n_R=n_R(\varepsilon,\rho)$, $n_I=n_I(\varepsilon,\rho)$ are real numbers with $|n_R(\varepsilon,\rho)|$ and $|n_I(\varepsilon,\rho)|$ uniformly bounded in (ε,ρ), and are independent of η. Using (5.11), we write the real and imaginary parts as

$$\begin{aligned} \mathrm{Re}\, t_- \;&=\; \eta^2+\rho_R+\left(1-\varepsilon\tau^{-1}\right)d_0 \\ &\quad+\frac{\left(b_0c_0-\varepsilon^2 d_0+\varepsilon\tau^{-1}d_0(a_0+\varepsilon^2 d_0)\right)\left(a_0+\varepsilon^2 d_0+\tau\varepsilon\rho_R\right)}{|m_0(\varepsilon,\rho)|^2} \\ &\quad+\varepsilon\left\{(a_0+\varepsilon^2 d_0+\tau\varepsilon\rho_R)n_R-\tau\varepsilon\rho_I n_R\right\}, \end{aligned} \tag{5.12}$$

and

$$\mathrm{Im} t_- = \rho_I-\frac{\tau\varepsilon\rho_I}{|m_0(\varepsilon,\rho)|^2}+\varepsilon\{(a_0+\varepsilon^2 d_0+\tau\varepsilon\rho_R)n_I+\tau\varepsilon\rho_I n_R)\}.$$

We will show that, $\mathrm{Im}\, t_-=0$ implies $\mathrm{Re}\, t_->0$. This leads to $t_-\in \boldsymbol{C}\backslash(-\infty,0]$. Assume $\mathrm{Im}\, t_-=0$. Then

$$\rho_I=-\frac{\varepsilon(a_0+\varepsilon^2 d_0+\tau\varepsilon\rho_R)n_I}{1-\varepsilon\tau|m_0(\varepsilon,\rho)|^{-2}+\tau\varepsilon^2 n_R}.$$

Substituting this relation into (5.12), we obtain

$$\begin{aligned} &\mathrm{Re}\, t_-=\eta^2+\left(1-\varepsilon\tau^{-1}\right)d_0 \\ &+\rho_R\left[1+\frac{\tau\varepsilon\{b_0c_0-\varepsilon^2 d_0+\varepsilon\tau^{-1}d_0(a_0+\varepsilon^2 d_0)\}}{|m_0(\varepsilon,\rho)|^2}+\tau\varepsilon^2 n_I+\frac{\tau\varepsilon^2 n_R\tau\varepsilon^2 n_I}{1-\varepsilon\tau|m_0(\varepsilon,\rho)|^{-2}+\tau\varepsilon^2 n_R}\right] \\ &\qquad+\frac{\{b_0c_0-\varepsilon^2 d_0+\varepsilon\tau^{-1}d_0(a_0+\varepsilon^2 d_0)\}(a_0+\varepsilon^2 d_0)}{|m_0(\varepsilon,\rho)|^2} \\ &\qquad\qquad+\varepsilon(a_0+\varepsilon^2 d_0)n_I+\tau\varepsilon^2 n_R\frac{\varepsilon(a_0+\varepsilon^2 d_0)n_I}{1-\varepsilon\tau|m_0(\varepsilon,\rho)|^{-2}+\tau\varepsilon^2 n_R}. \end{aligned}$$

The right-hand side equals

$$\eta^2+(1+O(\varepsilon))\rho_R+d_0+O(\varepsilon)+\frac{a_0b_0c_0+O(\varepsilon)}{|m_0(\varepsilon,\rho)|^2}.$$

Here the convergence of each $O(\varepsilon)$ is uniform in (ρ,η). Since $\rho_R>-2^{-1}d_0$, we obtain

$$\mathrm{Re}\, t_->2^{-1}d_0+O(\varepsilon)>0.$$

This argument shows $t_-\in \boldsymbol{C}\backslash(-\infty,0]$. Combining $t_+\in \boldsymbol{C}\backslash(-\infty,0]$, we complete the proof of Lemma 5.1. □

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