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Axially asymmetric traveling fronts in balanced bistable reaction-diffusion equations

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Abstract

For a balanced bistable reaction-diffusion equation, an axisymmetric traveling front has been well known. This paper proves that an axially asymmetric traveling front with any positive speed does exist in a balanced bistable reaction-diffusion equation. Our method is as follows. We use a pyramidal traveling front for an unbalanced reaction-diffusion equation whose cross section has a major axis and a minor axis. Preserving the ratio of the major axis and a minor axis to be a constant and taking the balanced limit, we obtain a traveling front in a balanced bistable reaction-diffusion equation. This traveling front is monotone decreasing with respect to the traveling axis, and its cross section is a compact set with a major axis and a minor axis when the constant ratio is not 1.

AMS Mathematical Classifications: 35C07, 35B20, 35K57
Key words: traveling front, reaction-diffusion equation, asymmetric, balanced

1 Introduction

In this paper we study a reaction-diffusion equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u - G'(u), && \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), && \boldsymbol{x} \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $n \geq 3$ is a given integer, and given $u_0 \in X$. Here X is the set of bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the norm

$$\|u_0\| = \sup_{\boldsymbol{x} \in \mathbb{R}^n} |u_0(\boldsymbol{x})|.$$

Now $G \in C^2[-1, 1]$ satisfies

$$G(1) = 0,\ G(-1) = 0,\ G'(1) = 0,\ G'(-1) = 0,\ G''(1) > 0,\ G''(-1) > 0,$$
$$G(s) > 0 \quad \text{if} \quad -1 < s < 1.$$

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For $G(s) = (1-s^2)^2/4$ and $-G'(s) = s - s^3$, (1.1) is called the Allen–Cahn equation, the scalar Ginzburg–Landau equation or the Nagumo equation.

The reaction term is called balanced when $G(1) = G(-1)$ and is called unbalanced when $G(1) \neq G(-1)$. When the reaction term is unbalanced with $G(1) < G(-1)$, multi-dimensional traveling fronts including axially asymmetric ones have been studied by [16, 17, 11, 12, 13, 18, 19, 20, 14, 24, 26, 15, 21, 22, 23] and so on. In this case, the propagation is mainly driven by the imbalance of the reaction kinetics and the curvature effect of an *interface*. Here a level set of a solution is often called an interface.

When the reaction term is balanced, one has no driven force caused by the reaction kinetics and the propagation is mainly driven by the curvature effect of an interfaces and is also driven by *interaction* between portions of an interface. For Equation (1.1), axisymmetric traveling fronts have been studied by Chen, Guo, Hamel, Ninomiya and Roquejoffre [4]. See del Pino, Kowalczyk and Wei [7] for a stationary solution, that is a traveling front with speed zero, related with De Giorgi's conjecture. See [8] for a traveling wave solution with two non-planar fronts and for a traveling wave solution with non-convex fronts. For a mean curvature flow, Wang [25] studied an axially asymmetric traveling front that lies between two parallel planes in $\mathbb{R}^n$. See [6] for other traveling waves in a mean curvature flow.

In this paper we prove the existence of an axially asymmetric traveling front solution to a balanced reaction-diffusion equation (1.1). This axially asymmetric traveling front solution is monotone decreasing in the traveling axis x_n and travels with any given positive speed.

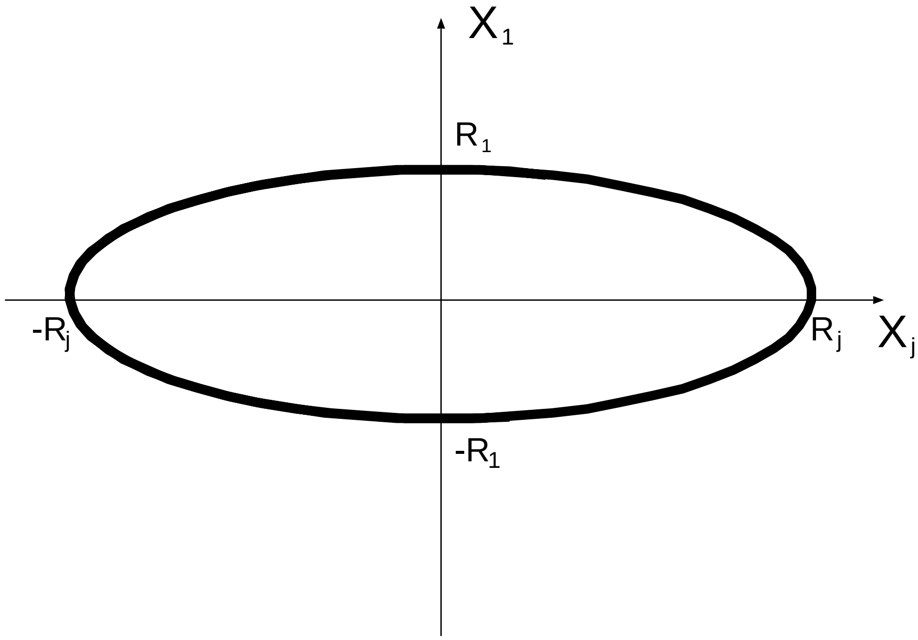

Figure 1: The cross section of $\{U(\boldsymbol{x}', x_n) = \theta_0\}$ at $x_n = \zeta$.

Let s_* be the largest zero point of G' in $(-1, 1)$, that is, $s_* \in (-1, 1)$ is defined by

$$s_* = \min\{s_0 \in (-1,1) \mid -G'(s) > 0 \quad \text{if } s_0 < s < 1\}.$$

We fix θ_0 with $s_* < \theta_0 < 1$ and have $-G'(\theta_0) > 0$.

Let

$$1 \le \alpha_2 \le \cdots \le \alpha_{n-1} \tag{1.2}$$

and $\zeta > 0$ be arbitrarily given. We put

$$\boldsymbol{\alpha}' = (1, \alpha_2, \ldots, \alpha_{n-1}). \tag{1.3}$$

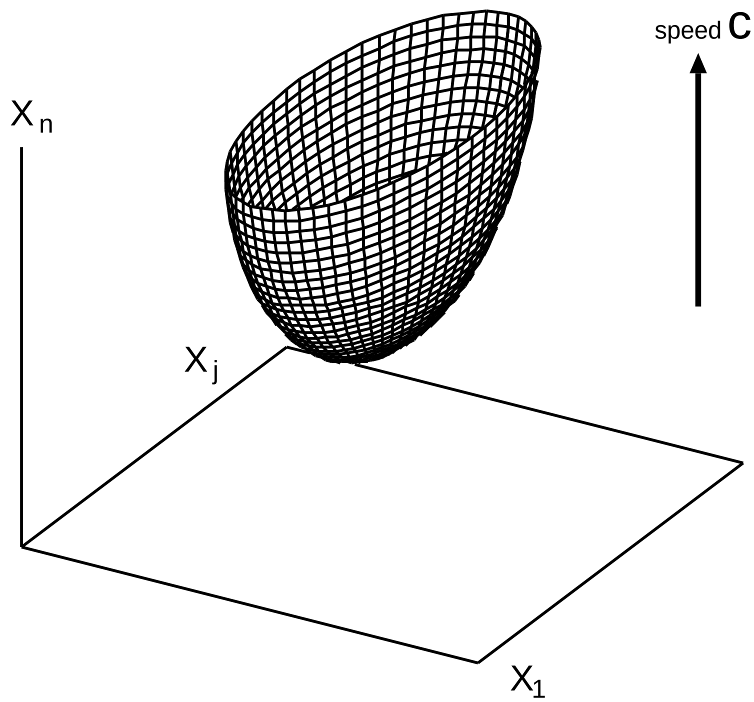

Figure 2: A level set $\{U(\boldsymbol{x}', x_n) = \theta_0\}$ of U.

The following is the main assertion in this paper.

Theorem 1 (Axially asymmetric traveling fronts) *Let $c > 0$ be an arbitrarily given number. Let $\zeta > 0$ be arbitrarily given and let $\boldsymbol{\alpha}'$ be given by (1.3) with (1.2). Then there exists $U(\boldsymbol{x}) = U(\boldsymbol{x}; \boldsymbol{\alpha}')$ such that one has the following. Let R_j be given by*

$$U(0, \ldots, 0, \overset{\overset{j}{\smile}}{R_j}, 0, \ldots, 0, \zeta; \boldsymbol{\alpha}') = \theta_0 \tag{1.4}$$

for $1 \le j \le n-1$. One has $U(\mathbf{0}) = \theta_0 \in (-1, 1)$ and

$$\Delta U + c\frac{\partial U}{\partial x_n} - G'(U) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \tag{1.5}$$

$$U(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = U(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n), \quad 1 \le j \le n-1,$$

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) < 0 \qquad \textit{if } \ \boldsymbol{x} \in \mathbb{R}^n,$$

$$\frac{\partial U}{\partial x_j}(\boldsymbol{x}) > 0 \qquad \textit{if } \ \boldsymbol{x} \in \mathbb{R}^n, x_j > 0,\ 1 \le j \le n-1,$$

$$\frac{R_j}{R_1} = \alpha_j, \qquad 1 \le j \le n-1.$$

For every $\theta \in (-1, 1)$, one has

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ |\nabla U(\boldsymbol{x})| \,\middle|\, U(\boldsymbol{x}) = \theta \right\} > 0, \tag{1.6}$$

and can define $q_\theta(\boldsymbol{x}') \in \mathbb{R}$ by $U(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) = \theta$ for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$. Here q_θ belongs to $C^1(\mathbb{R}^{n-1})$.

Remark 1 *For every $\theta \in (-1, 1)$, a level set $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, U(\boldsymbol{x}) = \theta\}$ is given by a graph of a function that is defined on the entire space $\mathbb{R}^{n-1}$.*

When $(\alpha_1, \ldots, \alpha_{n-1}) \neq (1, \ldots, 1)$, $\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \,|\, U(\boldsymbol{x}', \zeta; \boldsymbol{\alpha}') = \theta_0\}$ is a compact set with a major axis and a minor axis and is not a disk. Thus $U(\boldsymbol{x})$ is an axially asymmetric traveling front solution with the θ_0 level set whose cross section at $x_n = \zeta > 0$ is a compact set that is different from a disk. When $\boldsymbol{\alpha} = (\alpha_2, \ldots, \alpha_{n-1}) = (1, \ldots, 1)$ in (1.2), the author conjectures that U in Theorem 1 is axisymmetric with respect to the x_n-axis, and equals the traveling front studied by [4]. This is an interesting problem that should be studied in future.

Equation (1.1) and a mean curvature flow are closely related in the limit where $\varepsilon > 0$ goes to zero if

$$-G'(u) = \frac{1}{\varepsilon^2}(u - u^3).$$

See [3] for instance. The motion of an interface is driven by the curvature effect and attracting interaction between other portions of an interface. In the limit of $\varepsilon \to 0$, this interaction of interfaces disappears on given compact sets in $\mathbb{R}^n$. Thus, in the limit of $\varepsilon \to 0$, a solution in Equation (1.1) is approximated by that of a mean curvature flow on compact sets.

The cross section of $\{U(\boldsymbol{x}', x_n) = \theta_0\}$ at $x_n = \zeta$ in Figure 1 will be related to the Angenent Oval (Paper Clip) in a mean curvature flow. See [1, 5] for this oval. The relation between an axially asymmetric traveling front in Theorem 1 and that in [25] will be an interesting problem. We conjecture that an axially asymmetric traveling front in Theorem 1 converges to that in [25] in any compact set in $\mathbb{R}^n$ as ε goes to zero. This convergence cannot be uniform in $\mathbb{R}^n$. The reason is as follows. An axially asymmetric traveling front in a mean curvature flow in [25] lies between two parallel planes, while a level set of an axially asymmetric traveling front in Theorem 1 is defined on the whole $\mathbb{R}^{n-1}$, and has a shape as is seen in Figure 2. Thus the convergence cannot be uniform in $\mathbb{R}^n$. The reason of the difference of shapes is as follows. In a mean curvature flow, a solution propagates only by the curvature effect. While, a solution propagates by the curvature effect and by the interaction between portions of an interface in a balanced reaction-diffusion equation. For any fixed $x_n > 0$, U can be very close to $G(1) = 0$ in $\{(\boldsymbol{x}', x_n) \,|\, U(\boldsymbol{x}', x_n) > \theta_0\}$, while U cannot be so close to $G(-1) = 0$ in $\{(\boldsymbol{x}', x_n) \,|\, U(\boldsymbol{x}', x_n) < \theta_0\}$. Then, portions of an interface attract each other with time goes on. Because the shape of a traveling front remains unchanged up to phase shift, the portions of an interface have to be apart from each other as x_n goes to $+\infty$. Otherwise the portions of an interface attract each other and will collapse. Thus the interface of a traveling front cannot lie between two planes and has to be a graph of a function defined on the entire space $\mathbb{R}^{n-1}$ in a balanced reaction-diffusion equation. This shows a sharp contrast between traveling fronts in a balanced reaction-diffusion equation and those in a mean curvature flow.

This paper is organized as follows. In Section 2, we briefly explain the idea to approximate an axially asymmetric traveling front for a balanced reaction-diffusion by pyramidal traveling fronts for unbalanced reaction-diffusion equations. In Section 3, we make preparations. In Section 4, we show properties of pyramidal traveling fronts to unbalanced reaction-diffusion equations. In Section 5, we take the balanced limit of pyramidal traveling fronts, and prove Theorem 1.

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2 Approximation by pyramidal traveling fronts of unbalanced reaction-diffusion equations

In this section we briefly explain how to show the existence of axially asymmetric traveling fronts for balanced reaction-diffusion equations by approximating them by pyramidal traveling fronts for unbalanced reaction-diffusion equations.

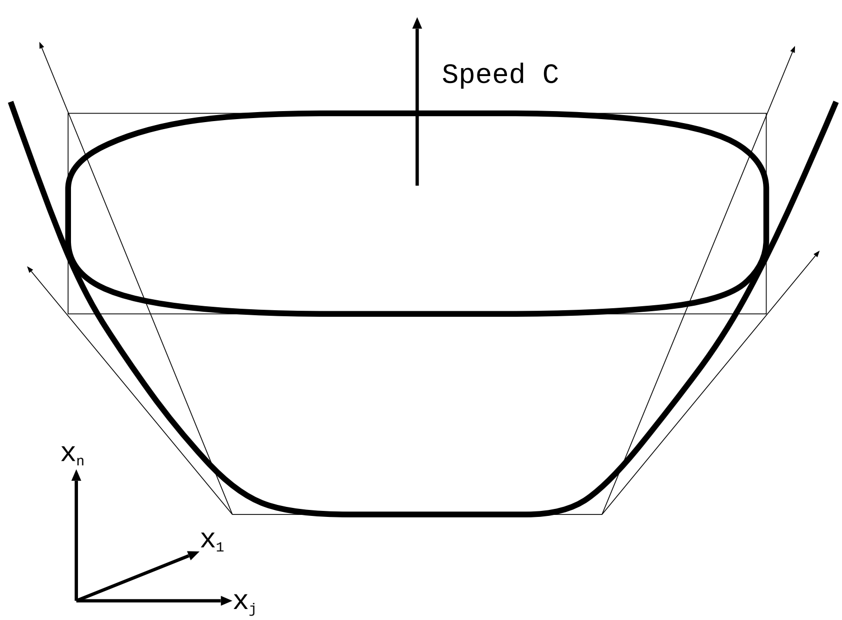

Figure 3: A level set $\{V_k(\boldsymbol{x}', x_n) = \theta_0\}$ of a pyramidal traveling front V_k.

Let $\boldsymbol{\alpha}' = (1, \alpha_2, \dots, \alpha_{n-1})$ be given by (1.3) with (1.2) and let $\zeta > 0$ be arbitrarily given. In Section 4, we introduce an unbalanced reaction-diffusion equation

$$\Delta V_k + c\frac{\partial V_k}{\partial x_n} - G'(V_k) + k\sqrt{2G(V_k)} = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n$$

for sufficiently small $k > 0$, and define a pyramidal traveling front solution V_k to this equation with

$$V_k(\mathbf{0}', z_k) = \theta_0,$$

$$V_k(0, \dots, 0, \underbrace{r_i(\boldsymbol{A}'(k))}_{i}, 0, \dots, 0, \zeta + z_k) = \theta_0, \qquad 2 \le i \le n-1,$$

$$\frac{r_i(\boldsymbol{A}'(k))}{r_1(\boldsymbol{A}'(k))} = \alpha_i, \qquad 2 \le i \le n-1,$$

where z_k is a real number and $\mathbf{0}' = (0, \dots, 0) \in \mathbb{R}^{n-1}$. Here $\boldsymbol{A}'(k_i)$ is given by (4.7) and a symbol $\underbrace{r_i(\boldsymbol{A}'(k))}_{i}$ implies that $r_i(\boldsymbol{A}'(k))$ is the i-th component. See Figure 3 for a level set of V_k.

In Section 5, we define

$$U(\boldsymbol{x}', x_n; \boldsymbol{\alpha}') = \lim_{i \to \infty} V_{k_i}(\boldsymbol{x}', x_n + z_{k_i}; \boldsymbol{A}'(k_i))$$

for all $(\boldsymbol{x}', x_n)$ in any compact set in $\mathbb{R}^n$. Here

$$k_1 > \cdots > k_i > \cdots \to 0$$

is a subsequence. We often write $U(\boldsymbol{x}; \boldsymbol{\alpha}')$ simply as $U(\boldsymbol{x})$. Then $U(\boldsymbol{x})$ satisfies Theorem 1. To take this limit, a uniform estimate on a pyramidal traveling front V_k for every small positive k is indispensable. We will introduce Proposition 1 that gives this uniform estimate, and carry on detailed discussions in Section 4 and Section 5.

3 Preliminaries

We extend $G \in C^2[-1, 1]$ as a function of $C^2(\mathbb{R})$ with

$$G(s) > 0 \qquad \text{if} \quad |s| \neq 1.$$

Let

$$\beta = \frac{1}{2}\min\{G''(1), G''(-1)\} > 0,$$

and let $\delta_* \in (0, 1/4)$ satisfy

$$\min_{|u+1| \le 2\delta_*} G''(u) > \beta, \qquad \min_{|u-1| \le 2\delta_*} G''(u) > \beta.$$

We put

$$M = 1 + \max_{|u| \le 1 + 2\delta_*} |G''(u)|.$$

Following to [17, 4, 18, 19, 20], we introduce a one-dimensional traveling front. For any k with

$$0 < k < \sqrt{G''(-1)},$$

let

$$\begin{aligned} f_k(s) &= -G'(s) + k\sqrt{2G(s)}, \quad s \in \mathbb{R}, \\ F_k(s) &= \int_{-1}^{s} f_k(\sigma')\,\mathrm{d}\sigma'. \end{aligned}$$

Then we have

$$\begin{aligned} & f_k'(1) = -G''(1) - k\sqrt{G''(1)} < 0, \\ & f_k'(-1) = -\sqrt{G''(-1)}\left(\sqrt{G''(-1)} - k\right) < 0, \\ & -F_k(-1) = 0,\ -F_k(1) = -k\int_{-1}^{1} \sqrt{2G(\sigma')}\,\mathrm{d}\sigma' < 0. \end{aligned}$$

Let $k_0 \in \left(0, \sqrt{G''(-1)}\right)$ be small enough such that one has

$$\min\left\{-F_k(s) \,|\, s \in (-1,1), f_k(s) = 0\right\} > 0$$

for every $k \in [0, k_0)$. We define Φ by

$$-x = \int_0^{\Phi(x)} \frac{\mathrm{d}s}{\sqrt{2G(s)}}, \qquad x \in \mathbb{R}.$$

Then we have $\Phi(0) = 0$ and

$$\begin{aligned} -\Phi'(x) &= \sqrt{2G(\Phi(x))}, \qquad x \in \mathbb{R}, \\ \Phi''(x) &= G'(\Phi(x)), \qquad x \in \mathbb{R}. \end{aligned}$$

Thus Φ satisfies

$$\begin{gathered} \Phi''(x) + k\Phi'(x) + f_k(\Phi(x)) = 0, \qquad x \in \mathbb{R}, \\ \Phi(-\infty) = 1,\ \Phi(\infty) = -1, \end{gathered}$$

and is a one-dimensional traveling front with speed $k \in (0, k_0)$. Now Φ also satisfies

$$\begin{gathered} \Phi''(x) - G'(\Phi(x)) = 0, \qquad x \in \mathbb{R}, \\ \Phi'(x) < 0, \qquad x \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(0) = 0, \quad \Phi(+\infty) = -1. \end{gathered}$$

Thus Φ is a planar stationary front to (1.1).

4 Properties of pyramidal traveling fronts to unbalanced reaction-diffusion equations

In this section we study properties of pyramidal traveling fronts for unbalanced reaction-diffusion equations. Two-dimensional V-form fronts and pyramidal traveling fronts in $\mathbb{R}^n$ have been studied by [16, 17, 11, 12, 13, 18, 19, 20, 14, 24, 26, 15] and so on.

Let $c > 0$ be arbitrarily given. For a given bounded and uniformly continuous function u_0 let $w(\boldsymbol{x}, t; u_0)$ be the solution of

$$\begin{aligned} \frac{\partial w}{\partial t} &= \Delta w + c\frac{\partial w}{\partial x_n} + f_k(w), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, t > 0, \\ w(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n. \end{aligned}$$

For any $k \in (0, \min\{k_0, c\})$, let

$$m_* = \frac{\sqrt{c^2 - k^2}}{k}.$$

For every $a_j \geq 0$ $(2 \leq j \leq n-1)$, we define

$$\begin{aligned} h_{2j-1}(\boldsymbol{x}') &= m_* \left(x_j - a_j\right), \\ h_{2j}(\boldsymbol{x}') &= m_* \left(x_j + a_j\right) \end{aligned}$$

for $1 \leq j \leq n-1$. We put

$$\boldsymbol{a}' = (0, a_2, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}.$$

Let

$$p(\boldsymbol{x}') = \max_{1 \leq j \leq 2n-2} h_j(\boldsymbol{x}') = m_* \max\{|x_1|, |x_2| - a_2, \ldots, |x_{n-1}| - a_{n-1}\} \tag{4.1}$$

for $\boldsymbol{x'} \in \mathbb{R}^{n-1}$, and let

$$p_i(\boldsymbol{x}') = m_* \max\{|x_1|, \max_{2 \leq j \leq n-1, j \neq i} (|x_j| - a_j)\} \tag{4.2}$$

for $\boldsymbol{x'} \in \mathbb{R}^{n-1}$ and $2 \leq i \leq n-1$.

Hereafter let h be either p or p_i for $2 \leq i \leq n-1$. We call $\{(\boldsymbol{x}', x_n) \,|\, x_n \geq h(\boldsymbol{x}')\}$ a pyramid in $\mathbb{R}^n$. For $1 \leq j \leq n-1$, we define

$$\Omega_j = \{\boldsymbol{x}' \in \mathbb{R}^{n-1} \,|\, h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\},$$

and have

$$\bigcup_{j=1}^{2n-2} \partial\Omega_j = \{h_{i_1}(\boldsymbol{x}') = h_{i_2}(\boldsymbol{x}') = h(\boldsymbol{x}') \text{ for some } i_1 \neq i_2\}.$$

The set of edges of a pyramid is given by

$$E = \left\{(\boldsymbol{x}', h(\boldsymbol{x}')) \,|\, \boldsymbol{x}' \in \cup_{j=1}^{2n-2} \partial\Omega_j\right\}.$$

For $\gamma > 0$, let

$$D(\gamma) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathrm{dist}(\boldsymbol{x}, E) > \gamma\}. \tag{4.3}$$

Hereafter let h be either p or p_i for $2 \le i \le n-1$. Following to [14, 24], let v_k be the pyramidal traveling front associated with

$$x_n = m_* \max_{1 \le j \le n-1} |x_j|$$

if $h = p$, and the pyramidal traveling front associated with

$$x_n = m_* \max_{1 \le j \le n-1, j \ne i} |x_j|$$

if $h = p_i$ for $2 \le i \le n-1$. Then v_k is a unique solution to

$$\Delta v_k + c\frac{\partial v_k}{\partial x_n} + f_k(v_k) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$
$$\Phi\left(\frac{k}{c}(x_n - h(\boldsymbol{x}'))\right) < v_k(\boldsymbol{x}', x_n), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$
$$\lim_{\gamma \to \infty} \sup_{(\boldsymbol{x}', x_n) \in D(\gamma)} \left| v_k(\boldsymbol{x}', x_n) - \Phi\left(\frac{k}{c}(x_n - h(\boldsymbol{x}'))\right) \right| = 0,$$

where $D(\gamma)$ is given by (4.3) with respect to h. Here v_k satisfies

$$\frac{\partial v_k}{\partial x_n} < 0 \qquad \text{for all } (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$
$$\frac{\partial v_k}{\partial x_j} > 0 \qquad \text{if } x_j > 0,\ 1 \le j \le n-1.$$

For $a_j \ge 0$ $(2 \le j \le n-1)$ we define

$$\underline{v}(\boldsymbol{x}', x_n) = \Phi\left(\frac{k}{c}(x_n - h(\boldsymbol{x}'))\right), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$
$$\overline{v}(\boldsymbol{x}', x_n) = \min_{1 \le j \le n-1} \min_{|s_j| \le a_j} v_k(x_1, \ldots, x_{j-1}, x_j - s_j, x_{j+1}, \ldots, x_n).$$

Then we have

$$\underline{v}(\boldsymbol{x}) < \overline{v}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$\lim_{R \to \infty} \sup_{|\boldsymbol{x}| \ge R} (\overline{v}(\boldsymbol{x}) - \underline{v}(\boldsymbol{x})) = 0.$$

Now $\underline{v}$ and $\overline{v}$ are a weak subsolution and a supersolution to

$$\Delta v + c\frac{\partial v}{\partial x_n} + f_k(v) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n,$$

respectively. We define

$$V_k(\boldsymbol{x}; h) = \lim_{t \to \infty} w(\boldsymbol{x}, t; \underline{v}) \tag{4.4}$$

for all $\boldsymbol{x}$ in any compact subset in $\mathbb{R}^n$. Then $V_k = V_k(\boldsymbol{x}; h)$ satisfies

$$\begin{aligned} &\Delta V_k + c\frac{\partial V_k}{\partial x_n} + f_k(V_k) = 0, \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \qquad (4.5)\\ &\lim_{\gamma\to\infty} \sup_{(\boldsymbol{x}',x_n)\in D(\gamma)} \left| V_k(\boldsymbol{x}', x_n) - \Phi\left(\frac{k}{c}(x_n - h(\boldsymbol{x}'))\right)\right| = 0,\\ &V_k(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = V_k(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n), \quad 1 \le j \le n-1,\\ &\frac{\partial V_k}{\partial x_n} < 0 \qquad \text{for all } (\boldsymbol{x}', x_n) \in \mathbb{R}^n,\\ &\frac{\partial V_k}{\partial x_j} > 0 \qquad \text{if } x_j > 0. \end{aligned}$$

Since $\underline{v}(\boldsymbol{x}', x_n; \boldsymbol{a}')$ depends continuously on $\boldsymbol{a}'$ in X, $V_k(\boldsymbol{x}; \boldsymbol{a}')$ depends continuously on $\boldsymbol{a}'$ in Z for each k by using (4.4). Using

$$\lim_{a_i\to\infty} p(\boldsymbol{x}') = p_i(\boldsymbol{x}') \qquad \text{in any compact set in } \mathbb{R}^{n-1}$$

uniformly in $a_j \ge 0, j \ne i$, we have

$$\lim_{a_i\to\infty} \Phi\left(\frac{k}{c}(x_n - p(\boldsymbol{x}'))\right) = \Phi\left(\frac{k}{c}(x_n - p_i(\boldsymbol{x}'))\right) \qquad \text{in any compact set in } \mathbb{R}^n$$

uniformly in $a_j \ge 0, j \ne i$. Combining this fact and (4.4), we have

$$\lim_{a_i\to\infty} V_k(\boldsymbol{x}; p) = V_k(\boldsymbol{x}; p_i) \qquad \text{in any compact set in } \mathbb{R}^n \qquad (4.6)$$

uniformly in $a_j \ge 0, j \ne i$.

Let $\zeta > 0$ be arbitrarily given and let $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1})$ satisfy (1.2). For every $k \in (0, \min\{k_0, c\})$ and every $a_j \ge 0$ $(2 \le j \le n-1)$, we choose $z_k = z_k(\boldsymbol{a}') \in \mathbb{R}$ by

$$V_k(\mathbf{0}', z_k; p) = \theta_0,$$

where p is given by (4.1). For $2 \le i \le n-1$ we define $r_i(\boldsymbol{a}') > 0$ by

$$V_k(0, \ldots, 0, \overset{i}{\smile}{r_i(\boldsymbol{a}')}, 0, \ldots, 0, \zeta + z_k; p) = \theta_0.$$

From (4.6), we have

$$\lim_{a_i\to\infty} \frac{r_i(\boldsymbol{a}')}{r_1(\boldsymbol{a}')} = \infty$$

uniformly in $a_j \ge 0, j \ne i$.

for $2 \le i \le n-1$. Now $r_i(\boldsymbol{a}')$ depends continuously on $\boldsymbol{a}'$. From symmetry we have

$$\left.\frac{r_i(\boldsymbol{a}')}{r_1(\boldsymbol{a}')}\right|_{a_i=0} = 1 \qquad \text{for every} \quad a_j \ge 0, j \ne i.$$

Lemma 1 *There exists* $A_i(k) \in [0,\infty)$ *such that one has*

$$\frac{r_i(A_1(k),\dots,A_{n-1}(k))}{r_1(A_1(k),\dots,A_{n-1}(k))} = \alpha_j, \qquad 2 \le i \le n-1.$$

Proof. In this proof we omit k for simplicity. First we consider the case $n=3$. Using

$$\left.\frac{r_2(a_2)}{r_1(a_2)}\right|_{a_2=0} = 1, \quad \left.\frac{r_2(a_2)}{r_1(a_2)}\right|_{a_2=\infty} = \infty,$$

we find $A_2 \in (0,\infty)$ with

$$\frac{r_2(A_2)}{r_1(A_2)} = \alpha_2.$$

Secondly we consider the case $n=4$. Using

$$\left.\frac{r_2(a_2,a_3)}{r_1(a_2,a_3)}\right|_{a_2=0} = 1, \quad \left.\frac{r_2(a_2,a_3)}{r_1(a_2,a_3)}\right|_{a_2=\infty} = \infty,$$

we can choose a continuous function $\overline{a}_2 : [0,\infty) \to [0,\infty)$ with

$$\frac{r_2(\overline{a}_2(a_3),a_3)}{r_1(\overline{a}_2(a_3),a_3)} = \alpha_2.$$

Using

$$\left.\frac{r_2(\overline{a}_2(a_3),a_3)}{r_1(\overline{a}_2(a_3),a_3)}\right|_{a_3=0} = 1, \quad \left.\frac{r_2(\overline{a}_2(a_3),a_3)}{r_1(\overline{a}_2(a_3),a_3)}\right|_{a_3=\infty} = \infty,$$

we can choose $A_3 \in (0,\infty)$ with

$$\frac{r_3(\overline{a}_2(A_3),A_3)}{r_1(\overline{a}_2(A_3),A_3)} = \alpha_3.$$

Putting

$$A_2 = \overline{a}_2(A_3),$$

we obtain

$$\frac{r_2(A_2,A_3)}{r_1(A_2,A_3)} = \alpha_2, \quad \frac{r_3(A_2,A_3)}{r_1(A_2,A_3)} = \alpha_3.$$

Finally we consider the case $n \ge 5$. For every $a_j \ge 0$ $(3 \le j \le n-1)$, there exists a continuous function $\overline{a}_2(a_3,\dots,a_{n-1}) \in [0,\infty)$ with

$$\frac{r_2(\overline{a}_2(a_3,\dots,a_{n-1}),a_3,\dots,a_{n-1})}{r_1(\overline{a}_2(a_3,\dots,a_{n-1}),a_3,\dots,a_{n-1})} = \alpha_2.$$

For every $a_j \ge 0$ $(4 \le j \le n-1)$, there exists a continuous function $\overline{a}_3(a_4,\dots,a_{n-1}) \in [0,\infty)$ with

$$\frac{r_3(\overline{a}_2(\overline{a}_3(a_4,\dots,a_{n-1}),\dots,a_{n-1}),\overline{a}_3(a_4,\dots,a_{n-1}),\dots,a_{n-1})}{r_1(\overline{a}_2(\overline{a}_3(a_4,\dots,a_{n-1}),\dots,a_{n-1}),\overline{a}_3(a_4,\dots,a_{n-1}),\dots,a_{n-1})} = \alpha_3.$$

Continuing this argument, we finally find a continuous function $\overline{a}_{n-2}(a_{n-1})$ and $A_{n-1} \in \mathbb{R}$. It suffices to put $A_{n-2} = \overline{a}_{n-2}(A_{n-1})$, $A_{n-3} = \overline{a}_{n-3}(A_{n-2}, A_{n-1})$ and so on. This completes the proof. ∎

Let

$$\boldsymbol{A}'(k) = (0, A_2(k), \ldots, A_{n-1}(k)) \tag{4.7}$$

for $k \in (0, \min\{k_0, c\})$. Let p be as in (4.1) for $\boldsymbol{A}'(k)$. Define $z_k = z_k(\boldsymbol{A}'(k)) \in \mathbb{R}$ by

$$V_k(\mathbf{0}', z_k; p) = \theta_0.$$

Hereafter we write $V_k(\boldsymbol{x}; p)$ simply as $V_k(\boldsymbol{x})$. We have

$$V_k(\mathbf{0}', z_k) = \theta_0,$$

$$V_k(0, \ldots, 0, \underbrace{r_i(\boldsymbol{A}'(k))}_{i}, 0, \ldots, 0, \zeta + z_k) = \theta_0, \qquad 2 \leq i \leq n-1,$$

$$\frac{r_i(\boldsymbol{A}'(k))}{r_1(\boldsymbol{A}'(k))} = \alpha_i, \qquad 2 \leq i \leq n-1.$$

We will study the limit of $V_k(\boldsymbol{x}', x_n + z_k; \boldsymbol{A}'(k))$ as $k \to 0$ in Section 5. By the Schauder estimate [10, Theorem 9.11], there exists a positive constant B such that

$$\|V_k\|_{L^\infty(\mathbb{R}^n)} < B$$

holds true for all $0 < k < k_0$.

Let s_1 and θ be arbitrarily given with

$$-1 < s_1 < \theta < 1,$$

$$0 < -F_k(s_1) < -F_k(\theta) \qquad \text{for all } k \in (0, \min\{k_0, c\}).$$

We choose and fix $R \in (0, \infty)$ with

$$(n-1)B(1+\theta) < \left(G(\theta) - k_0 \int_{-1}^{1} \sqrt{2G(\sigma')}\, \mathrm{d}\sigma' \right) R. \tag{4.8}$$

For arbitrarily given $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we define

$$\mathcal{D} = (\xi_1 - R, \xi_1 + R) \times (\xi_2 - R, \xi_2 + R) \times \cdots \times (\xi_{n-1} - R, \xi_{n-1} + R) \subset \mathbb{R}^{n-1}. \tag{4.9}$$

The volume of $\mathcal{D}$ is given by $(2R)^{n-1}$, and the surface area of the boundary of $\mathcal{D}$ is given by $2(n-1)(2R)^{n-2}$. Now we have

$$B(1+\theta)|\partial\mathcal{D}| < \left(G(\theta) - k_0 \int_{-1}^{1} \sqrt{2G(\sigma')}\, \mathrm{d}\sigma' \right) |\mathcal{D}| \tag{4.10}$$

for every $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Then we have

$$B(1+\theta)|\partial\mathcal{D}| < \left(G(\theta) - k \int_{-1}^{1} \sqrt{2G(\sigma')}\, \mathrm{d}\sigma' \right) |\mathcal{D}| \qquad \text{for all } k \in (0, k_0).$$

We define

$$\Omega(k) = \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, s_1 < V_k(\boldsymbol{x}', x_n) < \theta\}.$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be the outward normal vector on $\partial\Omega$. We have

$$\partial\Omega = \mathit{\Gamma}_\theta(k) \cup \mathit{\Gamma}_1(k) \cup \mathit{\Gamma}_{\mathrm{f}}(k),$$

where

$$\begin{aligned}
\mathit{\Gamma}_\theta(k) &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, V_k(\boldsymbol{x}', x_n) = \theta\}, \\
\mathit{\Gamma}_1(k) &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \mathcal{D}, V_k(\boldsymbol{x}', x_n) = s_1\}, \\
\mathit{\Gamma}_{\mathrm{f}}(k) &= \{(\boldsymbol{x}', x_n) \,|\, \boldsymbol{x}' \in \partial\mathcal{D}, s_1 \le V_k(\boldsymbol{x}', x_n) \le \theta\}.
\end{aligned}$$

Lemma 2 *For every $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, let $\mathcal{D}$ be given by (4.9). Then one has*

$$\frac{1}{2}\int_{\mathit{\Gamma}_\theta(k)} |\nabla V_k| \left(-\frac{\partial V_k}{\partial x_n}\right) \mathrm{d}s \ge \left(G(\theta) - k\int_{-1}^{1} \sqrt{2G(\sigma')}\,\mathrm{d}\sigma'\right) |\mathcal{D}| - B(1+\theta)|\partial\mathcal{D}| > 0$$

for all $k \in (0, \min\{k_0, c\})$.

Proof. We write V_k, $\Omega(k)$, $\mathit{\Gamma}_\theta(k)$, $\mathit{\Gamma}_1(k)$, $\mathit{\Gamma}_{\mathrm{f}}(k)$ simply as V, Ω, $\mathit{\Gamma}_\theta$, $\mathit{\Gamma}_1$, $\mathit{\Gamma}_{\mathrm{f}}$, respectively. Then we have

$$\operatorname{div}\left(\frac{\partial V}{\partial x_n}\nabla V\right) = \frac{\partial V}{\partial x_n}\Delta V + \frac{1}{2}\frac{\partial}{\partial x_n}\left(|\nabla V|^2\right).$$

Multiplying (4.5) by $\partial V/\partial x_n$, we have

$$-\operatorname{div}\left(\frac{\partial V}{\partial x_n}\nabla V\right) + \frac{1}{2}\frac{\partial}{\partial x_n}\left(|\nabla V|^2\right) - c\left(\frac{\partial V}{\partial x_n}\right)^2 - f_k(V)\frac{\partial V}{\partial x_n} = 0.$$

Integrating the both hand sides over Ω and using the Gauss divergence theorem, we get

$$\int_{\partial\Omega}\left(-\frac{\partial V}{\partial x_n}(\nabla V, \nu)\,\mathrm{d}s + \frac{1}{2}|\nabla V|^2\nu_n\right)\mathrm{d}s - c\int_\Omega \left(\frac{\partial V}{\partial x_n}\right)^2 \mathrm{d}\boldsymbol{x} - \int_\Omega \frac{\partial}{\partial x_n}\left(F_k(V)\right)\mathrm{d}\boldsymbol{x} = 0.$$

Using

$$\boldsymbol{\nu} = \frac{\nabla V}{|\nabla V|} \qquad \text{on } \mathit{\Gamma}_\theta,$$

we get

$$-\frac{\partial V}{\partial x_n}(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2\nu_n = -\frac{1}{2}|\nabla V|\frac{\partial V}{\partial x_n} \qquad \text{on } \mathit{\Gamma}_\theta.$$

Similarly, using

$$\boldsymbol{\nu} = -\frac{\nabla V}{|\nabla V|} \qquad \text{on } \mathit{\Gamma}_1,$$

we get

$$-\frac{\partial V}{\partial x_n}(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2\nu_n = \frac{1}{2}|\nabla V|\frac{\partial V}{\partial x_n} \qquad \text{on } \mathit{\Gamma}_1.$$

Using $\nu_n = 0$ on $\mathit{\Gamma}_{\mathrm{f}}$, we have

$$-\frac{\partial V}{\partial x_n}(\nabla V, \boldsymbol{\nu}) + \frac{1}{2}|\nabla V|^2 \nu_n = -\frac{\partial V}{\partial x_n}(\nabla V, \boldsymbol{\nu}) \qquad \text{on } \mathit{\Gamma}_{\mathrm{f}}.$$

We have

$$\int_{\Omega} \frac{\partial}{\partial x_n} \left(F_k(V)\right) \mathrm{d}\boldsymbol{x} = \int_{\mathcal{D}} \left(F_k(s_1) - F_k(\theta)\right) \mathrm{d}\boldsymbol{x} > 0.$$

Now we calculate

$$\left| \int_{\mathit{\Gamma}_{\mathrm{f}}} (\nabla V, \boldsymbol{\nu}) \frac{\partial V}{\partial x_n} \, \mathrm{d}s \right| \le \left(\max_{\mathbb{R}^n} |\nabla V| \right) \int_{\mathit{\Gamma}_{\mathrm{f}}} \left(-\frac{\partial V}{\partial x_n} \right) \mathrm{d}s.$$

Using

$$\int_{\mathit{\Gamma}_{\mathrm{f}}} \left(-\frac{\partial V}{\partial x_n} \right) \mathrm{d}s = \int_{\partial \mathcal{D}} (\theta - s_1) \, \mathrm{d}s \le (\theta - s_1)|\partial \mathcal{D}|.$$

Then we obtain

$$\begin{aligned}
& \frac{1}{2} \int_{\mathit{\Gamma}_{\theta}} |\nabla V| \left(-\frac{\partial V}{\partial x_n} \right) \mathrm{d}s \\
\ge \quad & \frac{1}{2} \int_{\mathit{\Gamma}_{1}} |\nabla V| \left(-\frac{\partial V}{\partial x_n} \right) \mathrm{d}s + c \int_{\Omega} \left(\frac{\partial V}{\partial x_n} \right)^2 \mathrm{d}\boldsymbol{x} + \left(F_k(s_1) - F_k(\theta)\right) |\mathcal{D}| - B(\theta - s_1)|\partial \mathcal{D}| \\
\ge \quad & \frac{1}{2} \int_{\mathit{\Gamma}_{1}} |\nabla V| \left(-\frac{\partial V}{\partial x_n} \right) \mathrm{d}s + c \int_{\Omega} \left(\frac{\partial V}{\partial x_n} \right)^2 \mathrm{d}\boldsymbol{x} \\
& + \left(G(\theta) - G(s_1) - k \int_{-1}^{1} \sqrt{2G(\sigma')} \, \mathrm{d}\sigma' \right) |\mathcal{D}| - B(1 + \theta)|\partial \mathcal{D}|.
\end{aligned}$$

Sending $s_1 \to -1$, we complete the proof. ∎

We define $g_\theta(\boldsymbol{x}'; k) \in \mathbb{R}$ by

$$V_k(\boldsymbol{x}', g_\theta(\boldsymbol{x}'; k)) = \theta.$$

Then $g_\theta(\boldsymbol{x}'; k)$ is of class $C^1(\mathbb{R}^{n-1})$ for each $k \in (0, \min\{c, k_0\})$, and satisfies

$$\begin{gathered}
g_\theta(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{n-1}; k) = g_\theta(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{n-1}; k), \\
\frac{\partial g_\theta}{\partial x_j}(\boldsymbol{x}'; k) > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, x_j > 0
\end{gathered}$$

for every $1 \le j \le n - 1$.

Proposition 1 *For each $k \in (0, \min\{k_0, c\})$ one has*

$$\begin{aligned}
\frac{1}{2} \int_{\mathcal{D}} |\nabla V_k(\boldsymbol{x}', g_\theta(\boldsymbol{x}'; k))|^2 \, \mathrm{d}\boldsymbol{x}' \quad &= \quad \frac{1}{2} \int_{\mathit{\Gamma}_\theta(k)} |\nabla V_k(\boldsymbol{x})| \left(-\frac{\partial V_k}{\partial x_n} \right) \mathrm{d}s \\
&\ge \quad \left(G(\theta) - k \int_{-1}^{1} \sqrt{2G(\sigma')} \, \mathrm{d}\sigma' \right) |\mathcal{D}| - B(1 + \theta)|\partial \mathcal{D}| > 0.
\end{aligned}$$

Proof. We write V_k and $g_\theta(\boldsymbol{x}'; k)$ simply as V and $g_\theta(\boldsymbol{x}')$, respectively. Since $\boldsymbol{\nu}$ is the outward normal vector at $\partial\Omega$, we have

$$\boldsymbol{\nu} = \frac{\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))}{|\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))|} = -\frac{1}{\sqrt{1+|\nabla g_\theta(\boldsymbol{x}')|^2}} \begin{pmatrix} -\nabla g_\theta(\boldsymbol{x}') \\ 1 \end{pmatrix} \qquad \text{on } \mathit{\Gamma}_\theta,$$

where

$$\nabla g_\theta(\boldsymbol{x}') = \left(\frac{\partial g_\theta}{\partial x_1}(\boldsymbol{x}'), \ldots, \frac{\partial g_\theta}{\partial x_n}(\boldsymbol{x}') \right)$$

Then we have

$$\nu_n = \frac{\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))}{|\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))|} = -\frac{1}{\sqrt{1+|\nabla g_\theta(\boldsymbol{x}')|^2}} \qquad \text{on } \mathit{\Gamma}_\theta.$$

Thus we obtain

$$-\frac{\partial V}{\partial x_n}(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))\sqrt{1+|\nabla g_\theta(\boldsymbol{x}')|^2} = |\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))| \qquad \text{on } \mathit{\Gamma}_\theta,$$

and

$$\begin{aligned} & \int_{\mathit{\Gamma}_\theta} |\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))| \left(-\frac{\partial V}{\partial x_n}(\boldsymbol{x}', g_\theta(\boldsymbol{x}')) \right) \mathrm{d}s \\ = & \int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))| \left(-\frac{\partial V}{\partial x_n}(\boldsymbol{x}', g_\theta(\boldsymbol{x}')) \right) \sqrt{1+|\nabla g_\theta(\boldsymbol{x}')|^2}\, \mathrm{d}\boldsymbol{x}' \\ = & \int_{\mathcal{D}} |\nabla V(\boldsymbol{x}', g_\theta(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}'. \end{aligned}$$

Combining Lemma 2, we complete the proof. ∎

5 Balanced limits of pyramidal traveling fronts

In this section we study the limits of pyramidal traveling fronts for unbalanced reaction-diffusion equations as the reaction term approaches to a balanced one.

Taking a sequence

$$k_1 > \cdots > k_i > \cdots \to 0,$$

we define

$$U(\boldsymbol{x}', x_n; \boldsymbol{\alpha}') = \lim_{i\to\infty} V_{k_i}(\boldsymbol{x}', x_n + z_{k_i}; \boldsymbol{A}'(k_i)) \tag{5.1}$$

for all $(\boldsymbol{x}', x_n)$ in any compact set in $\mathbb{R}^n$. Here $\boldsymbol{A}'(k_i)$ is given by (4.7). We often write $U(\boldsymbol{x}; \boldsymbol{\alpha}')$ simply as $U(\boldsymbol{x})$.

Then $U(\boldsymbol{x}) = U(\boldsymbol{x}; \boldsymbol{\alpha}')$ satisfies the profile equation (1.5), $U(\mathbf{0}; \boldsymbol{\alpha}') = \theta_0$ and

$$\frac{\partial U}{\partial x_n} \le 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$U(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = U(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n), \quad 1 \le j \le n-1,$$
$$\frac{\partial U}{\partial x_j} \ge 0 \qquad \text{if } x_j > 0.$$

For each $1 \le j \le n-1$, we define $R_j > 0$ by (1.4). Then we have

$$\frac{R_j}{R_1} = \alpha_j, \qquad 1 \le j \le n-1.$$

By (1.4), we have

$$\frac{\partial U}{\partial x_j} > 0 \qquad \text{if } x_j > 0.$$

Now we have

$$\frac{\partial^2 U}{\partial x_j^2}(\mathbf{0}', x_n) \ge 0, \qquad 1 \le j \le n-1, \quad x_n \in \mathbb{R}. \tag{5.2}$$

We will show

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) < 0, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

If $\partial U/\partial x_n = 0$ at some point in $\mathbb{R}^n$, we have

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

from the maximum principle. Then $U(\boldsymbol{x}', x_n)$ is independent of x_n and is a function of $\boldsymbol{x}'$. By (1.5) and (5.2), we have

$$\sum_{j=1}^{n-1} \frac{\partial^2 U}{\partial x_j^2}(\mathbf{0}', x_n) - G'(U(\mathbf{0}', x_n)) = -\frac{\partial^2 U}{\partial x_n^2}(\mathbf{0}', x_n) - c\frac{\partial U}{\partial x_n}(\mathbf{0}', x_n) = 0, \qquad x_n \in \mathbb{R}.$$

Combining this equality and (5.2), we find $-G'(U(\mathbf{0}', x_n)) \le 0$ for all $x_n \in \mathbb{R}$. This contradicts $U(\mathbf{0}) = \theta_0$ and $-G'(\theta_0) > 0$. Thus we have

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) < 0 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Lemma 3 *One has*

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) < 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$U(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n) = U(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n), \quad 1 \le j \le n-1,$$
$$\frac{\partial U}{\partial x_j} > 0 \qquad \textit{if } x_j > 0,$$
$$\lim_{x_n \to \infty} U(\mathbf{0}', x_n) \in [-1, s_*],$$
$$\lim_{x_n \to -\infty} U(\mathbf{0}', x_n) = 1.$$

Proof. It suffices to prove the last two equalities. Using

$$\frac{\partial U}{\partial x_n}(\boldsymbol{x}) < 0 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n,$$

we have

$$\begin{aligned}\lim_{x_n\to\infty} U(\mathbf{0}', x_n) &= \omega \in [-1, \theta_0),\\ \lim_{x_n\to\infty} U(\mathbf{0}', x_n) &= \alpha \in (\theta_0, 1].\end{aligned}$$

Using (1.5) and (5.2), we have

$$\frac{\partial^2 U}{\partial x_n^2}(\mathbf{0}', x_n) + c\frac{\partial U}{\partial x_n}(\mathbf{0}', x_n) - G'(U(\mathbf{0}', x_n)) = -\sum_{j=1}^{n-1} \frac{\partial^2 U}{\partial x_j^2}(\mathbf{0}', x_n) \leq 0, \qquad x_n \in \mathbb{R}.$$

Sending $x_n \to \infty$ or $x_n \to -\infty$, we have $-G'(\alpha) \leq 0$ and $-G'(\omega) \leq 0$. From the definition of θ_0 and s_*, we obtain $\alpha = 1$ and $\omega \in [-1, \theta_0)$. This completes the proof. ∎

Since U satisfies (1.5), we have the following lemma.

Lemma 4 *Let U be given by (5.1). One has*

$$\|U\|_{C^{2,\alpha_0}(\mathbb{R}^n)} < \infty$$

for some $\alpha_0 \in (0, 1)$.

Proof. This lemma follows from general regularity theory for elliptic equations. See [10] for instance. ∎

Let $1 \leq j \leq n-1$ and we define

$$\begin{aligned}K(\boldsymbol{x}, \boldsymbol{y}, t) = &\frac{e^{-Mt}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{\sum_{1\leq i\leq n-1, i\neq j}(x_i - y_i)^2}{4t}\right)\\ &\times \left(\exp\left(-\frac{(x_j - y_j)^2}{4t}\right) - \exp\left(-\frac{(x_j + y_j)^2}{4t}\right)\right) \exp\left(-\frac{(x_n + ct - y_n)^2}{4t}\right)\end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^n$, $x_j > 0$, $y_j > 0$ and $t > 0$.

Lemma 5 (the Harnack inequality) *Let U be given by (5.1). For every $r_1 > 0$, there exists $C_1 = C_1(r_1)$ such that one has*

$$\sup_{B(\boldsymbol{x}_0; r_1)} \left(-\frac{\partial U}{\partial x_n}\right) \leq C_1 \inf_{B(\boldsymbol{x}_0; r_1)} \left(-\frac{\partial U}{\partial x_n}\right)$$

for all $\boldsymbol{x}_0 \in \mathbb{R}^n$. Here C_1 is independent of $\boldsymbol{x}_0$. For $1 \leq j \leq n-1$ one has

$$0 < \int_{\{\boldsymbol{x}\in\mathbb{R}^n \,|\, x_j > 0\}} K(\boldsymbol{x}, \boldsymbol{y}, 1)\frac{\partial U}{\partial x_j}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \leq \frac{\partial U}{\partial x_j}(\boldsymbol{x}), \qquad \text{if} \quad x_j > 0.$$

Proof. For the proof the former half, one can see [10]. Here we prove the latter half. We see that $\partial U/\partial x_j(\boldsymbol{x})$ is a stationary solution to

$$\begin{aligned} \frac{\partial W}{\partial t} &= \Delta W + c\frac{\partial W}{\partial x_n} + f'(U(\boldsymbol{x}))W, \qquad x_j > 0, t > 0, \\ W(\boldsymbol{x}, 0) &= \frac{\partial U}{\partial x_j}(\boldsymbol{x}) > 0 \qquad \text{if} \quad x_j > 0, \\ W(\boldsymbol{x}, t) &= 0 \qquad \text{if} \quad x_j = 0, t > 0. \end{aligned}$$

Let $\widetilde{W}(\boldsymbol{x}, t)$ be given by

$$\begin{aligned} \frac{\partial \widetilde{W}}{\partial t} &= \Delta \widetilde{W} + c\frac{\partial \widetilde{W}}{\partial x_n} - M\widetilde{W}, \qquad x_j > 0, t > 0, \\ \widetilde{W}(\boldsymbol{x}, 0) &= \frac{\partial U}{\partial x_j}(\boldsymbol{x}) > 0 \qquad \text{if} \quad x_j > 0, \\ \widetilde{W}(\boldsymbol{x}, t) &= 0 \qquad \text{if} \quad x_j = 0, t > 0. \end{aligned}$$

Then we have

$$\widetilde{W}(\boldsymbol{x}, t) = \int_{\{\boldsymbol{x} \in \mathbb{R}^n \mid x_j > 0\}} K(\boldsymbol{x}, \boldsymbol{y}, t)\frac{\partial U}{\partial x_j}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \qquad x_j > 0,$$

$$0 < \widetilde{W}(\boldsymbol{x}, t) \le W(\boldsymbol{x}, t) \qquad \text{if} \quad x_j > 0, t > 0.$$

Setting $t = 1$, we have

$$0 < \widetilde{W}(\boldsymbol{x}, 1) \le W(\boldsymbol{x}, 1) = \frac{\partial U}{\partial x_j}(\boldsymbol{x}), \qquad x_j > 0.$$

This completes the proof. ∎

For every $s \in (-1, 1)$ we define $q_s(\boldsymbol{x}') \in \mathbb{R}$ by

$$U(\boldsymbol{x}', q_s(\boldsymbol{x}')) = s, \tag{5.3}$$

if it exists. If $q_s(\boldsymbol{x}')$ exists, it is of class C^1 in some open set in $\mathbb{R}^{n-1}$ and satisfies

$$q_s(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{n-1}) = q_s(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{n-1}), \tag{5.4}$$

$$\frac{\partial q_s}{\partial x_j}(\boldsymbol{x}') > 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, x_j > 0 \tag{5.5}$$

for every $1 \le j \le n-1$. In view of Lemma 3, q_s is defined for $\boldsymbol{x}'_0 \in \mathbb{R}^{n-1}$ if and only if

$$\lim_{x_n \to \infty} U(\boldsymbol{x}'_0, x_n) < s.$$

q_s cannot be defined for $\boldsymbol{x}'_0 \in \mathbb{R}^{n-1}$ if and only if

$$\lim_{x_n \to \infty} U(\boldsymbol{x}'_0, x_n) \ge s.$$

Similarly, for every $s \in (-1,1)$ and $1 \le j \le n-1$, we define $q_s^j(\boldsymbol{x}'', x_n)$ by

$$U(\boldsymbol{x}'', q_s^j(\boldsymbol{x}'', x_n), x_n) = s,$$

if it exists, where

$$\boldsymbol{x}'' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}).$$

Let $\theta_1 \in (s_*, 1)$ be arbitrarily given. Then q_{θ_1} exists from Lemma 3.

Lemma 6 *Let $\theta_1 \in (s_*, 1)$ be arbitrarily given. Let $v(\boldsymbol{x}')$ satisfy*

$$\sum_{j=1}^{n-1} \frac{\partial^2 v}{\partial x_j^2}(\boldsymbol{x}') - G'(v(\boldsymbol{x}')) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$
$$-1 \le v(\boldsymbol{x}') \le 1, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$
$$v(\boldsymbol{x}') \ge \theta_1 \qquad \textit{if} \quad |(\boldsymbol{x}', \boldsymbol{u}')| \ge A,$$

where A is a positive number and $\boldsymbol{u}'$ is a unit vector in $\mathbb{R}^{n-1}$. Then one has

$$v(\boldsymbol{x}') = 1 \qquad \textit{for all } \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Proof. First we prove

$$\lim_{m\to\infty} \sup \left\{ |v(\boldsymbol{x}') - 1| \mid \boldsymbol{x}' \in \mathbb{R}^{n-1},\ |(\boldsymbol{x}', \boldsymbol{u}')| \ge m \right\} = 0.$$

Let $W(\boldsymbol{x}', t; W_0)$ be the solution of

$$\frac{\partial W}{\partial t}(\boldsymbol{x}', t) - \sum_{j=1}^{n-1} \frac{\partial^2 W}{\partial x_j^2}(\boldsymbol{x}', t) - G'(W(\boldsymbol{x}', t)) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0,$$
$$W(\boldsymbol{x}', 0) = W_0(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$

where W_0 is any bounded and uniformly continuous function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Then we have

$$W(\boldsymbol{x}', t; v) = v(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0.$$

$W(\boldsymbol{x}', t; \theta_1)$ is independent of $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ and satisfies

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}' \in \mathbb{R}^{n-1}} |W(\boldsymbol{x}', t; \theta_1) - 1| = 0. \tag{5.6}$$

Putting $\overline{w}(\boldsymbol{x}', t) = W(\boldsymbol{x}', t; \theta_1) - W(\boldsymbol{x}', t; v)$, we have

$$\frac{\partial \overline{w}}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 \overline{w}}{\partial x_j^2} - \int_0^1 G''(\tau W(\boldsymbol{x}', t; \theta_1) + (1-\tau) W(\boldsymbol{x}', t; v))\, \mathrm{d}\tau\, \overline{w} = 0, \quad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0,$$
$$\overline{w}(\boldsymbol{x}', 0) = \theta_1 - v(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Especially we have

$$\overline{w}(\boldsymbol{x}',0) \le 0 \qquad \text{if} \quad |(\boldsymbol{x}',\boldsymbol{u}')| \ge A,$$
$$|\overline{w}(\boldsymbol{x}',0)| \le 2 \qquad \text{if} \quad |(\boldsymbol{x}',\boldsymbol{u}')| \le A,$$

Let $\widehat{w}(\boldsymbol{x}',t)$ be the solution of

$$\frac{\partial \widehat{w}}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 \widehat{w}}{\partial x_j^2} - M\overline{w} = 0, \quad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0,$$
$$\overline{w}(\boldsymbol{x}',0) = \begin{cases} 0 & \text{if} \quad |(\boldsymbol{x}',\boldsymbol{u}')| \ge A, \\ 2 & \text{if} \quad |(\boldsymbol{x}',\boldsymbol{u}')| \le A. \end{cases}$$

Then we have

$$\widehat{w}(\boldsymbol{x}',t) = e^{Mt} \int_{\mathbb{R}^{n-1}} \frac{1}{(4\pi t)^{\frac{n-1}{2}}} \exp\left(-\frac{|\boldsymbol{x}'-\boldsymbol{y}'|^2}{4t}\right) \widehat{w}(\boldsymbol{y}',0)\, \mathrm{d}\boldsymbol{y}', \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0,$$

and

$$W(\boldsymbol{x}',t;\theta_1) - \widehat{w}(\boldsymbol{x}',t) \le W(\boldsymbol{x}',t;v) = v(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0.$$

Combining this inequality and (5.6), we obtain

$$\lim_{m\to\infty} \sup \left\{ |v(\boldsymbol{x}') - 1| \mid \boldsymbol{x}' \in \mathbb{R}^{n-1},\ |(\boldsymbol{x}',\boldsymbol{u}')| \ge m \right\} = 0.$$

Putting

$$v_1(\boldsymbol{x}') = 1 + v(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$

and using $G'(-1) = 0$, we have

$$-\sum_{j=1}^{n-1} \frac{\partial^2 v_1}{\partial x_j^2} - \int_0^1 G''(\tau v(\boldsymbol{x}') - 1 + \tau)\, \mathrm{d}\tau\, v_1(\boldsymbol{x}') = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$
$$v_1(\boldsymbol{x}') \ge 1 + \theta_1 \qquad \text{if} \quad |(\boldsymbol{x}',\boldsymbol{u}')| \ge A.$$

Let $w_1(\boldsymbol{x}',t)$ be the solution of

$$\frac{\partial w_1}{\partial t} - \sum_{j=1}^{n-1} \frac{\partial^2 w_1}{\partial x_j^2} - \int_0^1 G''(\tau v(\boldsymbol{x}') - 1 + \tau)\, \mathrm{d}\tau\, w_1 = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0.$$
$$w_1(\boldsymbol{x}',0) = v_1(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Then we have

$$\inf \{ w_1(\boldsymbol{x}',1) \,|\, (\boldsymbol{x}',\boldsymbol{u}') = 0 \} \ge \varepsilon_1$$

for some $\varepsilon_1 \in (0, \delta_*)$. Since $v_1(\boldsymbol{x}')$ is a stationary solution of this parabolic equation, we have $v_1(\boldsymbol{x}') = w_1(\boldsymbol{x}',1)$ and

$$\inf \{ v_1(\boldsymbol{x}') \,|\, (\boldsymbol{x}',\boldsymbol{u}') = 0 \} \ge \varepsilon_1.$$

Now we choose $\widetilde{f}$ such that we have

$$\begin{gathered} -G(u) \leq \widetilde{f}(u), \qquad -1 \leq u \leq 1, \\ \widetilde{f}(u) = -G(u) \qquad \text{if} \quad u \in [-1+\varepsilon_1, 1], \\ \int_{-1}^{1} \widetilde{f}(u)\,\mathrm{d}u > 0, \\ \widetilde{f}(-1) = 0, \quad \widetilde{f}'(-1) = -G'(-1) < 0, \\ \max_{|u| \leq 1+2\delta_*} \left| \widetilde{f}'(u) \right| \leq M. \end{gathered}$$

Then there exists a one-dimensional traveling front solution φ to

$$\begin{gathered} \varphi''(x) + \widetilde{c}\varphi'(x) + \widetilde{f}(\varphi(x)) = 0, \qquad x \in \mathbb{R}, \\ \varphi(-\infty) = 1, \quad \varphi(+\infty) = -1, \\ \varphi'(x) < 0, \qquad x \in \mathbb{R}. \end{gathered} \tag{5.7}$$

Here $\widetilde{c} \in (0, \infty)$ is the speed. We choose $\sigma > 0$ with

$$\sigma\beta \min\left\{-\varphi'(x) \mid -1+\delta \leq \varphi(x) \leq 1-\delta\right\} > \beta + M.$$

Now we consider the following parabolic equation for $\widetilde{w}(\boldsymbol{x}', t)$ given by

$$\frac{\partial \widetilde{w}}{\partial t} = \sum_{j=1}^{n-1} \frac{\partial^2 \widetilde{w}}{\partial x_j^2} + \widetilde{f}(\widetilde{w}), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0. \tag{5.8}$$

Now $v(\boldsymbol{x}')$ is a stationary solution of this parabolic equation. Let $\delta \in (0, \delta_*)$. Following to [9, 2],

$$\varphi(x - \widetilde{c}t + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta t},$$

becomes a subsolution to (5.8). Taking $\xi_0 > 0$ large enough, we have

$$\begin{aligned} \varphi((\boldsymbol{x}', \boldsymbol{u}') - \xi_0) - \delta \leq v(\boldsymbol{x}'), \qquad & \boldsymbol{x}' \in \mathbb{R}^{n-1}, \\ \varphi(-(\boldsymbol{x}', \boldsymbol{u}') - \xi_0) - \delta \leq v(\boldsymbol{x}'), \qquad & \boldsymbol{x}' \in \mathbb{R}^{n-1}, \end{aligned}$$

Then we find

$$\begin{aligned} \varphi((\boldsymbol{x}', \boldsymbol{u}') - \widetilde{c}t - \xi_0 + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta t} \leq v(\boldsymbol{x}'), \qquad & \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0, \\ \varphi(-(\boldsymbol{x}', \boldsymbol{u}') - \widetilde{c}t - \xi_0 + \sigma\delta(1 - e^{-\beta t})) - \delta e^{-\beta t} \leq v(\boldsymbol{x}'), \qquad & \boldsymbol{x}' \in \mathbb{R}^{n-1}, t > 0. \end{aligned}$$

Sending $t \to \infty$, we obtain

$$1 - \delta \leq v(\boldsymbol{x}') \leq 1, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Since we can choose $\delta \in (0, \delta_*)$ arbitrarily small, we find $v \equiv 1$. This completes the proof. □

Lemma 7 *Let U be define by (5.1). Then one has either (a) or (b). Here (a) and (b) are as follows.*

(a) One has

$$\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \,|\, U(\boldsymbol{x}', x_n) = \theta_1\} \cap \{\boldsymbol{x}' \,|\, |(\boldsymbol{x}', \boldsymbol{u}')| \geq A\} \neq \emptyset$$

if $x_n > 0$ is large enough. Here $A > 0$ be an arbitrarily given number and $\boldsymbol{u}'$ is any unit vector in $\mathbb{R}^{n-1}$.

(b) there exists $\zeta_1 > 0$ such that one has

$$\left\{\boldsymbol{x}' \in \mathbb{R}^{n-1} \,|\, U(\boldsymbol{x}', x_n) = \theta_1\right\} = \emptyset \qquad \text{if} \quad x_n \geq \zeta_1.$$

Proof. We will get a contradiction by assuming that both (a) and (b) are false. Using Lemma 3, we set

$$v(\boldsymbol{x}') = \lim_{x_n \to \infty} U(\boldsymbol{x}', x_n) \qquad \text{for} \quad \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Since (b) does not hold true,

$$\{\boldsymbol{x}' \in \mathbb{R}^{n} \,|\, U(\boldsymbol{x}', x_n) = \theta_1\} \neq \emptyset$$

if $x_n \geq 0$ is large enough. Thus we have

$$\{\boldsymbol{x}' \in \mathbb{R}^{n} \,|\, v(\boldsymbol{x}') = \theta_1\} \neq \emptyset.$$

Now v satisfies

$$\sum_{j=1}^{n-1} \frac{\partial^2 v}{\partial x_j^2}(\boldsymbol{x}') - G'(v(\boldsymbol{x}')) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1},$$

$$\frac{\partial v}{\partial x_j}(\boldsymbol{x}') \geq 0 \qquad \text{if} \quad x_j > 0, 1 \leq j \leq n-1.$$

Then v satisfies the assumptions of Lemma 6 and we have $v \equiv 1$. Then we have (b) and we get a contradiction. This completes the proof. □

Lemma 8 *Let $\theta_1 \in (s_*, 1)$ be arbitrarily given. For any $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ one has*

$$\lim_{x_n \to \infty} U(\boldsymbol{x}', x_n) < \theta_1.$$

A function $q_{\theta_1}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$.

Proof. If (b) in Lemma 7 holds true, q_{θ_1} is defined in $\mathbb{R}^{n-1}$ by an implicit function theorem. Thus it suffices to prove this lemma by assuming (a) in Lemma 7 holds true.

We will show

$$\lim_{x_n \to \infty} U(\mu_0, \ldots, \mu_0, x_n) < \theta_1 \tag{5.9}$$

for every $\mu_0 \in (0, \infty)$. Then, using

$$U(\boldsymbol{x}', x_n) \geq U(|\boldsymbol{x}_0'|, \ldots, |\boldsymbol{x}_0'|, x_n),$$

we have

$$\lim_{x_n\to\infty} U(\boldsymbol{x}'_0, x_n) \le \lim_{x_n\to\infty} U(|\boldsymbol{x}'_{\mathbf{0}}|, \dots, |\boldsymbol{x}'_{\mathbf{0}}|, x_n) < \theta_1$$

for every $\boldsymbol{x}'_0 \in \mathbb{R}^{n-1}$. Then $q_{\theta_1}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$.

Hereafter we assume the contrary of (5.9) and get a contradiction. Then there exists $\mu_1 \in (0,\infty)$ such that we have

$$\lim_{x_n\to\infty} U(\mu_1, \dots, \mu_1, x_n) \ge \theta_1.$$

From (a) in Lemma 7 and Lemma 3, there exists $X_j(x_n) \in (0,\infty)$ such that

$$U(0,\dots,0,\overset{j}{\smile}{X_j(x_n)},0,\dots,0,x_n) = \theta_1,$$
$$\lim_{x_n\to\infty} X_j(x_n) = \infty$$

for every $1 \le j \le n-1$. Now we define

$$v(\boldsymbol{x}') = \lim_{x_n\to\infty} U(x_1,\dots,x_{j-1},x_j + \frac{1}{2}X_j(x_n), x_{j+1},\dots,x_{n-1},x_n), \qquad \boldsymbol{x}' \in \mathbb{R}^{n-1}.$$

Then v satisfies

$$v(\mathbf{0}') < \theta_1,$$
$$v(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_{n-1}) = v(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_{n-1}), \qquad \text{if} \quad i \ne j.$$

Since v satisfies the assumption of Lemma 6 with

$$\boldsymbol{u}' = \frac{1}{\sqrt{n-2}}(1,\dots,1,\overset{j}{\smile}{0},1,\dots,1),$$

Lemma 6 gives $v \equiv 1$. This contradicts $v(\mathbf{0}') < \theta_1$. Now we complete the proof. □

Lemma 9 *Assume that $\theta_1 \in (s_*, 1)$ is arbitrarily given. Let $R \in (0,\infty)$ satisfy (4.8) and let $\mathcal{D} = (-R,R)^{n-1}$. One has*

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta_1}(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \ge G(\theta_1)|\mathcal{D}| - B(1+\theta_1)|\partial\mathcal{D}| > 0.$$

One can choose $R > 0$ that satisfies (4.8) for all θ_1 in any given compact interval in $(s_*, 1)$.

Proof. Following to Proposition 1, we have

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla V_k(\boldsymbol{x}', g_{\theta_1}(\boldsymbol{x}';k))|^2 \, \mathrm{d}\boldsymbol{x}' \ge \left(G(\theta_1) - k\int_{-1}^{1} \sqrt{2G(\sigma')}\,\mathrm{d}\sigma' \right) |\mathcal{D}| - B(1+\theta_1)|\partial\mathcal{D}| > 0.$$

Using (5.1) and passing to the limit $k \to 0$, we obtain

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta_1}(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \ge G(\theta_1)|\mathcal{D}| - B(1+\theta_1)|\partial\mathcal{D}| > 0.$$

This completes the proof. □

Proposition 2 *Let $\theta_1 \in (s_*, 1)$ be arbitrarily given. Then, for any $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ one has*

$$\lim_{x_n \to \infty} U(\boldsymbol{x}', x_n) < \theta_1.$$

A function $q_{\theta_1}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$. One has

$$\frac{1}{2} \int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta_1}(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \geq G(\theta_1)|\mathcal{D}| - B(1+\theta_1)|\partial \mathcal{D}| > 0.$$

Proof. This proposition follows from Lemma 8 and Lemma 9. ∎

Lemma 10 *Let $\tau_0 \in (0, \infty)$ satisfy*

$$\sqrt{n\tau_0} \|U\|_{C^2(\mathbb{R}^n)} < \frac{1}{2}. \tag{5.10}$$

Then one has

$$\|\nabla U(\boldsymbol{x} - \tau_0 \nabla U(\boldsymbol{x})) - \nabla U(\boldsymbol{x})\| \leq \frac{1}{2} |\nabla U(\boldsymbol{x})|, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Proof. Let $\boldsymbol{y} = \nabla U(\boldsymbol{x})$. For $1 \leq j \leq n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial U}{\partial x_j}(\boldsymbol{x} - \tau \boldsymbol{y}) \right) = -\sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_j}(\boldsymbol{x} - \tau \boldsymbol{y}) y_i.$$

Using

$$\frac{\partial U}{\partial x_j}(\boldsymbol{x} - \tau_0 \boldsymbol{y}) - \frac{\partial U}{\partial x_j}(\boldsymbol{x}) = -\int_0^{\tau_0} \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_j}(\boldsymbol{x} - \tau \boldsymbol{y}) y_i \mathrm{d}\tau,$$

we find

$$\begin{aligned} \left| \frac{\partial U}{\partial x_j}(\boldsymbol{x} - \tau_0 \boldsymbol{y}) - \frac{\partial U}{\partial x_j}(\boldsymbol{x}) \right| &\leq \sqrt{\left(\sum_{i=1}^{n} \int_0^{\tau_0} \left| \frac{\partial^2 U}{\partial x_i \partial x_j}(\boldsymbol{x} - \tau \boldsymbol{y}) \right|^2 \mathrm{d}\tau \right)} \sqrt{\sum_{i=1}^{n} y_i^2} \\ &\leq \sqrt{\tau_0} \|U\|_{C^2(\mathbb{R}^n)} |\boldsymbol{y}|. \end{aligned}$$

Thus we get

$$|\nabla U(\boldsymbol{x} - \tau_0 \boldsymbol{y}) - \nabla U(\boldsymbol{x})| \leq \sqrt{n\tau_0} \|U\|_{C^2(\mathbb{R}^n)} |\boldsymbol{y}|.$$

∎

Proposition 3 *Let J be any given compact connected set in $(-1,1)$ including θ_1 in Proposition 2. Let $R>0$ satisfy (4.8) for all $\theta' \in J$. For every given $(\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{n-1}$, let $\mathcal{D}=(\xi_1-R,\xi_1+R)\times\cdots\times(\xi_{n-1}-R,\xi_{n-1}+R)$. Then there exists a positive number κ_0 that depends only on f, c and J, and is independent of the choice of $\theta \in J$, such that one has the following. Let $\theta \in J$ satisfy $\theta+\kappa_0 \in J$ and $\theta-\kappa_0 \in J$. Assume that $q_{\theta+\kappa_0}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ and one has*

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta+\kappa_0}(\boldsymbol{x}'))|^2 \,\mathrm{d}\boldsymbol{x}' \geq G(\theta+\kappa_0)|\mathcal{D}| - B(1+\theta+\kappa_0)|\partial\mathcal{D}| > 0.$$

Then $q_{\theta-\kappa_0}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ and one has

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta-\kappa_0}(\boldsymbol{x}'))|^2 \,\mathrm{d}\boldsymbol{x}' \geq G(\theta-\kappa_0)|\mathcal{D}| - B(1+\theta-\kappa_0)|\partial\mathcal{D}| > 0.$$

Proof. Let $\tau_0 > 0$ satisfy (5.10). Let ρ_0 be small enough to satisfy

$$0 < 2\rho_0 < \frac{\tau_0}{2}\min_{u\in J}\frac{G(u)|\mathcal{D}| - B(1+u)|\partial\mathcal{D}|}{|\mathcal{D}|}, \tag{5.11}$$

and we choose $\kappa_0 \in (0, \rho_0/2]$ small enough to satisfy

$$0 < \kappa_0 < \min\{1-\max J, 1+\min J\}.$$

By the assumptions there exists $\boldsymbol{x}_1' \in \mathcal{D}$ with

$$|\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))|^2 \geq 2\min_{u\in J}\frac{G(u)|\mathcal{D}| - B(1+u)|\partial\mathcal{D}|}{|\mathcal{D}|}.$$

For any $0 \leq \xi \leq \tau_0$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\xi}U\left((\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) - \xi\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right)\\
&= -\nabla U\left((\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) - \xi\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right)\cdot\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\\
&\leq -\frac{1}{2}\left|\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right|^2
\end{aligned}$$

in view of Lemma 10. Then we have

$$\begin{aligned}
&U\left((\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) - \tau_0\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right) - U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\\
&\leq -\frac{1}{2}\tau_0\left|\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right|^2\\
&\leq -\tau_0\min_{u\in J}\frac{G(u)|\mathcal{D}| - B(1+u)|\partial\mathcal{D}|}{|\mathcal{D}|}
\end{aligned}$$

and

$$\begin{aligned}
&U\left((\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) - \tau_0\nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1'))\right)\\
&\leq \theta+\kappa_0 - \tau_0\min_{u\in J}\frac{G(u)|\mathcal{D}| - B(1+u)|\partial\mathcal{D}|}{|\mathcal{D}|} < \theta - 3\kappa_0.
\end{aligned} \tag{5.12}$$

By using

$$(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) - \tau_0 \nabla U(\boldsymbol{x}_1', q_{\theta+\kappa_0}(\boldsymbol{x}_1')) \in \overline{B((\xi_1, \ldots, \xi_{n-1}); R + \tau_0 \|U\|_{C^1(\mathbb{R}^n)})},$$

$q_{\theta-\kappa_0}(\boldsymbol{x}')$ is defined for some point in a closed ball $\overline{B((\xi_1, \ldots, \xi_{n-1}); R + \tau_0 \|U\|_{C^1(\mathbb{R}^n)})}$ for every $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Combining this fact and Lemma 3, we see that $q_{\theta-\kappa_0}(\boldsymbol{x}')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$. Sending $k \to 0$ in Proposition 1, we obtain

$$\frac{1}{2} \int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_{\theta-\kappa_0}(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \geq G(\theta - \kappa_0)|\mathcal{D}| - B(1 + \theta - \kappa_0)|\partial \mathcal{D}| > 0.$$

This completes the proof. □

In the following proposition we assert the contents of Theorem 1 except (1.6). We will prove (1.6) at the end of this section.

Proposition 4 *Let R be given by (4.8). For every given $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, let $\mathcal{D} = (\xi_1 - R, \xi_1 + R) \times \cdots \times (\xi_{n-1} - R, \xi_{n-1} + R)$. For every $\theta \in (-1, 1)$, one can define $q_\theta(\boldsymbol{x}') \in \mathbb{R}$ by $U(\boldsymbol{x}', q_\theta(\boldsymbol{x}')) = \theta$ for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$. Moreover, one has*

$$\frac{1}{2} \int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \geq G(\theta)|\mathcal{D}| - B(1 + \theta)|\partial \mathcal{D}| > 0.$$

Proof. Let κ_0 and$J$ be as in Proposition 3. Repeating the argument in Proposition 3 finite times, we see that $q_\theta(bmx')$ is defined for all $\boldsymbol{x}' \in \mathbb{R}^{n-1}$ and we obtain

$$\frac{1}{2} \int_{\mathcal{D}} |\nabla U(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' \geq G(\theta)|\mathcal{D}| - B(1 + \theta)|\partial \mathcal{D}| > 0$$

for every $\theta \in J$. Since J can be any compact connected set in $(-1, 1)$, this proposition holds true for every $\theta \in (-1, 1)$. This completes the proof. □

Let s_1 and θ be arbitrarily given with

$$-1 < s_1 < \theta < 1, \qquad 0 < G(s_1) < G(\theta). \tag{5.13}$$

For every $\boldsymbol{a}' = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$, we have $q_\theta(\boldsymbol{a}')$ using Proposition 4. Let $1 \leq j \leq n-1$ be arbitrarily given. We define $\boldsymbol{x}'' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1})$. For $(\xi_1, \ldots, \xi_{n-1}) = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n-1}, q_\theta(\boldsymbol{a}'))$, let $\mathcal{D}$ be given by (4.9). We define

$$\Omega^j = \{(\boldsymbol{x}'', x_j, x_n) \,|\, x_j > 0,\, (\boldsymbol{x}'', x_n) \in \mathcal{D}, s_1 < U(\boldsymbol{x}', x_n) < \theta\}.$$

For given s_1 and θ with (5.13), we can find $M(s_1, \theta) > 0$ such that, for every $\boldsymbol{a}' \in \mathbb{R}^{n-1}$ we have

$$\partial \Omega^j = \Gamma_\theta^j \cup \Gamma_1^j \cup \Gamma_{\mathrm{f}}^j \qquad \text{if } q_\theta(\boldsymbol{a}') > M(s_1, \theta),$$

where

$$\begin{aligned}
\Gamma_\theta^j &= \{(\boldsymbol{x}'', x_j, x_n) \,|\, x_j > 0,\, (\boldsymbol{x}'', x_n) \in \mathcal{D}, U(\boldsymbol{x}'', x_j, x_n) = \theta\}, \\
\Gamma_1^j &= \{(\boldsymbol{x}'', x_j, x_n) \,|\, x_j > 0,\, (\boldsymbol{x}'', x_n) \in \mathcal{D}, U(\boldsymbol{x}'', x_j, x_n) = s_1\}, \\
\Gamma_{\mathrm{f}}^j(k) &= \{(\boldsymbol{x}'', x_j, x_n) \,|\, x_j > 0,\, (\boldsymbol{x}'', x_n) \in \partial\mathcal{D}, s_1 \leq U(\boldsymbol{x}'', x_j, x_n) \leq \theta\}
\end{aligned}$$

by using Proposition 4. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be the outward normal vector on $\partial\Omega^j$.

The following lemma combined with Proposition 4 asserts that the width of the interface of U is bounded.

Lemma 11 *Let s_1 and θ satisfy (5.13). Let $1 \leq j \leq n-1$ be arbitrarily fixed. For every $\boldsymbol{a}' \in \mathbb{R}^{n-1}$, let $(\xi_1, \ldots, \xi_{n-1}) = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n-1}, q_\theta(\boldsymbol{a}'))$, and let $\mathcal{D}$ be given by (4.9). Assume $q_\theta(\boldsymbol{a}') > M(s_1, \theta)$. Then one has*

$$\frac{1}{2}\int_{\Gamma_\theta^j} |\nabla U| \frac{\partial U}{\partial x_j}\, \mathrm{d}s \geq (G(\theta) - G(s_1))\, |\mathcal{D}| - B(1+\theta)|\partial\mathcal{D}| > 0$$

for every $\boldsymbol{a}' \in \mathbb{R}^{n-1}$.

Proof. We have

$$\mathrm{div}\left(\frac{\partial U}{\partial x_j}\nabla U\right) = \frac{\partial U}{\partial x_j}\Delta U + \frac{1}{2}\frac{\partial}{\partial x_j}\left(|\nabla U|^2\right).$$

Multiplying (1.5) by $\partial U/\partial x_j$, we have

$$-\,\mathrm{div}\left(\frac{\partial U}{\partial x_j}\nabla U\right) + \frac{1}{2}\frac{\partial}{\partial x_j}\left(|\nabla U|^2\right) - c\frac{\partial U}{\partial x_j}\frac{\partial U}{\partial x_n} + G'(U)\frac{\partial U}{\partial x_j} = 0.$$

Integrating the both hand sides over Ω^j, we get

$$\int_{\partial\Omega^j}\left(-\frac{\partial U}{\partial x_j}(\nabla U, \nu)\,\mathrm{d}s + \frac{1}{2}|\nabla U|^2\nu_j\right)\mathrm{d}s - c\int_{\Omega^j}\frac{\partial U}{\partial x_j}\frac{\partial U}{\partial x_n}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^j}\frac{\partial}{\partial x_j}\left(G(U)\right)\mathrm{d}\boldsymbol{x} = 0.$$

Using

$$\boldsymbol{\nu} = \frac{\nabla U}{|\nabla U|} \qquad \text{on } \Gamma_\theta^j,$$

we get

$$-\frac{\partial U}{\partial x_j}(\nabla U, \boldsymbol{\nu}) + \frac{1}{2}|\nabla U|^2\nu_j = -\frac{1}{2}|\nabla U|\frac{\partial U}{\partial x_j} \qquad \text{on } \Gamma_\theta^j.$$

Similarly, using

$$\boldsymbol{\nu} = -\frac{\nabla U}{|\nabla U|} \qquad \text{on } \Gamma_1^j,$$

we get

$$-\frac{\partial U}{\partial x_j}(\nabla U, \boldsymbol{\nu}) + \frac{1}{2}|\nabla U|^2\nu_j = \frac{1}{2}|\nabla U|\frac{\partial U}{\partial x_j} \qquad \text{on } \Gamma_1^j.$$

Using $\nu_j = 0$ on Γ_{f}, we have

$$-\frac{\partial U}{\partial x_j}(\nabla U, \boldsymbol{\nu}) + \frac{1}{2}|\nabla U|^2\nu_j = -\frac{\partial U}{\partial x_j}(\nabla U, \boldsymbol{\nu}) \qquad \text{on } \Gamma_{\mathrm{f}}^j.$$

We have

$$\int_{\Omega^j}\frac{\partial}{\partial x_j}\left(G(U)\right)\mathrm{d}\boldsymbol{x} = (G(\theta) - G(s_1))\,|\mathcal{D}|.$$

Now we calculate

$$\left|\int_{\Gamma_{\mathrm{f}}^j} (\nabla U, \boldsymbol{\nu}) \frac{\partial U}{\partial x_j}\, \mathrm{d}s\right| \leq \left(\max_{\mathbb{R}^n} |\nabla U|\right) \int_{\Gamma_{\mathrm{f}}^j} \frac{\partial U}{\partial x_j}\, \mathrm{d}s.$$

Using

$$\int_{\Gamma_{\mathrm{f}}^j} \frac{\partial U}{\partial x_j}\, \mathrm{d}s = \int_{\partial \mathcal{D}} (\theta - s_1)\, \mathrm{d}s \leq (\theta - s_1)|\partial \mathcal{D}|.$$

Then we obtain

$$\begin{aligned} & \frac{1}{2}\int_{\Gamma_\theta^j} |\nabla U| \frac{\partial U}{\partial x_j}\, \mathrm{d}s \\ \geq\ & \frac{1}{2}\int_{\Gamma_1^j} |\nabla U| \frac{\partial U}{\partial x_j}\, \mathrm{d}s - c\int_{\Omega^j} \frac{\partial U}{\partial x_j}\frac{\partial U}{\partial x_n}\, \mathrm{d}\boldsymbol{x} + (G(\theta) - G(s_1))\, |\mathcal{D}| - B(\theta - s_1)|\partial \mathcal{D}| \\ \geq\ & (G(\theta) - G(s_1))\, |\mathcal{D}| - B(1+\theta)|\partial \mathcal{D}|. \end{aligned}$$

This completes the proof. □

Let θ and s_1 satisfy (5.13). Let $1 \leq j \leq n-1$ be arbitrarily fixed. For every $\boldsymbol{a}' \in \boldsymbol{R}^{n-1}$, let $(\xi_1, \ldots, \xi_{n-1}) = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n-1}, q_\theta(\boldsymbol{a}'))$, and let $\mathcal{D}$ be given by (4.9). If $q_\theta(\boldsymbol{a}') > M(s_1, \theta)$, we can define $q_\theta^j(\boldsymbol{x}'', x_n)$ by

$$U(\boldsymbol{x}'', q_\theta^j(\boldsymbol{x}'', x_n), x_n) = \theta \tag{5.14}$$

for every $(\boldsymbol{x}'', x_n) \in \mathcal{D}$.

Proposition 5 *Let θ and s_1 satisfy (5.13). Let $1 \leq j \leq n-1$ be fixed. For every $\boldsymbol{a}' \in \boldsymbol{R}^{n-1}$, let $(\xi_1, \ldots, \xi_{n-1}) = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n-1}, q_\theta(\boldsymbol{a}'))$, and let $\mathcal{D}$ be given by (4.9). Assume $q_\theta(\boldsymbol{a}') > M(s_1, \theta)$. Then one has*

$$\begin{aligned} & \frac{1}{2}\int_{\mathcal{D}} \left|\nabla U(\boldsymbol{x}'', q_\theta^j(\boldsymbol{x}'', x_n), x_n)\right|^2\, \mathrm{d}\boldsymbol{x}''\mathrm{d}x_n = \frac{1}{2}\int_{\Gamma_\theta^j} |U| \frac{\partial U}{\partial x_j}\, \mathrm{d}s \\ \geq\ & (G(\theta) - G(s_1))\, |\mathcal{D}| - B(1+\theta)|\partial \mathcal{D}| > 0. \end{aligned}$$

Proof. This proposition can be proved by a parallel argument as in the proof of Proposition 1 due to Lemma 11. □

Now we prove (1.6) as follows.

Proposition 6 *For every $\theta \in (-1, 1)$, one has*

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n} \left\{ |\nabla U(\boldsymbol{x})| \,\middle|\, U(\boldsymbol{x}) = \theta \right\} > 0.$$

Proof. Assume the contrary. Then there exists $(\boldsymbol{x}'_i)_{i\in\mathbb{N}}$ with

$$\lim_{i\to\infty} |\nabla U(\boldsymbol{x}'_i, q_\theta(\boldsymbol{x}'_i))| = 0.$$

Let $\boldsymbol{x}'_i = (x_1^{(i)}, \dots, x_{n-1}^{(i)})$ for $i \in \mathbb{N}$. First we consider the case $\limsup_{i\to\infty} |\nabla q_\theta(\boldsymbol{x}'_i)| < \infty$. Let $\mathcal{D}_i$ be given by (4.9) with $(\xi_1, \dots, \xi_{n-1}) = \boldsymbol{x}'_i$. Using Lemma 4, we have

$$\sup_{i\in\mathbb{N}} \max_{\boldsymbol{x}'\in\mathcal{D}_i} |\nabla q_\theta(\boldsymbol{x}')| < \infty.$$

By Proposition 4 we have

$$\inf_{i\in\mathbb{N}} \int_{\mathcal{D}_i} |\nabla U(\boldsymbol{x}', q_\theta(\boldsymbol{x}'))|^2 \, \mathrm{d}\boldsymbol{x}' > 0.$$

This contradicts the assumption in view of Lemma 5. Next we consider the case where we have $\limsup_{i\to\infty} |\nabla q_\theta(\boldsymbol{x}'_i)| = \infty$. Then we have $\lim_{i\to\infty} q_\theta(\boldsymbol{x}'_i) = \infty$. By taking a subsequence if necessary, we choose $1 \le j_0 \le n-1$ such that

$$\left\{ (\boldsymbol{x}'', q_\theta^{j_0}(\boldsymbol{x}'', x_n), x_n) \,\middle|\, (\boldsymbol{x}'', x_n) \in \mathcal{D}_i \right\}$$

is a part of the graph of q_θ, where $\boldsymbol{x}'' = (x_1, \dots, x_{j_0-1}, x_{j_0+1}, \dots, x_{n-1})$ and $\mathcal{D}_i$ is given by (4.9) with $(\xi_1, \dots, \xi_{n-1}) = (x_1^{(i)}, \dots, x_{j_0-1}^{(i)}, x_{j_0+1}^{(i)}, \dots, x_{n-1}^{(i)}, q_\theta(\boldsymbol{x}'_i))$, and we have

$$\sup_{i\in\mathbb{N}} \left| \nabla q_\theta^{j_0}(x_1^{(i)}, \dots, x_{j_0-1}^{(i)}, x_{j_0+1}^{(i)}, \dots, x_{n-1}^{(i)}, q_\theta(\boldsymbol{x}'_i)) \right| < \infty.$$

Then Lemma 4 gives

$$\sup_{i\in\mathbb{N}} \max_{(\boldsymbol{x}'', x_n)\in\mathcal{D}_i} \left| \nabla q_\theta^{j_0}(\boldsymbol{x}'', q_\theta^{j_0}(\boldsymbol{x}'', x_n), x_n) \right| < \infty.$$

Proposition 4 gives

$$\inf_{i\in\mathbb{N}} q_\theta^{j_0}(x_1^{(i)}, \dots, x_{j_0-1}^{(i)}, x_{j_0+1}^{(i)}, \dots, x_{n-1}^{(i)}, q_\theta(\boldsymbol{x}'_i)) > 0.$$

Using Proposition 5, we have

$$\frac{1}{2} \int_{\mathcal{D}_i} \left| \nabla U(\boldsymbol{x}'', q_\theta^{j_0}(\boldsymbol{x}'', x_n), x_n) \right|^2 \, \mathrm{d}\boldsymbol{x}'' \mathrm{d}x_n \ge (G(\theta) - G(s_1)) \, |\mathcal{D}_i| - B(1+\theta) |\partial \mathcal{D}_i|.$$

Using $\lim_{i\to\infty} q_\theta(\boldsymbol{x}'_i) = \infty$, we can choose s_1 to be arbitrarily close to -1 in (5.13). Taking the limit of $s_1 \to -1$, we have

$$\frac{1}{2} \int_{\mathcal{D}_i} \left| \nabla U(\boldsymbol{x}'', q_\theta^{j_0}(\boldsymbol{x}'', x_n), x_n) \right|^2 \, \mathrm{d}\boldsymbol{x}'' \mathrm{d}x_n \ge G(\theta)(2R)^{n-1} - B(1+\theta) 2(n-1)(2R)^{n-2} > 0.$$

This contradicts the assumption in view of Lemma 5. Now we complete the proof. ∎

Now our main assertion Theorem 1 follows from Proposition 4 and Proposition 6.

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