Several classes of plane partitions with the same generating function

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RIMS Workshop
Aspects of Combinatorial Representation Theory
October 11, 2018
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A plane partition is a two-dimensional array \((\pi_{ij})_{i,j \geq 1}\) of nonnegative integers that is nonincreasing both from left to right in each row and top to bottom in each column, i.e.,

\[
\pi_{i,j} \geq \pi_{i,j+1} \quad \text{and} \quad \pi_{i,j} \geq \pi_{i+1,j}
\]

for all \(i\) and \(j\), in which only finitely many of the entries are nonzero. A nonzero entries are called a \textit{part} and the sum \(|\pi| = \sum_{i,j \geq 1} \pi_{i,j}\) of parts is called \textit{weight} of the plane partition. The partition \(\lambda = (\lambda_1, \lambda_2, \ldots)\) defined by \(\lambda_i = \#\{j \mid \pi_{ij} \neq 0\}\) is called the \textit{shape} of \(\pi\) and denoted by \(\text{sh}(\pi)\).

**Example**

\[
\pi = \begin{pmatrix}
4 & 4 & 3 & 1 \\
4 & 2 & 1 \\
2 & 1
\end{pmatrix}
\]

is a plane partion with shape 432 and weight \(|\pi| = 22\).
Plane partitions

**Definition**

A plane partition \((\pi_{i,j})_{i,j \geq 1}\) is said to be **row-strict** (resp. **column-strict**) if \(\pi_{i+1,j} > \pi_{i,j}\) (resp. \(\pi_{i,j+1} > \pi_{i,j}\)) holds whenever the both sides nonzero. The **Ferrers graph** of \(\pi\) is defined to be

\[
F(\pi) = \{ (i, j, k) \mid i, j \geq 1, 1 \leq k \leq \pi_{ij} \}
\]

which is regarded as a subset of \(\mathbb{Z}^3\).

**Example**

\[
\pi = \begin{pmatrix}
4 & 4 & 3 & 1 \\
4 & 2 & 1 \\
2 & 1
\end{pmatrix}
\]
Plane partitions

Definition

If a plane partition \((\pi_{i,j})_{i,j \geq 1}\) has the shape \((n, n-1, \ldots, 1)\), we call it \(n\)-staircase. We set

\[
\mathcal{B}_{l,m,n} = \{ (i, j, k) \mid 1 \leq i \leq l, 1 \leq j \leq m, 1 \leq k \leq n \}.
\]

and we say \(\pi \in \mathcal{B}_{l,m,n}\) if \(\text{sh}(\pi) \subseteq I^m\) and \(\pi_{i,j} \leq n\).

Example

For example,

\[
\begin{array}{ccc}
4 & 3 & 1 \\
2 & 1 \\
1
\end{array}
\]

\(\pi = \begin{array}{ccc}
4 & 3 & 1 \\
2 & 1 \\
1
\end{array}\)

is 3-staircase column strict plane partition such that \(\pi \subseteq \mathcal{B}_{4,4,4}\).
Shifted plane partitions

**Definition**

A shifted plane partition is a two-dimensional array \((\pi_{i,j})_{1 \leq i \leq j}\) of nonnegative integers that is nonincreasing both from left to right in each row and top to bottom in each column, and the sum \(|\pi| = \sum_{i,j \geq 1} \pi_{i,j}\) of parts is called *weight* of the shifted plane partition. The strict partition \(\mu = (\mu_1, \mu_2, \ldots)\) defined by \(\mu_i = \#\{j \mid \pi_{ij} \neq 0\}\) is called the *shape* of \(\pi\) and denoted by \(\text{ssh}(\pi)\), and the strict partition \((\pi_{1,1}, \pi_{2,2}, \ldots)\), denoted by \(\text{pr}(\pi)\), is called the *profile* of \(\pi\).

**Example**

\[
\begin{array}{cccc}
4 & 4 & 3 & 1 \\
\pi = & 2 & 2 & \\
& & 1 \\
\end{array}
\]

is a shifted plane partition with shape 421 and weight \(|\pi| = 18\).
Shifted plane partition

**Definition**

The row-strictness (resp. column-strictness) of a shifted plane partition is defined similarly. The *Ferrers graph* of $\pi$ of a shifted plane partition is defined to be

$$F(\pi) = \{ (i, j, k) | 1 \leq i \leq j, 1 \leq k \leq \pi_{ij} \}.$$ 

**Example**

$$\pi = \begin{array}{cccc}
4 & 4 & 3 & 1 \\
2 & 2 & & \\
1 & & & \\
\end{array}$$

Diagram of the Ferrers graph corresponding to $\pi$. 
Shifted plane partitions

**Definition**

If a shifted plane partition \((\pi_{ij})_{i,j\geq 1}\) has the shape \((n, n-1, \ldots, 1)\), we say it is \textit{n-staircase}. We set

\[
\mathcal{SB}_{m,n} = \{(i, j, k) \mid 1 \leq i \leq j \leq m, 1 \leq k \leq n\}.
\]

and we say \(\pi \in \mathcal{SB}_{m,n}\) if ssh(\(\pi\)) \(\subseteq (m, m-1, \ldots, 1)\) and \(\pi_{i,j} \leq n\).

**Example**

For example,

\[
\pi = \begin{pmatrix}
4 & 4 & 3 \\
2 & 2 & \\
1
\end{pmatrix}
\]

is 3-staircase column-strict shifted plane partition such that \(\pi \subseteq \mathcal{SB}_{4,3}\).
Alternating sign matrices

**Definition**

An alternating sign matrix $A$ of size $n$ is an $n$ by $n$ square matrix of 0s, 1s, and $-1$s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign. Let $A_n$ denote the set of alternating sign matrices of size $n$.

**Example**

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} \in A_5$$

is an alternating sign matrix of size 5.
The weights for alternating sign matrices

Definition

For an alternating sign matrix $A = (A_{ij})_{1 \leq i, j \leq n}$ of size $n$, let $s(A)$ denote the number of $-1$s, set $p(A) = k - 1$ where the $1$ in the top row occurs in position $k$ and

$$\text{inv}(A) = \sum_{i<k} \sum_{j<l} A_{ij}A_{kl}.$$  

Example

An alternating sign matrix

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$$

has $s(A) = 2$, $p(A) = 1$ and $\text{inv}(A) = 5$. 
# Generating function for ASMs

## Definition

Let us define the generating function of $\mathcal{A}_n$ as

$$A(n; q, t, x) = \sum_{A \in \mathcal{A}_n} q^{\text{inv}(A)} t^{\text{p}(A)} x^{\text{s}(A)}.$$

## Example

$$A(2; q, t, x) = qt + 1,$$

$$A(3; q, t, x) = q^2 (q + 1) t^2 + \{q^2 x + q (q + 1)\} t + q + 1,$$

$$A(4; q, t, x) = q^3 \{q^2 x + (q + 1) (q^2 + q + 1)\} t^3$$

$$+ q^2 \{q^3 x^2 + q (q^2 + 4 q + 2) x + (q + 1) (q^2 + q + 1)\} t^2$$

$$+ q \{q^2 x^2 + q (2 q^2 + 4 q + 1) x + (q + 1) (q^2 + q + 1)\} t$$

$$+ q^2 x + (q + 1) (q^2 + q + 1).$$
Vertical-Symmetric alternating sign matrix

**Definition**

Let $\mathcal{A}^V_{2n+1}$ denote the set of $(2n + 1) \times (2n + 1)$ vertically symmetric ASMs (VSASMs). For a symmetric ASM, we set $s(A) = m$ if $A$ has $m$ of the orbits of the entries under symmetry excluding any $-1$s that are forced by symmetry. Set $p(A) = k - 1$ where the 1 in the leftmost column occurs in position $k$.

**Example**

An alternating sign matrix

$$A = \begin{pmatrix}
0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & +1 & -1 & +1 & 0 & 0 \\
+1 & 0 & -1 & +1 & -1 & 0 & +1 \\
0 & 0 & +1 & -1 & +1 & 0 & 0 \\
0 & +1 & -1 & +1 & -1 & +1 & 0 \\
0 & 0 & +1 & -1 & +1 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 & 0 & 0
\end{pmatrix}$$

has $s(A) = 2$ and $p(A) = 2$. 
Generating function for VSASMs

**Definition**

Let us define the generating function of $\mathcal{A}_n$ as

$$A_V(2n + 1; t, x) = \sum_{A \in \mathcal{A}_n} t^{p(A)} x^{s(A)}.$$ 

**Example**

We have $A_V(3; t, x) = 1$ and

$$A_V(5; t, x) = t^2 + xt + 1,$$

$$A_V(7; t, x) = (x + 2) t^4 + 2 x (x + 2) t^3 + (x + 1) \left(x^2 + x + 2\right) t^2 + 2 x (x + 2) t + x + 2,$$

$$A_V(9; t, x) = \left(x^3 + 6 x^2 + 13 x + 6\right) t^6 + 3 x \left(x^3 + 6 x^2 + 13 x + 6\right) t^5$$

$$+ \left(3 x^5 + 18 x^4 + 44 x^3 + 42 x^2 + 25 x + 6\right) t^4$$

$$+ x (x + 2)^2 \left(x^3 + 2 x^2 + 9 x + 6\right) t^3 + \left(3 x^5 + 18 x^4 + 44 x^3 + 42 x^2 + 25 x + 6\right) t^2$$

$$+ 3 x \left(x^3 + 6 x^2 + 13 x + 6\right) t + x^3 + 6 x^2 + 13 x + 6.$$
Relations between the generating functions

Theorem (Kuperberg)

There exists a polynomial $\tilde{A}_V(2n; t, x)$ such that the following identities hold:

$$A(2n; q, t, x, y)\bigg|_{q=y=1} = (t + 1)A_V(2n + 1; t, x)\tilde{A}_V(2n; 1, x),$$

$$A(2n - 1; q, t, x, y)\bigg|_{q=y=1} = A_V(2n - 1; 1, x)\tilde{A}_V(2n; t, x).$$

Example

We have $\tilde{A}_V(2; t, x) = 1$ and

\[
\begin{align*}
\tilde{A}_V(4; t, x) &= 2 t^2 + (x + 2) t + 2, \\
\tilde{A}_V(6; t, x) &= 2(x + 6)t^4 + (x + 6)(3x + 2)t^3 + (x^3 + 6x^2 + 26x + 12)t^2 + (x + 6)(3x + 2)t + 2(x + 6), \\
\tilde{A}_V(8; t, x) &= 2(x^3 + 12x^2 + 70x + 60)t^6 + (5x + 2)\left(x^3 + 12x^2 + 70x + 60\right)t^5 \\
&\quad + 2\left(2x^5 + 25x^4 + 161x^3 + 352x^2 + 310x + 60\right)t^4 + \left(x^6 + 12x^5 + 85x^4 + 452x^3 + 834x^2 + 680x + 120\right)t^3 \\
&\quad + 2\left(2x^5 + 25x^4 + 161x^3 + 352x^2 + 310x + 60\right)t^2 \\
&\quad + (5x + 2)\left(x^3 + 12x^2 + 70x + 60\right)t + 2(x^3 + 12x^2 + 70x + 60).
\end{align*}
\]
Six vertex model

Definition

A configuration in the six vertex model correspond to an alternating sign matrix:

\[
\begin{pmatrix}
1 & & & 3 & & & 5 \\
\uparrow & \searrow & \nearrow & \uparrow & \searrow & \nearrow & \\
0 & & & 0 & & & 1 \\
\searrow & \nearrow & \downarrow & & \searrow & \nearrow & 0 \\
5 & & & 3 & & & 1 \\
\uparrow & \nearrow & \downarrow & \uparrow & \nearrow & \downarrow & \\
0 & & & 0 & & & -1
\end{pmatrix}
\]

Example

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\uparrow & \searrow & \nearrow & \uparrow & \searrow & \nearrow & \\
\searrow & \nearrow & \downarrow & & \searrow & \nearrow & \\
\nearrow & \downarrow & \uparrow & \nearrow & \downarrow & \uparrow & \\
\searrow & \nearrow & \downarrow & \searrow & \nearrow & \downarrow & \\
\uparrow & \nearrow & \downarrow & \uparrow & \nearrow & \downarrow & \\
\downarrow & \nearrow & \searrow & \downarrow & \nearrow & \searrow & \\
\nearrow & \downarrow & \uparrow & \nearrow & \downarrow & \uparrow &
\end{pmatrix}
\]
Definition

The following figure shows the boundary condition of a UUASM. A UUASM is a $2n \times 2n$ matrix vertically just like an ASM, in which both the columns and the rows of a UUASM are like the rows of a UASM.

Let $\mathcal{A}_{4n}^{UU}$ denote the set of UUASMs of size $4n$. 
Definition

We define the $x$-weight $s$ of a UUASM be the number of $-1$s, as before. We define the $y$-weight of a UUASM to be $y^u$ if $u$ of the U-turns are oriented upward in the corresponding square ice state, and define the $z$-weight of a UUASM to be $z^r$ if $r$ of the U-turns on the top are oriented to the right. We set

$$A_{UU}(4n; x, y, z) = \sum_{A \in \mathcal{A}_{4n}^{UU}} x^s y^u z^r.$$ 

Example

For example, if $n = 1$, then there are 5 UUASMs:
UU-Turn Alternating Sign Matrices (UUASMs)

Example

We have

$$A_{UU}(4; x, y, z) = xz + (z + 1)(y + 1),$$

$$A_{UU}(8; x, y, z) = z^2 x^4 + z (2 yz + y + 6 z + 2) x^3$$

$$+ \left(2 y^2 z^2 + 3 y^2 z + 11 yz^2 + y^2 + 12 yz + 13 z^2 + 3 y + 11 z + 2\right) x^2$$

$$+ (z + 1)(y + 1)(5 yz + 3 y + 11 z + 5) x + 2 (z + 1)^2 (y + 1)^2.$$
**Definition**

Let $\mathcal{A}^{HTS}_{2n}$ denote the set of $2n \times 2n$ half-turn-symmetric ASMs (HTSASMs). Let $\mathcal{A}_{2n}^{HTS}$ denote the set of half-turn symmetric ASMs with size $2n$.

**Example**

We consider SVMs in $2n \times n$ rectangle with the following boundary condition:

```
   \_ o o . . o
   \_ o o . . o
   . . . . . .
   . . . . . .
   \_ o o . . o
   \_ o o . . o
```


Half-Turn-Symmetric alternating sign matrix

Example

For example, if $n = 2$ then the following 10 SVMs are with this boundary condition.
Half-Turn-Symmetric alternating sign matrix

**Example**

These SVM's correspond to the following ASM's.

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
Generating function for HTSASMs

**Definition**

We set $A_{HT}(2n; t, x, y, w) = \sum_{A \in \mathcal{A}_{2n}^{HTS}} t^{k-1} x^m y^u w^v$, where $u$ denotes the number of upward arrow on the east wall, and let $v$ denotes the number of nonzero entries in the upper half part, $A$ has $m$ of the orbits of the entries under symmetry, and the 1 in the first column occurs in position $k$.

**Example**

For example, we have $A_{HT}(2; t, x, y, w) = wy t + 1$ and

\[
A_{HT}(4; t, x, y, w) = yz (yz + 1) t^3 + \left\{yz^3 x + yz (yz + 1)\right\} t^2 + (yz x + yz + 1) t + yz + 1,
\]

\[
A_{HT}(6; t, x, y, w) = \left\{y^2 z^2 \left(z^2 + 1\right) x + 2 yz (yz + 1)^2\right\} t^5
\]
\[
+ \left\{y^2 z^4 \left(z^2 + 2\right) x^2 + yz \left(y^2 z^4 + 6 yz^3 + 2 yz + 4 z^2 + 1\right) x + 2 yz (yz + 1)^2\right\} t^4
\]
\[
+ \left\{x^3 y^2 z^6 + yz^3 \left(yz^3 + 3 yz + 2 z^2 + 2\right) x^2 + yz^2 \left(7 z^2 y + 3 y + 8 z\right) x + 2 yz (yz + 1)^2\right\} t^3
\]
\[
+ \left\{x^3 yz^3 + yz \left(2 yz^3 + 2 yz + 3 z^2 + 1\right) x^2 + yz \left(8 yz + 3 z^2 + 7\right) x + 2 \left(yz + 1\right)^2\right\} t^2
\]
\[
+ \left\{yz \left(2 z^2 + 1\right) x^2 + \left(y^2 z^4 + 4 y^2 z^2 + 2 yz^3 + 6 yz + 1\right) x + 2 \left(yz + 1\right)^2\right\} t
\]
\[
+ \left(z^2 + 1\right) yz x + 2 \left(yz + 1\right)^2.
\]
Relations between the generating functions

Theorem (Kuperberg)

There exists a polynomial $A^{(2)}_{HT}(2n; t, x, y)$ such that the following identities hold:

$$A_{HT}(2n; t, x, y, w) \bigg|_{w=1} = A(n; q, t, x, y) \bigg|_{q=y=1} A^{(2)}_{HT}(2n; t, x, y),$$

$$A_{HT}(2n; t, x, y, w) \bigg|_{w=1} = A(n; q, t, x, y) \bigg|_{q=y=1} A^{(2)}_{HT}(2n; t, x, -y).$$

Example

We have $A^{(2)}_{HT}(2; t, x, y) = yt + 1$ and

$$A^{(2)}_{HT}(4; t, x, y) = y (y + 1) t^2 + txy + y + 1,$$

$$A^{(2)}_{HT}(6; t, x, y) = y \{ xy + (y + 1)^2 \} t^3 + xy \{ xy + 2 (y + 1) \} t^2 + xy \{ x + 2 (y + 1) \} t + xy + (y + 1)^2,$$

$$A^{(2)}_{HT}(8; t, x, y) = y \{ y (y + 1) x^2 + 5 y (y + 1) x + (y + 1)^3 \} t^4$$

$$+ xy \{ y (y + 2) x^2 + 3 y (2 y + 3) x + 3 (y + 1)^2 \} t^3 + xy \{ x^3 y + 4 x^2 y + 3 (y^2 + 3 y + 1) x + 3 (y + 1)^2 \} t^2$$

$$+ xy \{ (2 y + 1) x^2 + 3 (3 y + 2) x + 3 (y + 1)^2 \} t + y (y + 1) x^2 + 5 y (y + 1) x + (y + 1)^3$$
Relations between the generating functions

**Theorem (Kuperberg)**

There exists a polynomial $\widetilde{A}^{(2)}_{UU}(4n; t, x)$ such that the following identities hold:

$$A^{(2)}_{HT}(4n + 2; t, x, y)\bigg|_{y=1} = (t + 1) A^{(2)}_{UU}(4n; x, y, z)\bigg|_{y=z=1} \widetilde{A}^{(2)}_{UU}(4n + 4; t, x)$$

**Example**

We have $\widetilde{A}^{(2)}_{UU}(4; t, x) = 1$ and

$$\widetilde{A}^{(2)}_{UU}(8; t, x) = t^2 + (x - 1) t + 1$$

$$\widetilde{A}^{(2)}_{UU}(12; t, x) = (x + 1) t^4 + (x + 1) (2x - 1) t^3 + (x^3 + x + 1) t^2 + (x + 1) (2x - 1) t + x + 1$$
Definition

We consider SVMs in \((2n + 1) \times n\) rectangle with the following boundary condition.

\[
\begin{array}{cccc}
\nwedge & \circ & \circ & \nwedge \\
\wedge & \circ & \circ & \wedge \\
\wedge & \circ & \circ & \wedge \\
\wedge & \circ & \circ & \wedge \\
\vdots & \vdots & \vdots & \\
\wedge & \circ & \circ & \nwedge \\
\nwedge & \nwedge & \nwedge & \\
\end{array}
\]

But this boundary condition has \(2n + n = 3n\) outgoing arrows and \((2n + 1) + (n + 1) = 3n + 2\) incoming arrows. Hence we have to change the direction of one of the \(n + 1\) incoming arrows on the east wall to the outgoing direction.
**Vertical Symmetric Alternating Sign Matrices with a defect**

**Example** (if \( n = 2 \), then there exists 11 SVMs.)

![Diagram of Vertical Symmetric Alternating Sign Matrices with a defect](image-url)
Example

The corresponding ASMs are

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}.
\]

Definition

Let $\mathcal{A}_{2n+1}^{dVS}$ denote the set of corresponding ASMs.
Generating function for dVSASMs

**Definition**

We set $A_{dV}(2n+1; t, x, z) = \sum_{A \in \mathcal{O}_{dV}^{2n+1}} t^{k-1} x^s z^{w-1}$, where $s$ be the number of $(-1)$'s. Assume that $k$th row of the first column has the unique 1, and Let $w$ denote the parameter such that $2w - 1$th vertex on the east wall has ourward arrow in the corresponding SVM.

**Example**

For example, we have $A_{dV}(3; t, x, z) = zt^2 + 1$ and

\[
A_{dV}(5; t, x, z) = z (z + 1) t^4 + z (z + 1) xt^3 + (z^2 + xz + 1) t^2 + x (z + 1) t + z + 1
\]

\[
A_{dV}(7; t, x, z) = z \left\{ (z^2 + 3 z + 1) x + 2 (z^2 + z + 1) \right\} t^6 + \left\{ 2 z (z^2 + 3 z + 1) x^2 + 4 z x (z^2 + z + 1) \right\} t^5
\]

\[
+ \left\{ z (z^2 + 3 z + 1) x^3 + 2 z (z^2 + 2 z + 2) x^2 + (3 z^3 + 5 z^2 + 4 z + 1) x + 2 z^3 + 2 z^2 + 2 \right\} t^4
\]

\[
+ 2 (z + 1) x \left\{ z x^2 + (z + 1) x + 2 (z^2 + 1) \right\} t^3 + \left\{ (z^2 + 3 z + 1) x^3 + (4 z^2 + 4 z + 2) x^2
\]

\[
+ (z^3 + 4 z^2 + 5 z + 3) x + 2 z^3 + 2 z + 2 \right\} t^2 + \left\{ (2 z^2 + 6 z + 2) x^2 + (4 z^2 + 4 z + 4) x \right\} t
\]

\[
+ (z^2 + 3 z + 1) x + 2 (z^2 + z + 1).
\]
Cardinalities

## Example

We have the following table:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(n;1,1,1,1)$</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>42</td>
<td>429</td>
<td>7436</td>
<td>A005130</td>
</tr>
<tr>
<td>$A_V(2n + 1;1,1)$</td>
<td>1</td>
<td>3</td>
<td>26</td>
<td>646</td>
<td>45885</td>
<td>9304650</td>
<td>A005156</td>
</tr>
<tr>
<td>$A_V(2n;1,1)$</td>
<td>1</td>
<td>7</td>
<td>143</td>
<td>8398</td>
<td>1411510</td>
<td>677688675</td>
<td></td>
</tr>
<tr>
<td>$A_{UU}(4n;1,1,1)$</td>
<td>5</td>
<td>198</td>
<td>63206</td>
<td>163170556</td>
<td>3410501048325</td>
<td>577465332522075000</td>
<td>A107445</td>
</tr>
<tr>
<td>$A_{UU}^{(2)}(4n;1,1,1)$</td>
<td>5</td>
<td>66</td>
<td>2431</td>
<td>252586</td>
<td>74327145</td>
<td>62062015500</td>
<td>A059489</td>
</tr>
<tr>
<td>$A_{HT}(2n;1,1,1)$</td>
<td>2</td>
<td>10</td>
<td>140</td>
<td>5544</td>
<td>622908</td>
<td>198846076</td>
<td>A059475</td>
</tr>
<tr>
<td>$A_{HT}^{(2)}(2n;1,1,1)$</td>
<td>2</td>
<td>5</td>
<td>20</td>
<td>132</td>
<td>1452</td>
<td>26741</td>
<td>A006366</td>
</tr>
<tr>
<td>$A_{UU}^{(2)}(4n;1,1,1)$</td>
<td>2</td>
<td>11</td>
<td>170</td>
<td>7429</td>
<td>920460</td>
<td>323801820</td>
<td>A051255</td>
</tr>
<tr>
<td>$A_{dV}(2n + 1;1,1,1)$</td>
<td>2</td>
<td>11</td>
<td>170</td>
<td>7429</td>
<td>920460</td>
<td>323801820</td>
<td>A051255</td>
</tr>
</tbody>
</table>
Cyclically \((m, n)\)-twisted shifted plane partition

**Definition**

If a \(n\)-staircase shifted plane partition \(\pi = (\pi_{ij})_{1 \leq i \leq j}\) contained in \(\mathcal{B}_{n,2m}\) satisfies

\[
(i, j, k) \in F(\pi) \Leftrightarrow (j, k - 2m, i) \in F(\pi)
\]

whenever \(1 \leq i \leq j \leq k - 2m \leq n\), we call \(\pi\) **cyclically \((m, n)\)-twisted**. (The case when \(m = 1\) is defined by Mills-Robbins-Rumsey.) Let \(\mathcal{C}_{m,n}\) denote the set of cyclically \((m, n)\)-twisted SPPs. If a part \(\pi_{ij}\) satisfies \(i + m \leq \pi_{ij} < j + m\), we call it a **special** part. Let \(s(\pi)\) denote the number of special parts, \(p(\pi)\) the number of the parts equal to \(n + 2m\) and in the first row, \(\text{inv}(\pi)\) the number of the parts such that \(\pi_{ij} \geq i + m\), and \(\text{des}(\pi)\) the number of the parts in the main diagonal such that \(\pi_{ii} \geq i + m\).

**Example**

<table>
<thead>
<tr>
<th>5</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

has 2 special part, 2 maximal parts in the first row, 5 parts \(\geq i + m\), 2 parts \(\geq i + m\) in the main diagonal. \((m = 1\) and \(n = 3\)\).
Cyclically \((m, n)\)-twisted shifted plane partition

### Example

If \(m = 0\) and \(n = 2\), there are 5 cyclically \((0, 2)\)-twisted SPPs:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>\langle 0, 0, 0, 0 \rangle</td>
<td>\langle 0, 0, 1, 1 \rangle</td>
<td>\langle 1, 1, 1, 2 \rangle</td>
<td>\langle 2, 0, 1, 2 \rangle</td>
<td>\langle 2, 0, 2, 3 \rangle</td>
</tr>
</tbody>
</table>

### Example

If \(m = 1\) and \(n = 2\), there are 7 cyclically \((1, 2)\)-twisted SPPs:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>\langle 0, 0, 0, 0 \rangle</td>
<td>\langle 0, 0, 1, 1 \rangle</td>
<td>\langle 1, 0, 1, 1 \rangle</td>
<td>\langle 1, 1, 1, 2 \rangle</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>\langle 1, 0, 1, 2 \rangle</td>
<td>\langle 2, 0, 1, 2 \rangle</td>
<td>\langle 2, 0, 2, 3 \rangle</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


\( (m, n) \)-profile-shape shifted plane partition

**Definition**

If a \( n \)-staircase column-strict shifted plane partition \( \pi = (\pi_{ij})_{1 \leq i \leq j} \) contained in \( \mathcal{B}_{n,n+2m} \) satisfies

\[
\pi_{ii} = \mu_i + 2m \quad \text{where } \mu \text{ is the shape of } \pi
\]

then we call \( \pi \) \( (m, n) \)-profile-shape column-strict shifted plane partition or \( (m, n) \)-profile-shape shifted plane partition in short. (The case when \( m = 1 \) is defined by Mills-Robbins-Rumsey.) Let \( \mathcal{D}_{m,n} \) denote the set of cyclically \( (m, n) \)-twisted SPPs. If a part \( \pi_{ij} \) satisfies \( 1 + m \leq \pi_{ij} \leq j - i + m \), then we call it special, and if a part satisfies \( \pi_{ij} = n + 2m \), we call it maximal. Let \( s(\pi) \) (resp. \( p(\pi) \)) denote the number of special parts (resp. maximal parts), let \( \text{inv}(\pi) \) denote the number of the parts greater than \( m \), and let \( \text{des}(\pi) \) the number of rows of \( \pi \).

**Example**

\[
\begin{array}{ccc}
5 & 5 & 3 \\
4 & 2 & \\
\end{array}
\]

has 2 special part, 2 maximal parts, 5 parts \( \geq 2 \), and 2 rows. (\( m = 1 \) and \( n = 3 \)
### Example

If $m = 0$ and $n = 2$, there are 5 $(0, 2)$-profile-shape SPPs:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 0, 0, 0, 0 \rangle$</td>
<td>$\langle 0, 0, 1, 1 \rangle$</td>
<td>$\langle 1, 1, 1, 2 \rangle$</td>
<td>$\langle 2, 0, 1, 2 \rangle$</td>
<td>$\langle 2, 0, 2, 3 \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Example

If $m = 1$ and $n = 2$, there are 7 $(1, 2)$-profile-shape SPPs:

<table>
<thead>
<tr>
<th>0</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>4</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 0, 0, 0, 0 \rangle$</td>
<td>$\langle 0, 0, 1, 1 \rangle$</td>
<td>$\langle 1, 0, 1, 1 \rangle$</td>
<td>$\langle 1, 1, 1, 2 \rangle$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$\langle 1, 0, 1, 2 \rangle$</td>
<td>$\langle 2, 0, 1, 2 \rangle$</td>
<td>$\langle 2, 0, 2, 3 \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A bijection

Theorem

A map from \( a = (a_{ij}) \in \mathcal{C}_{m,n} \) to \( b = (b_{ij}) = \Phi(a) \in \mathcal{D}_{m,n} \) defined by

\[
    b_{ij} = \begin{cases} 
    a_{ij} - i + 1 & \text{if } a_{ij} \geq i - 1 \\
    0 & \text{otherwise}
    \end{cases}
\]

gives a bijection from \( \mathcal{C}_{m,n} \) onto \( \mathcal{D}_{m,n} \). By this bijection all the statistics \( s(\pi), p(\pi), \text{inv}(\pi), \text{des}(\pi) \) are invariant.

Example

In the case of \( m = 1 \) and \( n = 3 \), this bijection is illustrated by

\[
    a = \begin{pmatrix} 
    5 & 5 & 3 \\
    5 & 3 & \\
    0 & \\
    \end{pmatrix} \quad \mapsto \quad b = \begin{pmatrix} 
    5 & 5 & 3 \\
    4 & 2 & \\
    \end{pmatrix}.
\]
A bijection

**Definition**

Let us define the generating function of $\mathcal{D}_{m,n}$ (or $\mathcal{C}_{m,n}$) as

$$F_{m,n}(q, t, x, y) = \sum_{\pi \in \mathcal{D}_{m,n}} q^{\text{inv}(\pi)} t^{\rho(\pi)} x^{s(\pi)} y^{\text{des}(\pi)}.$$

**Example**

In the case of $m = 0$ we have

$$F_{0,2}(q, t, x, y) = qt y + 1$$

$$F_{0,3}(q, t, x, y) = q^2 y (q y + 1) t^2 + q^2 t x y + q y + 1$$

$$F_{0,4}(q, t, x, y) = q^3 y \left\{ q^2 x y + (q y + 1) \left( q^2 y + 1 \right) \right\} t^3 + q^3 x y \left\{ q^2 x y + 2 (q y + 1) \right\} t^2$$

$$+ q^3 x y \left\{ x + 2 (q y + 1) \right\} t + q^2 x y + (q y + 1) \left( q^2 y + 1 \right)$$
Generating function

Example

In the case of $m = 1$ we have

$$F_{1,2}(q, t, x, y) = qty + 1$$
$$F_{1,3}(q, t, x, y) = q^2 y (qy + 1) t^2 + qy \{qx + (q + 1)\} t + qy + 1$$
$$F_{1,4}(q, t, x, y) = \left\{q^5 xy^2 + q^3 y \left(q^3 y^2 + 2 q^2 y + 2 qy + 1\right)\right\} t^3$$
$$+ q^2 y \left\{q^3 x^2 y + q \left(q^2 y + 4 qy + 2\right) x + (q + 1) \left(q^2 y + qy + 1\right)\right\} t^2$$
$$+ qy \left\{q^2 x^2 + q \left(2 q^2 y + 4 q + 1\right) x + (q + 1) \left(q^2 y + q + 1\right)\right\} t$$
$$+ q^2 xy + q^3 y^2 + 2 q^2 y + 2 qy + 1$$

Remark

When $m = 1$, there is a bijection between $\mathcal{D}_{1,n}$ and the set of descending plane partitions of of order $n + 1$. 
Theorem

(i) When \( m = 0 \), let us define the \( n \) by \( n \) matrix \( A_{0,n} = (a_{i,j}(q, t, x, y))_{1 \leq i, j \leq n} \) by

\[
a_{i,j}(q, t, x, y) = \begin{cases} 
q^i y \sum_{k=1}^{j} \binom{j-1}{k-1} \binom{j-1}{k-1} x^{j-k} & \text{if } i < n, \\
q^i y \sum_{v \leq k \leq j} \binom{j-v-1}{k-v} \binom{j-1}{k-1} t^v x^{j-k} & \text{if } i = n,
\end{cases}
\]

then we have

\[
F_{0,n}(q, t, x, y) = \det(I_n + A_{0,n}).
\]

(ii) When \( m > 0 \), let us define the \( n \) by \( n \) matrix \( A_{m,n} = (a_{i,j}(q, t, x, y))_{1 \leq i, j \leq n} \) by

\[
a_{i,j}(q, t, x, y) = \begin{cases} 
q^i y \sum_{k \leq l \leq j} \binom{i+m-1}{k-1} \binom{l-1}{j-l} \binom{j-l+m-1}{j-l} x^{l-k} & \text{if } i < n, \\
y \sum_{v \leq k \leq l \leq j} \binom{i+m-v-1}{k-v} \binom{l-1}{j-l} \binom{j-l+m-1}{j-l} t^v x^{l-k} & \text{if } i = n,
\end{cases}
\]

then we have

\[
F_{m,n}(q, t, x, y) = \det(I_n + A_{m,n}).
\]
Proof of the theorem

We use the lattice path method. For example, if \( m = 1 \) and \( n = 7 \), the \((1, 7)\)-profile-shape SPP

\[
\begin{array}{ccccccc}
9 & 9 & 9 & 7 & 5 & 4 & 2 \\
8 & 7 & 6 & 4 & 2 & 1 \\
6 & 5 & 3 & 1 \\
4 & 2 \\
\end{array}
\in \mathcal{D}_{1,7}
\]

correspond to the following lattice path
Proof of the theorem

When \( m = 0 \), we take the starting vertices \( \mathbf{u}_i = (1, i) \), and the ending vertices \( \mathbf{v}_j = (1, j) \) \((i, j = 1, \ldots, n)\).

When \( m \geq 1 \), we take the starting vertices \( \mathbf{u}_i = (1, i + 2m) \), and the ending vertices \( \mathbf{v}_j = (1, j) \) \((i, j = 1, \ldots, n)\).

The details are omitted because it is too much technical. \(\Box\)

Conjecture

If we put \( x = 0 \) and \( q = t = 1 \) then we obtain

\[
F_{m,n}(1,1,0,y) = (1 - y)^{n+1} \sum_{j=0}^{\infty} (j + 1)^n y^j
\]

which is the Eulerian polynomial \( \sum_{\sigma \in \Sigma_{n+1}} y^{\text{des}(\sigma)} \).

Proof. The left-hand side equals \( \det \left( \delta_{ij} + y \sum_{k=1}^{i} \binom{i}{k-1} \right) \) for \( 1 \leq i,j \leq n \)
Eulerian polynomial

Remark

The Eulerian polynomial $E_n(y) = \sum_{\sigma \in S_n} y^{\text{des}(\sigma)}$ has the generating function

$$\sum_{n=0}^{\infty} E_n(y) \frac{x^n}{n!} = \frac{(1 - y) e^{x(1-y)}}{1 - ye^{x(1-y)}}.$$ 

Meanwhile, the des and inv has the following simultaneous generating function:

$$\sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} y^{\text{des}(\sigma)} = \frac{(1 - y) \exp_q \{x(1 - y)\}}{1 - y \exp_q \{x(1 - y)\}},$$

where $[n]_q! = \frac{(q;q)_n}{(1-q)_n}$ and $\exp_q \{x\} = \sum_{n \geq 0} \frac{x^n}{[n]_q!}$. J. Striker gave a bijection which maps the descending plane partitions with no special part onto the permutation matrix. By her bijection the number of rows in a DPP does not correspond to the number of descents.
Cardinality

Example

The table of $F_{m,n}(1, 1, 1, 1)$ is as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>20</td>
<td>132</td>
<td>1452</td>
<td>26741</td>
<td>A006366</td>
</tr>
<tr>
<td>$m = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7</td>
<td>42</td>
<td>429</td>
<td>7436</td>
<td>218348</td>
<td>A005130</td>
</tr>
<tr>
<td>$m = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9</td>
<td>72</td>
<td>1040</td>
<td>26000</td>
<td>1130500</td>
<td></td>
</tr>
<tr>
<td>$m = 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11</td>
<td>110</td>
<td>2125</td>
<td>72250</td>
<td>4420255</td>
<td>A051255</td>
</tr>
</tbody>
</table>

Example

When $m = 1$, the special values are as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{1,n}(1, 1, 0, y)$</td>
<td>$y + 1$</td>
<td>$y^2 + 4y + 1$</td>
<td>$y^3 + 11y^2 + 11y + 1$</td>
<td>$y^4 + 26y^3 + 66y^2 + 26y + 1$</td>
</tr>
</tbody>
</table>
Conjecture

The following formulas are conjectured by Hiroyuki Tagawa:

\[ F_{m,2n}(1, 1, 1, 1) = \frac{1}{2^{2n(n+m-1)}} \]
\[ \times \prod_{i=1}^{n} \frac{(6i + 2m - 1)!!(6i + 2m - 7)!!(3i + 2m - 2)!^2}{(4i + 2m - 1)!!(4i + 2m - 3)!^2(4i + 2m - 5)!!(2i - 1)!!(2i - 3)!!(i + m - 1)!^2} \]

\[ F_{m,2n+1}(1, 1, 1, 1) = \frac{1}{2^{2n(n+m)-1}} \]
\[ \times \prod_{i=1}^{n} \frac{(6i + 2m - 1)!^2(3i + 2m + 1)!(3i + 2m - 2)!}{(4i + 2m + 1)!!(4i + 2m - 1)!!^2(4i + 2m - 3)!!(2i - 1)!!^2(i + m)!(i + m - 1)!} \]
Definition

For a cyclically \((m, n)\)-twisted SPP \(\pi = (\pi_{ij})_{1 \leq i \leq j} \in \mathcal{C}_{m,n}\), we define its \((m, n)\)-transpose complement \(\pi' = (\pi'_{ij})_{1 \leq i \leq j} \in \mathcal{C}_{m,n}\) by

\[(i, j, k) \in F(\pi') \iff (n + 1 - j, n + 1 - i, n + 2m + 1 - k) \notin F(\pi),\]

or, equivalently,

\[
\pi'_{ij} + \pi_{n+1-j,n+1-i} = n + 2m \quad \text{for} \quad 1 \leq i \leq j \leq n,
\]

Let \(\varphi_{m,n}\) denote the map \(\pi \mapsto \pi'\). This map is well-defined and clearly an involution.

Example \((m = 1 \text{ and } n = 3)\)

\[
\begin{array}{ccc}
5 & 5 & 3 \\
\pi = & 5 & 3 \\
& 0 & 0 \\
\end{array}
\quad \mapsto \quad
\begin{array}{ccc}
5 & 2 & 2 \\
\pi' = & 0 & 0 \\
& 0 & 0 \\
\end{array}
\]
(m, n)-transpose self-complement

Definition

A cyclically (m, 2n)-twisted SPP $\pi = (\pi_{ij})_{1 \leq i \leq j} \in \mathcal{C}_{m,2n}$ is said to be \textit{(m, 2n)-transpose self-complement} if $\varphi_{m,2n}(\pi) = \pi$ holds. Let $\mathcal{S}_{m,n}$ denote the set of all cyclically-twisted (m, 2n)-transpose self-complement SPPs.

Example

If $m = 1$ and $n = 3$ then we obtain the following 11 PPs:

\[
\begin{array}{ccccccccc}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \langle 0, 3 \rangle & \langle 1, 2 \rangle & \langle 2, 1 \rangle & \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 2, 2 \rangle \\
\langle 0, 3 \rangle & \langle 1, 2 \rangle & \langle 2, 1 \rangle & \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 3, 2 \rangle & \langle 4, 1 \rangle & \langle 4, 1 \rangle & \langle 5, 0 \rangle
\end{array}
\]
(m, n)-restricted plane partition

Definition

A plane partition \( \pi = (\pi_{ij})_{i,j \geq 1} \) is said to be \( m \)-bounded if it satisfies

\[
0 \leq \pi_{ij} \leq m - i + 1.
\]

Let \( \mathcal{T}_{m,n} \) denote the set of all \( (n + m) \)-bounded \( n \)-staircase PPs. We call an element of \( \mathcal{T}_{m,n} \) an \( (m, n) \)-restricted plane partition. A part \( \pi_{ij} \) is said to be special if it satisfies \( \pi_{ij} < j \).

Example

If \( m = 0 \) and \( n = 3 \) then we obtain the following 11 PPs:

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
\langle 0, 3 \rangle & \langle 1, 2 \rangle & \langle 2, 1 \rangle & \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 2, 2 \rangle \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
\langle 3, 1 \rangle & \langle 3, 2 \rangle & \langle 4, 1 \rangle & \langle 4, 1 \rangle & \langle 5, 0 \rangle
\end{array}
\]
Another bijection

**Theorem**

A map $\Psi$ from $a = (a_{ij}) \in \mathcal{I}_{m,n}$ to $b = (b_{ij}) = \Phi(a) \in \mathcal{T}_{m,n}$ defined by

$$b_{ij} = a_{i,j+n} - (n + m) \quad (1 \leq i \leq n - 1, 1 \leq j \leq n - i)$$

gives a bijection from $\mathcal{I}_{m,n}$ onto $\mathcal{T}_{m,n}$. By this bijection the statistics $s(\pi)$ is kept invariant.

**Definition**

Let us define the generating function $G_{m,n}(x)$ of $\mathcal{T}_{m,n}$ by

$$G_{m,n}(x) = \sum_{\pi \in \mathcal{T}_{m,n}} x^{s(\pi)}.$$
Generating Function

Example

In the case of $m = 0$ we have $G_{0,0}(x) = 1$ and

\[
G_{0,1}(x) = x + 1,
\]
\[
G_{0,2}(x) = x^3 + 4 x^2 + 5 x + 1,
\]
\[
G_{0,3}(x) = x^6 + 9 x^5 + 34 x^4 + 62 x^3 + 49 x^2 + 14 x + 1.
\]

Example

In the case of $m = 1$ we have $G_{1,0}(x) = 1$ and

\[
G_{1,1}(x) = x + 2,
\]
\[
G_{1,2}(x) = x^3 + 6 x^2 + 13 x + 6,
\]
\[
G_{1,3}(x) = x^6 + 12 x^5 + 63 x^4 + 176 x^3 + 234 x^2 + 136 x + 24.
\]
Generating Function

Example

In the case of \( m = 2 \) we have \( G_{2,0}(x) = 1 \) and

\[
G_{2,1}(x) = x + 3,
\]

\[
G_{2,2}(x) = x^3 + 8x^2 + 24x + 17,
\]

\[
G_{2,3}(x) = x^6 + 15x^5 + 100x^4 + 366x^3 + 666x^2 + 559x + 155.
\]

Example

The table of \( G_{m,n}(1) \) is as follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 0 )</td>
<td>2</td>
<td>11</td>
<td>170</td>
<td>7429</td>
<td>920460</td>
<td>323801820</td>
<td>A051255</td>
</tr>
<tr>
<td>( m = 1 )</td>
<td>3</td>
<td>26</td>
<td>646</td>
<td>45885</td>
<td>9304650</td>
<td>5382618660</td>
<td>A005156</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>4</td>
<td>50</td>
<td>1862</td>
<td>202860</td>
<td>64080720</td>
<td>1130500</td>
<td></td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>5</td>
<td>85</td>
<td>4508</td>
<td>720360</td>
<td>340695828</td>
<td>471950744980</td>
<td></td>
</tr>
</tbody>
</table>
Generating Function

Theorem
We have

\[ G_{m,n}(x) = \det \left( \sum_{k \geq 0} \left( \begin{array}{c} i + m \\ k - i + 1 \end{array} \right) \left( \begin{array}{c} j \\ k - j + 1 \end{array} \right) x^{2j-k-1} \right) \quad 1 \leq i, j \leq n. \]

Remark
How can we introduce \( t \)?
 Generating Function

**Definition**

If an $n$-staircase column-strict plane partition $\pi = (\pi_{ij})_{1 \leq i \leq n, 1 \leq j \leq n+1-i}$ satisfies

$$\pi_{ij} \leq m + n + 1 - j,$$

then we say $\pi$ is a $(m, n)$-constrained column-strict plane partition. Let $\mathcal{U}_{m,n}$ denote the set of $(m, n)$-constrained column-strict plane partition. For $\pi = (\pi_{ij})_{1 \leq i \leq n, 1 \leq j \leq n+1-i} \in \mathcal{U}_{m,n}$, a part is said to be saturated if $\pi_{ij} = n + m + 1 - j$. Let $U_r(\pi)$ denote the number of parts equal to $r$ plus the number of saturated parts smaller than $r$.

**Example**

If $m = 0$ and $n = 2$ then we have the following 7 PPs.

$$\begin{array}{ccccccccc}
0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\
1 & & & & & & & & & \\
\end{array}$$
## Future work

### Definition

Let $V_{m,n}$ denote the pair $(a, b)$ of plane partitions such that $a \in U_{m,n}$ and $b \in U_{0,n}$ with the same shape, i.e.,

$$V_{m,n} = \{(a, b) \mid a \in U_{m,n}, b \in U_{0,n}, \text{sh}(a) = \text{sh}(b) \}$$

### Example

If $m = 0$ and $n = 2$ then we have the following 11 pairs of PPs.

\[
\begin{bmatrix}
0&0 \\
2&2 \\
2&1&2&1
\end{bmatrix}
\begin{bmatrix}
1&1 \\
1&1&1 \\
2&2
\end{bmatrix}
\begin{bmatrix}
1&2 \\
1&2&1 \\
1&1
\end{bmatrix}
\begin{bmatrix}
2&1
\end{bmatrix}
\begin{bmatrix}
2&1
\end{bmatrix}
\begin{bmatrix}
1&1
\end{bmatrix}
\]
Future work

Conjecture

If we put \( p_r(a, b) = U_r(a) + U_r(b) \), then this statistics is independent of \( r \) and give the statistics corresponding to the position of 1. For example, if \( m = 0, \)

\[
\sum_{(a,b) \in \mathcal{U}_{0,n}} t^{p_r(a,b)} = A_{dV}(2n + 1; t, x, z) \bigg|_{x=z=1}
\]

Remark

There is a bijection between the set of totally symmetric self-complementary plane partitions and \( \mathcal{U}_{0,n} \).

Remark

The pair \( \mathcal{U}_{m,n} \) of plane partition can be restated in the word of domino plane partitions.
Reference


