# Several classes of plane partitions with the same generating function

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# Plane partition

## Definition

A plane partition is a two-dimensional array $(\pi_{i,j})_{i,j\geq 1}$ of nonnegative integers that is nonincreasing both from left to right in each row and top to bottom in each column, i.e.,

$$\pi_{i,j} \geq \pi_{i,j+1} \quad \text{and} \quad \pi_{i,j} \geq \pi_{i+1,j} \quad \text{for all } i \text{ and } j,$$

in which only finitely many of the entries are nonzero. A nonzero entries are called a *part* and the sum $|\pi| = \sum_{i,j\geq 1} \pi_{i,j}$ of parts is called *weight* of the plane partition. The partition $\lambda = (\lambda_1, \lambda_2, \dots)$ defined by $\lambda_i = \sharp\{j \mid \pi_{ij} \neq 0\}$ is called the *shape* of $\pi$ and denoted by $\mathrm{sh}(\pi)$.

## Example

$$\pi = \begin{matrix} 4 & 4 & 3 & 1 \\ 4 & 2 & 1 & \\ 2 & 1 & & \end{matrix}$$

is a plane partion with shape 432 and weight $|\pi| = 22$

# Plane partitions

## Definition

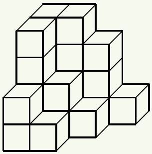
A plane partition $(\pi_{i,j})_{i,j\geq 1}$ is said to be *row-strict* (resp. *column-strict*) if $\pi_{i+1,j} > \pi_{i,j}$ (resp. $\pi_{i,j+1} > \pi_{i,j}$) holds whenever the both sides nonzero. The *Ferrers graph* of $\pi$ is defined to be

$$F(\pi) = \{\, (i,j,k) \mid i,j \geq 1, 1 \leq k \leq \pi_{ij} \,\}$$

which is regarded as a subset of $\mathbb{Z}^3$.

## Example

$$\pi = \begin{array}{cccc} 4 & 4 & 3 & 1 \\ 4 & 2 & 1 & \\ 2 & 1 & & \end{array}$$



# Plane partitions

## Definition

If a plane partition $(\pi_{i,j})_{i,j\geq 1}$ has the shape $(n, n-1, \dots, 1)$, we call it *n-staircase*. We set

$$\mathfrak{B}_{l,m,n} = \{ (i,j,k) \mid 1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq k \leq n \}.$$

and we say $\pi \in \mathfrak{B}_{l,m,n}$ if $\mathrm{sh}(\pi) \subseteq l^m$ and $\pi_{i,j} \leq n$.

## Example

For example,

$$\pi = \begin{matrix} 4 & 3 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix}$$

is 3-staircase column strict plane partition such that $\pi \subseteq \mathfrak{B}_{4,4,4}$

# Shifted plane partitions

## Definition

A shifted plane partition is a two-dimensional array $(\pi_{i,j})_{1\leq i\leq j}$ of nonnegative integers that is nonincreasing both from left to right in each row and top to bottom in each column, and the sum $|\pi| = \sum_{i,j\geq 1} \pi_{i,j}$ of parts is called *weight* of the shifted plane partition. The strict partition $\mu = (\mu_1, \mu_2, \dots)$ defined by $\mu_i = \sharp\{j \mid \pi_{ij} \neq 0\}$ is called the *shape* of $\pi$ and denoted by $\mathrm{ssh}(\pi)$, and the strict partition $(\pi_{1,1}, \pi_{2,2}, \dots)$, denoted by $\mathrm{pr}(\pi)$, is called the *profile* of $\pi$.

## Example

$$\pi = \begin{array}{cccc} 4 & 4 & 3 & 1 \\ & 2 & 2 & \\ & & 1 & \end{array}$$

is a shifted plane partion with shape 421 and weight $|\pi| = 18$

# Shifted plane partition

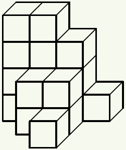
## Definition

The row-strictness (resp. column-strictness) of a shifted plane partition is defined similarly. The *Ferrers graph* of $\pi$ of a shifted plane partition is defined to be

$$F(\pi) = \{ (i, j, k) \mid 1 \le i \le j,\ 1 \le k \le \pi_{ij} \}.$$

## Example

$$\pi = \begin{array}{cccc} 4 & 4 & 3 & 1 \\ & 2 & 2 & \\ & & 1 & \end{array}$$



# Shifted plane partitions

## Definition

If a shifted plane partition $(\pi_{i,j})_{i,j\geq 1}$ has the shape $(n, n-1, \dots, 1)$, we say it is *n-staircase*. We set

$$\mathfrak{S}\mathfrak{B}_{m,n} = \{\,(i,j,k)\,|\,1 \leq i \leq j \leq m,\ 1 \leq k \leq n\,\}.$$

and we say $\pi \in \mathfrak{S}\mathfrak{B}_{m,n}$ if $\mathrm{ssh}(\pi) \subseteq (m, m-1, \dots, 1)$ and $\pi_{i,j} \leq n$.

## Example

For example,

$$\pi = \begin{matrix} 4 & 4 & 3 \\ & 2 & 2 \\ & & 1 \end{matrix}$$

is 3-staircase column-strict shifted plane partition such that $\pi \subseteq \mathfrak{S}\mathfrak{B}_{4,3}$

# Alternating sign matrices

## Definition

An *alternating sign matrix* $A$ of size $n$ is an $n$ by $n$ square matrix of 0s, 1s, and $-1$s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign. Let $\mathscr{A}_n$ denote the set of alternating sign matrices of size $n$.

## Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathscr{A}_5$$

is an alternating sign matrix of size 5.

# The weights for alternating sign matrices

## Definition

For an alternating sign matrix $A = (A_{ij})_{1 \le i,j \le n}$ of size $n$, let $s(A)$ denote the number of $-1$s, set $p(A) = k - 1$ where the 1 in the top row occurs in position $k$ and

$$\operatorname{inv}(A) = \sum_{i<k} \sum_{j>l} A_{ij} A_{kl}.$$

## Example

An alternating sign matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

has $s(A) = 2$, $p(A) = 1$ and $\operatorname{inv}(A) = 5$.

# Generating function for ASMs

## Definition

Let us define the generating function of $\mathscr{A}_n$ as

$$A(n;q,t,x)=\sum_{A\in\mathscr{A}_n} q^{\mathrm{inv}(A)}t^{p(A)}x^{s(A)}.$$

## Example

$$\begin{aligned}
A(2;q,t,x)&=qt+1,\\
A(3;q,t,x)&=q^2(q+1)t^2+\left\{q^2x+q(q+1)\right\}t+q+1,\\
A(4;q,t,x)&=q^3\left\{q^2x+(q+1)\left(q^2+q+1\right)\right\}t^3\\
&\quad+q^2\left\{q^3x^2+q\left(q^2+4q+2\right)x+(q+1)\left(q^2+q+1\right)\right\}t^2\\
&\quad+q\left\{q^2x^2+q\left(2q^2+4q+1\right)x+(q+1)\left(q^2+q+1\right)\right\}t\\
&\quad+q^2x+(q+1)\left(q^2+q+1\right).
\end{aligned}$$

# Vertical-Symmetric alternating sign matrix

## Definition

Let $\mathscr{A}^{V}_{2n+1}$ denote the set of $(2n+1)\times(2n+1)$ vertically symmetric ASMs (VSASMs). For a symmetric ASM, we set $s(A)=m$ if $A$ has $m$ of the orbits of the entries under symmetry excluding any $-1$s that are forced by symmetry. Set $p(A)=k-1$ where the 1 in the leftmost column occurs in position $k$.

## Example

An alternating sign matrix

$$A=\begin{pmatrix} 0 & 0 & 0 & +1 & 0 & 0 & 0\\ 0 & 0 & +1 & -1 & +1 & 0 & 0\\ +1 & 0 & -1 & +1 & -1 & 0 & +1\\ 0 & 0 & +1 & -1 & +1 & 0 & 0\\ 0 & +1 & -1 & +1 & -1 & +1 & 0\\ 0 & 0 & +1 & -1 & +1 & 0 & 0\\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \end{pmatrix}$$

has $s(A)=2$ and $p(A)=2$.

# Generating function for VSASMs

## Definition

Let us define the generating function of $\mathscr{A}^{\mathrm{V}}_{2n+1}$ as

$$A_{\mathrm{V}}(2n+1;t,x)=\sum_{A\in\mathscr{A}^{\mathrm{V}}_{2n+1}} t^{p(A)}x^{s(A)}.$$

## Example

We have $A_{\mathrm{V}}(3;t,x)=1$ and

$$\begin{aligned}
A_{\mathrm{V}}(5;t,x)&=t^2+xt+1,\\
A_{\mathrm{V}}(7;t,x)&=(x+2)\,t^4+2\,x\,(x+2)\,t^3+(x+1)\left(x^2+x+2\right)t^2+2\,x\,(x+2)\,t+x+2,\\
A_{\mathrm{V}}(9;t,x)&=\left(x^3+6\,x^2+13\,x+6\right)t^6+3\,x\left(x^3+6\,x^2+13\,x+6\right)t^5\\
&\quad+\left(3\,x^5+18\,x^4+44\,x^3+42\,x^2+25\,x+6\right)t^4\\
&\quad+x\,(x+2)^2\left(x^3+2\,x^2+9\,x+6\right)t^3+\left(3\,x^5+18\,x^4+44\,x^3+42\,x^2+25\,x+6\right)t^2\\
&\quad+3\,x\left(x^3+6\,x^2+13\,x+6\right)t+x^3+6\,x^2+13\,x+6.
\end{aligned}$$

# Relations between the generating functions

## Theorem (Kuperberg)

There exists a polynomial $\widetilde{A}_{\mathrm{V}}(2n; t, x)$ such that the following identities hold:

$$A(2n; q, t, x, y)\Big|_{q=y=1} = (t+1)A_{\mathrm{V}}(2n+1; t, x)\widetilde{A}_{\mathrm{V}}(2n; 1, x),$$
$$A(2n-1; q, t, x, y)\Big|_{q=y=1} = A_{\mathrm{V}}(2n-1; 1, x)\widetilde{A}_{\mathrm{V}}(2n; t, x).$$

## Example

We have $\widetilde{A}_{\mathrm{V}}(2; t, x) = 1$ and

$$\widetilde{A}_{\mathrm{V}}(4; t, x) = 2\,t^2 + (x+2)t + 2,$$

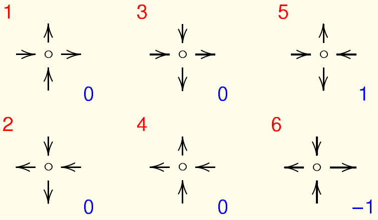
$$\widetilde{A}_{\mathrm{V}}(6; t, x) = 2(x+6)t^4 + (x+6)(3x+2)t^3 + (x^3+6\,x^2+26\,x+12)t^2 + (x+6)(3x+2)t + 2(x+6),$$

$$\widetilde{A}_{\mathrm{V}}(8; t, x) = 2(x^3+12\,x^2+70\,x+60)t^6 + (5\,x+2)\left(x^3+12\,x^2+70\,x+60\right)t^5$$
$$+2\left(2\,x^5+25\,x^4+161\,x^3+352\,x^2+310\,x+60\right)t^4 + \left(x^6+12\,x^5+85\,x^4+452\,x^3+834\,x^2+680\,x+120\right)t$$
$$+2\left(2\,x^5+25\,x^4+161\,x^3+352\,x^2+310\,x+60\right)t^2$$
$$+(5\,x+2)\left(x^3+12\,x^2+70\,x+60\right)t + 2(x^3+12\,x^2+70\,x+60).$$

# Six vertex model

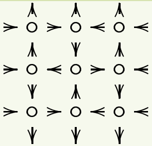
## Definition

A configuration in the six vertex model correspond to an alternating sign matrix:




## Example

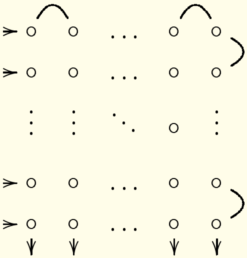
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



# UU-Turn Alternating Sign Matrices (UUASMs)

## Definition

The following figure shows the boundary condition of a UUASM. A UUASM is a $2n \times 2n$ matrix vertically just like an ASM. in which both the columns and the rows of a UUASM are like the rows of a UASM.



Let $\mathscr{A}_{4n}^{\mathrm{UU}}$ denote the set of UUASMs of size $4n$.

# UU-Turn Alternating Sign Matrices (UUASMs)

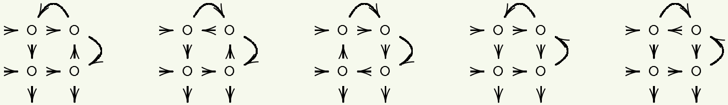
## Definition

We define the $x$-weight $s$ of a UUASM be the number of $-1$s, as before. We define the $y$-weight of a UUASM to be $y^u$ if $u$ of the U-turns are oriented upward in the corresponding square ice state, and define the $z$-weight of a UUASM to be $z^r$ if $r$ of the U-turns on the top are oriented to the right. We set

$$A_{\mathrm{UU}}(4n; x, y, z) = \sum_{A \in \mathscr{A}_{4n}^{\mathrm{UU}}} x^s y^u z^r.$$

## Example

For example, if $n = 1$, then there are 5 UUASMs:



# UU-Turn Alternating Sign Matrices (UUASMs)

## Example

We have

$$A_{\mathrm{UU}}(4;x,y,z) = xz + (z+1)(y+1),$$

$$A_{\mathrm{UU}}(8;x,y,z) = z^2x^4 + z(2yz + y + 6z + 2)x^3$$
$$+ \left(2y^2z^2 + 3y^2z + 11yz^2 + y^2 + 12yz + 13z^2 + 3y + 11z + 2\right)x^2$$
$$+ (z+1)(y+1)(5yz + 3y + 11z + 5)x + 2(z+1)^2(y+1)^2.$$

## Theorem (Kuperberg)

There exists a polynomial $A^{(2)}_{\mathrm{UU}}(4n;x,y,z)$ such that

$$A_{\mathrm{UU}}(4n;x,y,z) = A_{\mathrm{V}}(2n+1;t,x)\Big|_{t=1} A^{(2)}_{\mathrm{UU}}(4n;x,y,z)$$

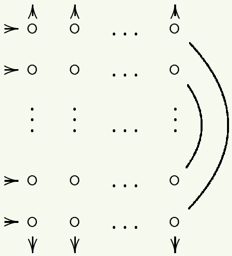
# Half-Turn-Symmetric alternating sign matrix

## Definition

Let $\mathscr{A}_{2n}^{\mathrm{HTS}}$ denote the set of $2n \times 2n$ half-turn-symmetric ASMs (HTSASMs). Let $\mathscr{A}_{2n}^{\mathrm{HTS}}$ denote the set of half-turn symmetric ASMs with size $2n$.

## Example

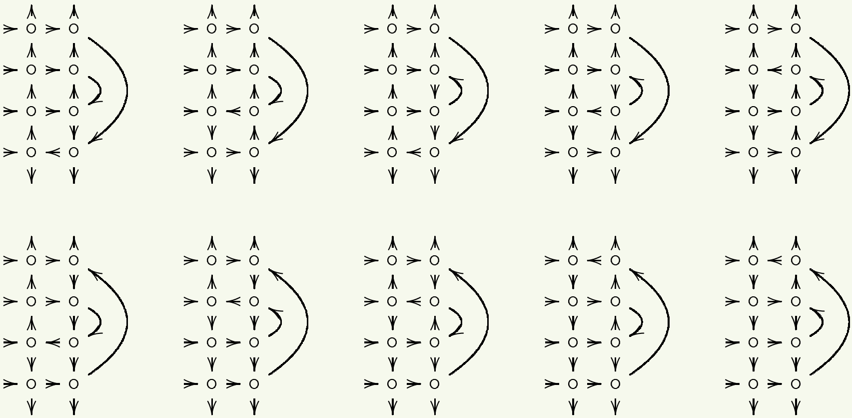
We consider SVMs in $2n \times n$ rectangle with the following boundary condition:



# Half-Turn-Symmetric alternating sign matrix

## Example

For example, if $n = 2$ then the following 10 SVMs are with this boundary condition.



# Half-Turn-Symmetric alternating sign matrix

## Example

These SVM's correspond to the following ASM's.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Generating function for HTSASMs

## Definition

We set $A_{\mathrm{HT}}(2n;t,x,y,w)=\sum_{A\in\mathscr{A}_{2n}^{\mathrm{HTS}}} t^{k-1}x^m y^u w^v$, where $u$ denotes the number of upward arrow on the east wall, and let $v$ denotes the number of nonzero entries in the upper half part, $A$ has $m$ of the orbits of the entries under symmetry, and the 1 in the first column occurs in position $k$.

## Example

For example, we have $A_{\mathrm{HT}}(2;t,x,y,w)=wyt+1$ and

$$A_{\mathrm{HT}}(4;t,x,y,w)=yz\,(yz+1)\,t^3+\left\{yz^3x+yz\,(yz+1)\right\}t^2+(yzx+yz+1)\,t+yz+1,$$

$$\begin{aligned}A_{\mathrm{HT}}(6;t,x,y,w)&=\left\{y^2z^2\left(z^2+1\right)x+2\,yz\,(yz+1)^2\right\}t^5\\&+\left\{y^2z^4\left(z^2+2\right)x^2+yz\left(y^2z^4+6\,yz^3+2\,yz+4\,z^2+1\right)x+2\,yz\,(yz+1)^2\right\}t^4\\&+\left\{x^3y^2z^6+yz^3\left(yz^3+3\,yz+2\,z^2+2\right)x^2+yz^2\left(7\,z^2y+3\,y+8\,z\right)x+2\,yz\,(yz+1)^2\right\}t^3\\&+\left\{x^3yz^3+yz\left(2\,yz^3+2\,yz+3\,z^2+1\right)x^2+yz\left(8\,yz+3\,z^2+7\right)x+2\,(yz+1)^2\right\}t^2\\&+\left\{yz\left(2\,z^2+1\right)x^2+\left(y^2z^4+4\,y^2z^2+2\,yz^3+6\,yz+1\right)x+2\,(yz+1)^2\right\}t\\&+\left(z^2+1\right)yzx+2\,(yz+1)^2.\end{aligned}$$

# Relations between the generating functions

## Theorem (Kuperberg)

There exists a polynomial $A_{\mathrm{HT}}^{(2)}(2n;t,x,y)$ such that the following identities hold:

$$A_{\mathrm{HT}}(2n;t,x,y,w)\Big|_{w=1} = A(n;q,t,x,y)\Big|_{q=y=1} A_{\mathrm{HT}}^{(2)}(2n;t,x,y),$$

$$A_{\mathrm{HT}}(2n;t,x,y,w)\Big|_{w=1} = A(n;q,t,x,y)\Big|_{q=y=1} A_{\mathrm{HT}}^{(2)}(2n;t,x,-y).$$

## Example

We have $A_{\mathrm{HT}}^{(2)}(2;t,x,y) = yt+1$ and

$$A_{\mathrm{HT}}^{(2)}(4;t,x,y) = y\,(y+1)\,t^2 + txy + y + 1,$$

$$A_{\mathrm{HT}}^{(2)}(6;t,x,y) = y\left\{xy + (y+1)^2\right\}t^3 + xy\,\{xy + 2\,(y+1)\}\,t^2 + xy\,\{x + 2\,(y+1)\}\,t + xy + (y+1)^2\,,$$

$$A_{\mathrm{HT}}^{(2)}(8;t,x,y) = y\left\{y\,(y+1)\,x^2 + 5\,y\,(y+1)\,x + (y+1)^3\right\}t^4$$
$$+ xy\left\{y\,(y+2)\,x^2 + 3\,y\,(2\,y+3)\,x + 3\,(y+1)^2\right\}t^3 + xy\left\{x^3y + 4\,x^2y + 3\,\left(y^2 + 3\,y + 1\right)x + 3\,(y+1)^2\right\}t^2$$
$$+ xy\left\{(2\,y+1)\,x^2 + 3\,(3\,y+2)\,x + 3\,(y+1)^2\right\}t + y\,(y+1)\,x^2 + 5\,y\,(y+1)\,x + (y+1)^3$$

# Relations between the generating functions

## Theorem (Kuperberg)

There exists a polynomial $\widetilde{A}^{(2)}_{\mathrm{UU}}(4n; t, x)$ such that the following identities hold:

$$A^{(2)}_{\mathrm{HT}}(4n+2; t, x, y)\Big|_{y=1} = (t+1)\, A^{(2)}_{\mathrm{UU}}(4n; x, y, z)\Big|_{y=z=1} \widetilde{A}^{(2)}_{\mathrm{UU}}(4n+4; t, x)$$

## Example

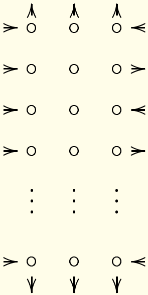
We have $\widetilde{A}^{(2)}_{\mathrm{UU}}(4; t, x) = 1$ and

$$\begin{aligned}\widetilde{A}^{(2)}_{\mathrm{UU}}(8; t, x) &= t^2 + (x-1)\, t + 1 \\ \widetilde{A}^{(2)}_{\mathrm{UU}}(12; t, x) &= (x+1)\, t^4 + (x+1)\,(2\,x-1)\, t^3 + \left(x^3 + x + 1\right) t^2 \\ &\quad + (x+1)\,(2\,x-1)\, t + x + 1\end{aligned}$$

# Vertcal Symmetric Alternating Sign Matrices with a defect

## Definition

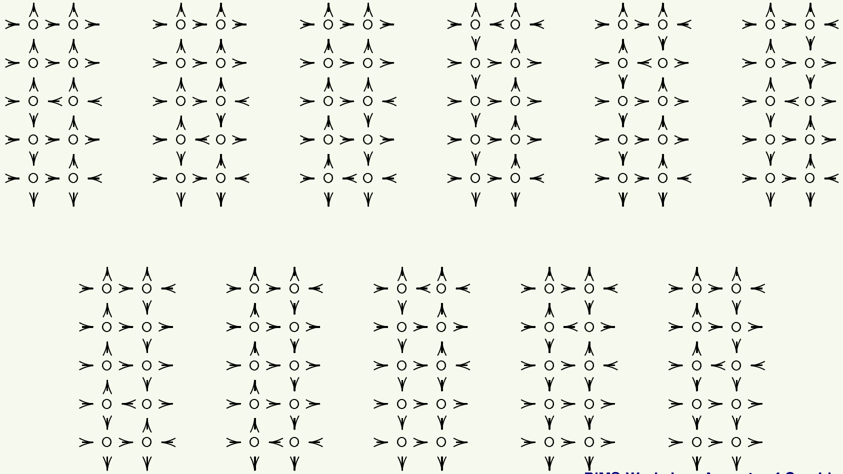
We consider SVMs in $(2n+1) \times n$ rectangle with the following boundary condition.



But this boundary condition has $2n + n = 3n$ outgoing arrows and $(2n+1) + (n+1) = 3n+2$ incoming arrows. Hence we have to change the direction of one of the $n+1$ incoming arrows on the east wall to the outgong direction.

# Vertcal Symmetric Alternating Sign Matrices with a defect

## Example (if $n = 2$, then there exists 11 SVMs.)



# Vertcal Symmetric Alternating Sign Matrices with a defect

## Example

The corresponding ASMs are

$$
\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

## Definition

Let $\mathscr{A}_{2n+1}^{\mathrm{dVS}}$ denote the set of corresponding ASMs.

# Generating function for dVSASMs

## Definition

We set $A_{\mathrm{dV}}(2n+1;t,x,z) = \sum_{A\in\mathscr{A}^{\mathrm{dVS}}_{2n+1}} t^{k-1}x^{s}z^{w-1}$, where Let $s$ be the number of $(-1)$'s. Assume that $k$th row of the first column has the unique 1, and Let $w$ denote the parameter such that $2w-1$th vertex on the east wall has ourward arrow in the corresponding SVM.

## Example

For example, we have $A_{\mathrm{dV}}(3;t,x,z) = zt^2+1$ and

$$
\begin{aligned}
A_{\mathrm{dV}}(5;t,x,z) &= z\,(z+1)\,t^4 + z\,(z+1)\,xt^3 + \left(z^2+xz+1\right)t^2 + x\,(z+1)\,t + z + 1\\
A_{\mathrm{dV}}(7;t,x,z) &= z\left\{\left(z^2+3\,z+1\right)x + 2\left(z^2+z+1\right)\right\}t^6 + \left\{2\,z\left(z^2+3\,z+1\right)x^2 + 4\,zx\left(z^2+z+1\right)\right\}t^5\\
&+\left\{z\left(z^2+3\,z+1\right)x^3 + 2\,z\left(z^2+2\,z+2\right)x^2 + \left(3\,z^3+5\,z^2+4\,z+1\right)x + 2\,z^3+2\,z^2+2\right\}t^4\\
&+2\,(z+1)\,x\left\{zx^2+(z+1)^2\,x+2\,\left(z^2+1\right)\right\}t^3 + \{\left(z^2+3\,z+1\right)x^3 + \left(4\,z^2+4\,z+2\right)x^2\\
&+\left(z^3+4\,z^2+5\,z+3\right)x + 2\,z^3+2\,z+2\}t^2 + \left\{\left(2\,z^2+6\,z+2\right)x^2 + \left(4\,z^2+4\,z+4\right)x\right\}t\\
&+\left(z^2+3\,z+1\right)x + 2(z^2+z+1).
\end{aligned}
$$

# Cardinalities

## Example

We have the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | OEIS |
|---|---|---|---|---|---|---|---|
| $A(n;1,1,1,1)$ | 1 | 2 | 7 | 42 | 429 | 7436 | A005130 |
| $A_{\mathrm{V}}(2n+1;1,1)$ | 1 | 3 | 26 | 646 | 45885 | 9304650 | A005156 |
| $\widetilde{A}_{\mathrm{V}}(2n;1,1)$ | 1 | 7 | 143 | 8398 | 1411510 | 677688675 | |
| $A_{\mathrm{UU}}(4n;1,1,1)$ | 5 | 198 | 63206 | 163170556 | 3410501048325 | 577465332522075000 | A107445 |
| $A^{(2)}_{\mathrm{UU}}(4n;1,1,1)$ | 5 | 66 | 2431 | 252586 | 74327145 | 62062015500 | A059489 |
| $A_{\mathrm{HT}}(2n;1,1,1)$ | 2 | 10 | 140 | 5544 | 622908 | 198846076 | A059475 |
| $A^{(2)}_{\mathrm{HT}}(2n;1,1,1)$ | 2 | 5 | 20 | 132 | 1452 | 26741 | A006366 |
| $\widetilde{A}^{(2)}_{\mathrm{UU}}(4n;1,1,1)$ | 2 | 11 | 170 | 7429 | 920460 | 323801820 | A051255 |
| $A_{\mathrm{dV}}(2n+1;1,1,1)$ | 2 | 11 | 170 | 7429 | 920460 | 323801820 | A051255 |

# Cyclically $(m, n)$-twisted shifted plane partition

## Definition

If a $n$-staircase shifted plane partition $\pi = (\pi_{ij})_{1\leq i\leq j}$ contained in $\mathfrak{S}\mathfrak{B}_{n,2m}$ satisfies

$$(i, j, k) \in F(\pi) \Leftrightarrow (j, k-2m, i) \in F(\pi)$$

whenever $1 \leq i \leq j \leq k-2m \leq n$, we call $\pi$ *cyclically $(m, n)$-twisted*. (The case when $m = 1$ is defined by Mills-Robbins-Rumsey.) Let $\mathscr{C}_{m,n}$ denote the set of cyclically $(m, n)$-twisted SPPs. If a part $\pi_{ij}$ satisfies $i + m \leq \pi_{ij} < j + m$, we call it a *special* part. Let $s(\pi)$ denote the number of special parts, $p(\pi)$ the number of the parts equal to $n + 2m$ and in the first row, $\mathrm{inv}(\pi)$ the number of the parts such that $\pi_{ij} \geq i + m$, and $\mathrm{des}(\pi)$ the number of the parts in the main diagonal such that $\pi_{ii} \geq i + m$.

## Example

$$\begin{array}{ccc} 5 & 5 & 3 \\ & 5 & 3 \\ & & 0 \end{array}$$

has 2 special part, 2 maximal parts in the first row,
5 parts $\geq i + m$, 2 parts $\geq i + m$ in the main diagonal.
($m = 1$ and $n = 3$)

# Cyclically $(m, n)$-twisted shifted plane partition

## Example

If $m = 0$ and $n = 2$, there are 5 cyclically $(0, 2)$-twisted SPPs:

$$\begin{matrix} 0 & 0 \\ & 0 \end{matrix} \quad \begin{matrix} 1 & 0 \\ & 0 \end{matrix} \quad \begin{matrix} 2 & 1 \\ & 0 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 1 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 2 \end{matrix}$$

$$\langle 0, 0, 0, 0 \rangle \quad \langle 0, 0, 1, 1 \rangle \quad \langle 1, 1, 1, 2 \rangle \quad \langle 2, 0, 1, 2 \rangle \quad \langle 2, 0, 2, 3 \rangle$$

## Example

If $m = 1$ and $n = 2$, there are 7 cyclically $(1, 2)$-twisted SPPs:

$$\begin{matrix} 0 & 0 \\ & 0 \end{matrix} \quad \begin{matrix} 3 & 0 \\ & 0 \end{matrix} \quad \begin{matrix} 4 & 1 \\ & 0 \end{matrix} \quad \begin{matrix} 4 & 2 \\ & 0 \end{matrix}$$

$$\langle 0, 0, 0, 0 \rangle \quad \langle 0, 0, 1, 1 \rangle \quad \langle 1, 0, 1, 1 \rangle \quad \langle 1, 1, 1, 2 \rangle$$

$$\begin{matrix} 4 & 3 \\ & 0 \end{matrix} \quad \begin{matrix} 4 & 4 \\ & 1 \end{matrix} \quad \begin{matrix} 4 & 4 \\ & 4 \end{matrix}$$

$$\langle 1, 0, 1, 2 \rangle \quad \langle 2, 0, 1, 2 \rangle \quad \langle 2, 0, 2, 3 \rangle$$

# $(m, n)$-profile-shape shifted plane partition

## Definition

If a $n$-staircase column-strict shifted plane partition $\pi = (\pi_{ij})_{1 \le i \le j}$ contained in $\mathfrak{S}\mathfrak{B}_{n,n+2m}$ satisfies

$$\pi_{ii} = \mu_i + 2m \quad \text{where } \mu \text{ is the shape of } \pi$$

then we call $\pi$ *$(m, n)$-profile-shape column-strict shifted plane partition* or *$(m, n)$-profile-shape shifted plane partition* in short. (The case when $m = 1$ is defined by Mills-Robbins-Rumsey.) Let $\mathscr{D}_{m,n}$ denote the set of cyclically $(m, n)$-twisted SPPs. If a part $\pi_{ij}$ satisfies $1 + m \le \pi_{ij} \le j - i + m$, then we call it *special*, and if a part satisfies $\pi_{ij} = n + 2m$, we call it *maximal*. Let $s(\pi)$ (resp. $p(\pi)$) denote the number of special parts (resp. maximal parts), let $\mathrm{inv}(\pi)$ denote the number of the parts greater than $m$, and let $\mathrm{des}(\pi)$ the number of rows of $\pi$.

## Example

$$\begin{matrix} 5 & 5 & 3 \\ & 4 & 2 \end{matrix}$$

has 2 special part, 2 maximal parts, 5 parts $\ge 2$, and 2 rows. $(m = 1 \text{ and } n = 3)$

# $(m, n)$-profile-shape shifted plane partition

## Example

If $m = 0$ and $n = 2$, there are 5 $(0, 2)$-profile-shape SPPs:

| | | | | |
|---|---|---|---|---|
| $\emptyset$ | $1$ | $\begin{matrix} 2 & 1 \end{matrix}$ | $\begin{matrix} 2 & 2 \end{matrix}$ | $\begin{matrix} 2 & 2 \\ & 1 \end{matrix}$ |
| $\langle 0, 0, 0, 0 \rangle$ | $\langle 0, 0, 1, 1 \rangle$ | $\langle 1, 1, 1, 2 \rangle$ | $\langle 2, 0, 1, 2 \rangle$ | $\langle 2, 0, 2, 3 \rangle$ |

## Example

If $m = 1$ and $n = 2$, there are 7 $(1, 2)$-profile-shape SPPs:

| | | | |
|---|---|---|---|
| $\emptyset$ | $3$ | $\begin{matrix} 4 & 1 \end{matrix}$ | $\begin{matrix} 4 & 2 \end{matrix}$ |
| $\langle 0, 0, 0, 0 \rangle$ | $\langle 0, 0, 1, 1 \rangle$ | $\langle 1, 0, 1, 1 \rangle$ | $\langle 1, 1, 1, 2 \rangle$ |
| $\begin{matrix} 4 & 3 \end{matrix}$ | $\begin{matrix} 4 & 4 \end{matrix}$ | $\begin{matrix} 4 & 4 \\ & 3 \end{matrix}$ | |
| $\langle 1, 0, 1, 2 \rangle$ | $\langle 2, 0, 1, 2 \rangle$ | $\langle 2, 0, 2, 3 \rangle$ | |

# A bijection

## Theorem

A map from $a = (a_{ij}) \in \mathscr{C}_{m,n}$ to $b = (b_{ij}) = \Phi(a) \in \mathscr{D}_{m,n}$ defined by

$$b_{ij} = \begin{cases} a_{ij} - i + 1 & \text{if } a_{ij} \geq i - 1 \\ 0 & \text{otherwise} \end{cases}$$

gives a bijection from $\mathscr{C}_{m,n}$ onto $\mathscr{D}_{m,n}$. By this bijection all the statistics $s(\pi)$, $p(\pi)$, $\operatorname{inv}(\pi)$, $\operatorname{des}(\pi)$ are invariant.

## Example

In the case of $m = 1$ and $n = 3$, this bijection is illustrated by

$$a = \begin{array}{ccc} 5 & 5 & 3 \\ & 5 & 3 \\ & & 0 \end{array} \quad \mapsto \quad b = \begin{array}{ccc} 5 & 5 & 3 \\ & 4 & 2 \end{array}.$$

# A bijection

## Definition

Let us define the generating function of $\mathscr{D}_{m,n}$ (or $\mathscr{C}_{m,n}$) as

$$F_{m,n}(q,t,x,y) = \sum_{\pi \in \mathscr{D}_{m,n}} q^{\mathrm{inv}(\pi)} t^{p(\pi)} x^{s(\pi)} y^{\mathrm{des}(\pi)}.$$

## Example

In the case of $m = 0$ we have

$$\begin{aligned}
F_{0,2}(q,t,x,y) &= qty + 1 \\
F_{0,3}(q,t,x,y) &= q^2 y\,(qy+1)\,t^2 + q^2 txy + qy + 1 \\
F_{0,4}(q,t,x,y) &= q^3 y \left\{q^2 xy + (qy+1)\left(q^2 y + 1\right)\right\} t^3 + q^3 xy \left\{q^2 xy + 2\,(qy+1)\right\} t^2 \\
&\quad + q^3 xy \left\{x + 2\,(qy+1)\right\} t + q^2 xy + (qy+1)\left(q^2 y + 1\right)
\end{aligned}$$

# Generating function

## Example

In the case of $m = 1$ we have

$$
\begin{aligned}
F_{1,2}(q,t,x,y) &= qty + 1 \\
F_{1,3}(q,t,x,y) &= q^2 y\,(qy+1)\,t^2 + qy\,\{qx + (q+1)\}\,t + qy + 1 \\
F_{1,4}(q,t,x,y) &= \left\{q^5 xy^2 + q^3 y\left(q^3 y^2 + 2\,q^2 y + 2\,qy + 1\right)\right\} t^3 \\
&\quad + q^2 y\left\{q^3 x^2 y + q\left(q^2 y + 4\,qy + 2\right)x + (q+1)\left(q^2 y + qy + 1\right)\right\} t^2 \\
&\quad + qy\left\{q^2 x^2 + q\left(2\,q^2 y + 4\,q + 1\right)x + (q+1)\left(q^2 y + q + 1\right)\right\} t \\
&\quad + q^2 xy + q^3 y^2 + 2\,q^2 y + 2\,qy + 1
\end{aligned}
$$

## Remark

When $m = 1$, there is a bijection between $\mathscr{D}_{1,n}$ and the set of descending plane partitions of of order $n+1$.

# Determinantal formula

## Theorem

(i) When $m = 0$, let us define the $n$ by $n$ matrix $A_{0,n} = \left(a_{i,j}(q,t,x,y)\right)_{1\le i,j\le n}$ by

$$a_{i,j}(q,t,x,y) = \begin{cases} q^j y \sum_{k=1}^{j} \binom{i-1}{k-1}\binom{j-1}{k-1} x^{j-k} & \text{if } i < n, \\ q^j y \sum_{1\le\nu\le k\le j} \binom{i-\nu-1}{k-\nu}\binom{j-1}{k-1} t^{\nu} x^{j-k} & \text{if } i = n, \end{cases}$$

then we have

$$F_{0,n}(q,t,x,y) = \det(I_n + A_{0,n}).$$

(ii) When $m > 0$, let us define the $n$ by $n$ matrix $A_{m,n} = (a_{i,j}(q,t,x,y))_{1\le i,j\le n}$ by

$$a_{i,j}(q,t,x,y) = \begin{cases} y \sum_{1\le k\le l\le j} \binom{i+m-1}{k-1}\binom{l-1}{k-1}\binom{j-l+m-1}{j-l} q^l x^{l-k} & \text{if } i < n, \\ y \sum_{1\le\nu\le k\le l\le j} \binom{i+m-\nu-1}{k-\nu}\binom{l-1}{k-1}\binom{j-l+m-1}{j-l} q^l t^{\nu} x^{l-k} & \text{if } i = n, \end{cases}$$

then we have

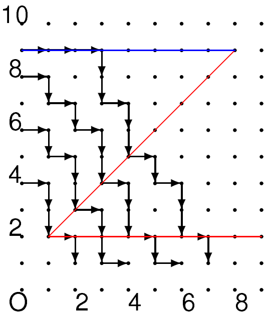
$$F_{m,n}(q,t,x,y) = \det(I_n + A_{m,n}).$$

# Proof of the theorem

We use the lattice path method. For example, if $m = 1$ and $n = 7$, the $(1, 7)$-profile-shape SPP

$$\begin{array}{ccccccc} 9 & 9 & 9 & 7 & 5 & 4 & 2 \\ & 8 & 7 & 6 & 4 & 2 & 1 \\ & & 6 & 5 & 3 & 1 & \\ & & & 4 & 2 & & \end{array} \in \mathscr{D}_{1,7}$$

correspond to the following lattice path



# Proof of the theorem

When $m = 0$, we take the starting vertices $\boldsymbol{u}_i = (1, i)$, and the ending vertices $\boldsymbol{v}_j = (1, j)$ $(i, j = 1, \dots, n)$.

When $m \geq 1$, we take the starting vertices$\boldsymbol{u}_i = (1, i + 2m)$, nd the ending vertices $\boldsymbol{v}_j = (1, j)$ $(i, j = 1, \dots, n)$.

The details are omitted because it is too much technical. $\square$

## Conjecture

If we put $x = 0$ and $q = t = 1$ then we obtain

$$F_{m,n}(1, 1, 0, y) = (1 - y)^{n+1} \sum_{j=0}^{\infty} (j + 1)^n y^j$$

which is the Eulerian polynomial $\sum_{\sigma \in \mathfrak{S}_{n+1}} y^{\mathrm{des}(\sigma)}$.

**Proof.** The left-hand side equals $\det \left( \delta_{ij} + y \sum_{k=1}^{j} \binom{i}{k-1} \right)_{1 \leq i,j \leq n}$.

# Eulerian polynomial

## Remark

The Eulerian polynomial $E_n(y) = \sum_{\sigma \in \mathfrak{S}_n} y^{\operatorname{des}(\sigma)}$ has the generating function

$$\sum_{n=0}^{\infty} E_n(y) \frac{x^n}{n!} = \frac{(1-y)e^{x(1-y)}}{1-ye^{x(1-y)}}.$$

Meanwhile, the des and inv has the following simultaneous generating function:

$$\sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} y^{\operatorname{des}(\sigma)} = \frac{(1-y)\exp_q\{x(1-y)\}}{1-y\exp_q\{x(1-y)\}},$$

where $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$ and $\exp_q\{x\} = \sum_{n \geq 0} \frac{x^n}{[n]_q!}$. J. Striker gave a bijection which maps the descending plane partitions with no special part onto the permutation matrix. By her bijection the number of rows in a DPP does not correspond to the number of descents.

# Cardinality

## Example

The table of $F_{m,n}(1, 1, 1, 1)$ is as follows:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | OEIS |
|---|---|---|---|---|---|---|---|
| $m = 0$ | 2 | 5 | 20 | 132 | 1452 | 26741 | A006366 |
| $m = 1$ | 2 | 7 | 42 | 429 | 7436 | 218348 | A005130 |
| $m = 2$ | 2 | 9 | 72 | 1040 | 26000 | 1130500 | |
| $m = 3$ | 2 | 11 | 110 | 2125 | 72250 | 4420255 | A051255 |

## Example

When $m = 1$, the special values are as follows:

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $F_{1,n}(1, 1, 0, y)$ | $y + 1$ | $y^2 + 4y + 1$ | $y^3 + 11y^2 + 11y + 1$ | $y^4 + 26y^3 + 66y^2 + 26y + 1$ |

# Cardinality

## Conjecture

The following formulas are conjectured by Hiroyuki Tagawa:

$$F_{m,2n}(1,1,1,1) = \frac{1}{2^{2n(n+m-1)}} \times \prod_{i=1}^{n} \frac{(6i+2m-1)!!(6i+2m-7)!!(3i+2m-2)!^2}{(4i+2m-1)!!(4i+2m-3)!!^2(4i+2m-5)!!(2i-1)!!(2i-3)!!(i+m-1)!^2}$$

$$F_{m,2n+1}(1,1,1,1) = \frac{1}{2^{2n(n+m)-1}} \times \prod_{i=1}^{n} \frac{(6i+2m-1)!!^2(3i+2m+1)!(3i+2m-2)!}{(4i+2m+1)!!(4i+2m-1)!!^2(4i+2m-3)!!(2i-1)!!^2(i+m)!(i+m-1)!}$$

# $(m, n)$-transpose complement

## Definition

For a cyclically $(m, n)$-twisted SPP $\pi = (\pi_{ij})_{1 \le i \le j} \in \mathscr{C}_{m,n}$, we define its *$(m, n)$-transpose complement* $\pi' = (\pi'_{ij})_{1 \le i \le j} \in \mathscr{C}_{m,n}$ by

$$(i, j, k) \in F(\pi') \iff (n + 1 - j, n + 1 - i, n + 2m + 1 - k) \notin F(\pi),$$

or, equivalently,

$$\pi'_{ij} + \pi_{n+1-j,n+1-i} = n + 2m \qquad \text{for} \quad 1 \le i \le j \le n,$$

Let $\varphi_{m,n}$ denote the map $\pi \mapsto \pi'$. This map is well-defined and clearly an involution.

## Example ($m = 1$ and $n = 3$)

$$\pi = \begin{matrix} 5 & 5 & 3 \\ & 5 & 3 \\ & & 0 \end{matrix} \quad \mapsto \quad \pi' = \begin{matrix} 5 & 2 & 2 \\ & 0 & 0 \\ & & 0 \end{matrix}$$

# $(m, n)$-transpose self-complement

## Definition

A cyclically $(m, 2n)$-twisted SPP $\pi = (\pi_{ij})_{1 \le i \le j} \in \mathscr{C}_{m,2n}$ is said to be *$(m, 2n)$-transpose self-complement* if $\varphi_{m,2n}(\pi) = \pi$ holds. Let $\mathscr{S}_{m,n}$ denote the set of all cyclically-twisted $(m, 2n)$-transpose self-complement SPPs.

## Example

If $m = 1$ and $n = 3$ then we obtain the following 11 PPs:

$$\begin{array}{cccccc} \begin{matrix} 0 & 0 \\ 0 & \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & \end{matrix} & \begin{matrix} 1 & 0 \\ 1 & \end{matrix} & \begin{matrix} 1 & 1 \\ 0 & \end{matrix} & \begin{matrix} 1 & 1 \\ 1 & \end{matrix} & \begin{matrix} 2 & 0 \\ 0 & \end{matrix} \\ \langle 0, 3 \rangle & \langle 1, 2 \rangle & \langle 2, 1 \rangle & \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 2, 2 \rangle \\ & & & & & \\ \begin{matrix} 2 & 0 \\ 1 & \end{matrix} & \begin{matrix} 2 & 1 \\ 0 & \end{matrix} & \begin{matrix} 2 & 1 \\ 1 & \end{matrix} & \begin{matrix} 2 & 2 \\ 0 & \end{matrix} & \begin{matrix} 2 & 2 \\ 1 & \end{matrix} & \\ \langle 3, 1 \rangle & \langle 3, 2 \rangle & \langle 4, 1 \rangle & \langle 4, 1 \rangle & \langle 5, 0 \rangle & \end{array}$$

# $(m,n)$-restricted plane partition

## Definition

A plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is said to be *$m$-bounded* if it satisfies

$$0 \leq \pi_{ij} \leq m - i + 1.$$

Let $\mathscr{T}_{m,n}$ denote the set of all $(n+m)$-bounded $n$-staircase PPs. We call an element of $\mathscr{T}_{m,n}$ an *$(m,n)$-restricted* plane partition. A part $\pi_{ij}$ is said to be *special* if it satisfies $\pi_{ij} < j$.

## Example

If $m = 0$ and $n = 3$ then we obtain the following 11 PPs:

| | | | | | |
|---|---|---|---|---|---|
| $\begin{matrix} 0 & 0 \\ 0 & \end{matrix}$ | $\begin{matrix} 1 & 0 \\ 0 & \end{matrix}$ | $\begin{matrix} 1 & 0 \\ 1 & \end{matrix}$ | $\begin{matrix} 1 & 1 \\ 0 & \end{matrix}$ | $\begin{matrix} 1 & 1 \\ 1 & \end{matrix}$ | $\begin{matrix} 2 & 0 \\ 0 & \end{matrix}$ |
| $\langle 0, 3\rangle$ | $\langle 1, 2\rangle$ | $\langle 2, 1\rangle$ | $\langle 2, 2\rangle$ | $\langle 3, 1\rangle$ | $\langle 2, 2\rangle$ |
| $\begin{matrix} 2 & 0 \\ 1 & \end{matrix}$ | $\begin{matrix} 2 & 1 \\ 0 & \end{matrix}$ | $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$ | $\begin{matrix} 2 & 2 \\ 0 & \end{matrix}$ | $\begin{matrix} 2 & 2 \\ 1 & \end{matrix}$ | |
| $\langle 3, 1\rangle$ | $\langle 3, 2\rangle$ | $\langle 4, 1\rangle$ | $\langle 4, 1\rangle$ | $\langle 5, 0\rangle$ | |

# Another bijection

## Theorem

A map $\Psi$ from $a = (a_{ij}) \in \mathscr{S}_{m,n}$ to $b = (b_{ij}) = \Phi(a) \in \mathscr{T}_{m,n}$ defined by

$$b_{ij} = a_{i,j+n} - (n+m) \qquad (1 \le i \le n-1, 1 \le j \le n-i)$$

gives a bijection from $\mathscr{S}_{m,n}$ onto $\mathscr{T}_{m,n}$. By this bijection the statistics $s(\pi)$ is kept invariant.

## Definition

Let us define the generating function $G_{m,n}(x)$ of $\mathscr{T}_{m,n}$ by

$$G_{m,n}(x) = \sum_{\pi \in \mathscr{T}_{m,n}} x^{s(\pi)}.$$

# Generating Function

## Example

In the case of $m = 0$ we have $G_{0,0}(x) = 1$ and

$$G_{0,1}(x) = x + 1,$$
$$G_{0,2}(x) = x^3 + 4x^2 + 5x + 1,$$
$$G_{0,3}(x) = x^6 + 9x^5 + 34x^4 + 62x^3 + 49x^2 + 14x + 1.$$

## Example

In the case of $m = 1$ we have $G_{1,0}(x) = 1$ and

$$G_{1,1}(x) = x + 2,$$
$$G_{1,2}(x) = x^3 + 6x^2 + 13x + 6,$$
$$G_{1,3}(x) = x^6 + 12x^5 + 63x^4 + 176x^3 + 234x^2 + 136x + 24$$

# Generating Function

## Example

In the case of $m = 2$ we have $G_{2,0}(x) = 1$ and

$$G_{2,1}(x) = x + 3,$$
$$G_{2,2}(x) = x^3 + 8x^2 + 24x + 17,$$
$$G_{2,3}(x) = x^6 + 15x^5 + 100x^4 + 366x^3 + 666x^2 + 559x + 155.$$

## Example

The table of $G_{m,n}(1)$ is as follows:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | OEIS |
|---|---|---|---|---|---|---|---|
| $m = 0$ | 2 | 11 | 170 | 7429 | 920460 | 323801820 | A051255 |
| $m = 1$ | 3 | 26 | 646 | 45885 | 9304650 | 5382618660 | A005156 |
| $m = 2$ | 4 | 50 | 1862 | 202860 | 64080720 | 1130500 | |
| $m = 3$ | 5 | 85 | 4508 | 720360 | 340695828 | 471950744980 | |

# Generating Function

**Theorem**

We have

$$G_{m,n}(x) = \det\left(\sum_{k\geq 0}\binom{i+m}{k-i+1}\binom{j}{k-j+1}x^{2j-k-1}\right)_{1\leq i,j\leq n}.$$

**Remark**

How can we intriduce $t$?

# Generating Function

## Definition

If an $n$-staircase column-strict plane partition $\pi = (\pi_{ij})_{1 \le i \le n, 1 \le j \le n+1-i}$ satisfies

$$\pi_{ij} \le m + n + 1 - j,$$

then we say $\pi$ is a *$(m, n)$-constrained* column-strict plane partition. Let $\mathscr{U}_{m,n}$ denote the set of $(m, n)$-constrained column-strict plane partition. For $\pi = (\pi_{ij})_{1 \le i \le n, 1 \le j \le n+1-i} \in \mathscr{U}_{m,n}$, a part is siad to be *saturated* if $\pi_{ij} = n + m + 1 - j$. Let $U_r(\pi)$ denote the number of parts equal to $r$ plus the number of saturated parts smaller than $r$.

## Example

If $m = 0$ and $n = 2$ then we have the following 7 PPs.

$$\emptyset \qquad 1 \qquad \begin{matrix} 1 & ① \end{matrix} \qquad ② \qquad \begin{matrix} ② & ① \end{matrix} \qquad \begin{matrix} ② \\ 1 \end{matrix} \qquad \begin{matrix} ② & ① \\ 1 & \end{matrix}$$

# Future work

## Definition

Let $\mathscr{V}_{m,n}$ denote the pair $(a,b)$ of palne partitions such that $a \in \mathscr{U}_{m,n}$ and $b \in \mathscr{U}_{0,n}$ with the same shape,i.e.,

$$\mathscr{V}_{m,n} = \{(a,b) \mid a \in \mathscr{U}_{m,n}, b \in \mathscr{U}_{0,n}, \mathrm{sh}(a) = \mathrm{sh}(b)\}$$

## Example

If $m = 0$ and $n = 2$ then we have the following 11 pairs of PPs.

$$\left(\emptyset\ ,\ \emptyset\right) \quad \left(1\ ,\ 1\right) \quad \left(1\ ,\ 2\right) \quad \left(2\ ,\ 1\right)$$

$$\left(2\ ,\ 2\right) \quad \left(1\ 1\ ,\ 1\ 1\right) \quad \left(1\ 1\ ,\ 2\ 1\right) \quad \left(2\ 1\ ,\ 1\ 1\right)$$

$$\left(2\ 1\ ,\ 2\ 1\right) \quad \left(\begin{matrix}2\\1\end{matrix}\ ,\ \begin{matrix}2\\1\end{matrix}\right) \quad \left(\begin{matrix}2 & 1\\1 & \end{matrix}\ ,\ \begin{matrix}2 & 1\\1 & \end{matrix}\right)$$

# Future work

## Conjecture

If we put $p_r(a,b) = U_r(a) + U_r(b)$, then this statistics is independent of $r$ and give the statistics corresponding to the position of 1. For example, if $m = 0$,

$$\sum_{(a,b)\in\mathscr{V}_{0,n}} t^{p_r(a,b)} = A_{\mathrm{dV}}(2n+1;t,x,z)\Big|_{x=z=1}$$

## Remark

There is a bijection between the set of totally symmetric self-complementary plane partitions and $\mathscr{U}_{0,n}$.

## Remark

The pair $\mathscr{U}_{m,n}$ of plane partition can be restated in the word of domino plane partitions.

# Reference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