Let $C_n = \binom{2n}{n}$ denote the Catalan number, $M_n = \sum_{i=0}^{n} \binom{n}{i} C_i$ the Motzkin number, $D_n = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i} C_i$ the Delannoy number, $S_n = \sum_{k=0}^{n} \binom{n}{k} C_k$ the Schröder number. Finally, the number $N(a, b, \gamma) = \binom{2n}{n} C_n$ is known as a Narayana number, and

$$N_n(a) = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} a^{k-1}$$

is known as the nth Narayana polynomial.

The Selberg integral is introduced and proven by Atle Selberg [Selberg(1944)]:

$$S_n(a, b, \gamma) = \int_{\mathbb{R}} \prod_{k=1}^{n-1} (1 + x_k) \prod_{k=1}^{n} |t_k - x_k| dt = \prod_{k=1}^{n} b^{1/2} (a + x_k - \gamma) + \gamma (1 + x_k)$$

The following theorem was conjectured in [Ishikawa et al.(2013)] and first proved in [Ishikawa and Koutschan(2012)].

Let $n \geq 1$ be an integer. Then the following identities hold:

$$\text{Pr}((j-i)M_{j-i}) = \prod_{k=1}^{j}(4k+1),$$

$$\text{Pr}((j-i)D_{j-i}) = \prod_{k=1}^{j-i} (2^{i-1}(2n-1 - \frac{i-1}{2}) (4k-1),$$

$$\text{Pr}((j-i)S_{j-i}) = \prod_{k=1}^{j-i} (2^{n-i}(4k+1),$$

$$\text{Pr}((j-i)N_{j-i}(-a)) = a^{n-j} \prod_{k=1}^{j-i} (4k+1).$$

Hence, we obtain the following identities:

$$\text{Pr}((j-i)M_{j-i}) = \frac{1}{(2n)!} \int_{\mathbb{R}} \prod_{k=1}^{n} \prod_{k=1}^{j} (x_k - x_j)^{j} \sqrt{x_k + \frac{1}{j} (x_k + 1) (x_j - x_k)} dx,$$

$$\text{Pr}((j-i)D_{j-i}) = \frac{1}{(2n)!} \int_{\mathbb{R}} \prod_{k=1}^{n} \prod_{k=1}^{j-i} (x_k - x_j)^{j-i} \sqrt{a - x_j} dx,$$

$$\text{Pr}((j-i)S_{j-i}) = \frac{1}{(2n)!} \int_{\mathbb{R}} \prod_{k=1}^{n} \prod_{k=1}^{j-i} (x_k - x_j)^{j-i} \sqrt{x_k - x_j} dx.$$

From Selberg’s formula we derive the following special cases:

$$\int_{\mathbb{R}} \prod_{k=1}^{n} \prod_{k=1}^{j} (x_k - x_j)^{j} \prod_{k=1}^{n} \sqrt{x_k + \frac{1}{j} (x_k + 1) (x_j - x_k)} dx = (b - a)^{n-1} S_{n}(\frac{3}{2}, 2),$$

$$\int_{\mathbb{R}} \prod_{k=1}^{n} \prod_{k=1}^{j-i} (x_k - x_j)^{j-i} \sqrt{x_k - x_j} dx = (b - a)^{n-1-j} S_{n}(\frac{3}{2}, 2).$$

In this section we use the standard q-exponential functions:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad E_q(x) = 1 + \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}$$

Al-Salam and Carlitz [Al-Salam and Carlitz(1965)], Chihara[1978] defined the sequences $U_{i}^{(m)}(q; y)$ and $V_{i}^{(m)}(x; q)$ of orthogonal polynomials by

$$\rho_i(x) e_q(x) = \sum_{k=0}^{\infty} U_{i}^{(m)}(q; y) e_q(x), \quad \rho_i(x) E_q(x) = \sum_{k=0}^{\infty} V_{i}^{(m)}(x; q) E_q(x),$$

where

$$\rho_i(x) = (1 - ax)^{n-i} (1 - ax)^{i},$$

These sequences $U_{i}^{(m)}(q; y)$ and $V_{i}^{(m)}(x; q)$ are the Al-Salam and Carlitz I polynomials and the Al-Salam and Carlitz II polynomials, respectively. The orthogonality relations of these polynomials are given by

$$\int_{\mathbb{R}} U_{i}^{(m)}(q; y) U_{j}^{(m)}(q; y) w_{m}(q; y) dx = (1 - q)^i e_{n-i}(q; y) \delta_{i,j},$$

$$\int_{\mathbb{R}} V_{i}^{(m)}(x; q) V_{j}^{(m)}(x; q) w_{m}(x; q) dx = (1 - q)^{-i} e_{-i}(q; y) \delta_{i,j},$$

where the weight functions $w_{m}(q; y)$ and $w_{m}(x; q)$ are defined by

$$w_{m}(q; y) = \frac{(a; q)_{n-i}(a; q)^{i}}{(1 - q)^i (1 - q)^{n-i}(a; q)_{n-i} (a; q)^{i}},$$

$$w_{m}(x; q) = \frac{(a; q)_{n-i}(a; q)^{i}}{(1 - q)^i (1 - q)^{n-i}(a; q)_{n-i} (a; q)^{i}}.$$

Theorem

Let $F_{i,j}^{(m)}(x; q)$ and $G_{i,j}^{(m)}(x; q)$ be as above. Then we have

$$\text{Pr}((q^{j-i} F_{i,j}^{(m)}(x; q))) = a^{n-i} q^{i-i} (1-a)^{n-i} \prod_{k=1}^{n} (q; q)_{a-1},$$

$$\text{Pr}((q^{j-i} G_{i,j}^{(m)}(x; q))) = a^{n-i} q^{i-i} (1-a)^{n-i} \prod_{k=1}^{n} (q; q)_{a-1}.$$