Generalized carry process and riffle shuffle

Fumihiko NAKANO¹  Taizo SADAHIRO²

¹Gakushuin University
²Tsuda College

2018 年 2 月
Carries in addition

Adding 2 numbers with randomly chosen digits,
Carries in addition

Adding 2 numbers with randomly chosen digits,

<table>
<thead>
<tr>
<th></th>
<th>01111</th>
<th>00001</th>
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<th>01101</th>
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and 1 seem to appear at equal rate.
Carries in addition

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0 and 1 seem to appear at equal rate.

Adding 3 numbers,
Carries in addition

Adding 2 numbers with randomly chosen digits,

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0 and 1 seem to appear at equal rate.

Adding 3 numbers,

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<td>11102</td>
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<td>01011</td>
<td>11210</td>
<td>2112</td>
<td></td>
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<td>38426</td>
<td></td>
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<tr>
<td>00171</td>
<td>55077</td>
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<td>95932</td>
<td>91116</td>
<td>17255</td>
<td>19649</td>
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<td>70267</td>
<td>68885</td>
<td>98147</td>
<td>70311</td>
<td>43856</td>
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<td>92954</td>
<td>32426</td>
<td>84728</td>
<td>09380</td>
<td>26070</td>
<td>34608</td>
<td>95451</td>
<td></td>
</tr>
</tbody>
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then 1 seems to appear frequently. \( \frac{\#0}{\#1} : \frac{\#2}{\#1} = 7 : 20 : 7 \).
Transition Probability 1

\[ P_{ij} := P(C_{k+1} = j \mid C_k = i), \quad i, j \in \{0, 1, \cdots, n - 1\} \]
Transition Probability 1

\[ P_{ij} := P(C_{k+1} = j \mid C_k = i), \quad i, j \in \{0, 1, \cdots, n - 1\} \]

Example 1 \((b = 2, n = 2)\)

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[ \implies (P_{0,0}, P_{0,1}) = \frac{1}{2^2} (3, 1) \]
Transition Probability 1

\[ P_{ij} := \mathbf{P} \left( C_{k+1} = j \mid C_k = i \right), \quad i, j \in \{0, 1, \ldots, n - 1\} \]

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\[ \implies (P_{0,0}, P_{0,1}) = \frac{1}{2^2} (3, 1) \]

For \(b = 2, n = 2\)

\[
P = \frac{1}{2^2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \implies \text{Stationary dist. } \pi = \left( \frac{1}{2}, \frac{1}{2} \right)\]
Transition Probability 2

Example 2 \((b = 2, n = 3)\)
Transition Probability 2

Example 2 \((b = 2, n = 3)\)

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\end{array}
\]

\[\Rightarrow \quad (P_{0,0}, P_{0,1}, P_{0,2}) = \frac{1}{2^3} \cdot (4, 4, 0)\]
Transition Probability 2

Example 2 \((b = 2, n = 3)\)

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{array}
\]

\[
(P_{0,0}, P_{0,1}, P_{0,2}) = \frac{1}{2^3} \cdot (4, 4, 0)
\]

For \(b = 2, n = 3\)

\[
P = \frac{1}{2^3} \begin{pmatrix}
4 & 4 & 0 \\
1 & 6 & 1 \\
0 & 4 & 4 \\
\end{pmatrix}
\implies \pi = \frac{1}{3!} \cdot (1, 4, 1)
\]
Carries Process

Add $n$ base-$b$ numbers $(b, n \in \mathbb{N}, b, n \geq 2)$
**Carries Process**

Add $n$ base-$b$ numbers ($b, n \in \mathbb{N}, b, n \geq 2$)

<table>
<thead>
<tr>
<th>Carry</th>
<th>$C_{k+1}$</th>
<th>$C_k$</th>
<th>$C_1$</th>
<th>$C_0 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addends</td>
<td>$X_{1,k}$</td>
<td>$X_{1,1}$</td>
<td>$X_{1,0}$</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$X_{n,k}$</td>
<td>$X_{n,1}$</td>
<td>$X_{n,0}$</td>
<td></td>
</tr>
<tr>
<td>Sum</td>
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Choose $X_{j,k}$ uniformly at random from $D_b := \{0, 1, \cdots, b-1\}$. Given $C_k$, $C_{k+1}$ is determined by
Carries Process

Add $n$ base-$b$ numbers ($b, n \in \mathbb{N}, b, n \geq 2$)

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$$C_k + X_{1,k} + \cdots + X_{n,k} = C_{k+1}b + S_k, \quad S_k \in D_b$$
Carries Process

Add \( n \) base- \( b \) numbers \((b, n \in \mathbb{N}, b, n \geq 2)\)

\[
\begin{array}{cccccc}
\text{Carry} & C_{k+1} & C_k & \cdots & C_1 & C_0 = 0 \\
\text{Addends} & X_{1,k} & \cdots & X_{1,1} & X_{1,0} \\
 & \vdots & & \vdots & & \vdots \\
 & X_{n,k} & \cdots & X_{n,1} & X_{n,0} \\
\text{Sum} & S_k & \cdots & S_1 & S_0 \\
\end{array}
\]

Choose \( X_{j,k} \) uniformly at random from \( D_b := \{0, 1, \ldots, b-1\} \).
Given \( C_k \), \( C_{k+1} \) is determined by

\[
C_k + X_{1,k} + \cdots + X_{n,k} = C_{k+1}b + S_k, \quad S_k \in D_b
\]

\( \{C_k\}_{k=0}^{\infty} \) \((C_k \in \{0, \cdots, n-1\})\) is called the carries process.
Amazing Matrix : Holte (1997)

\[ P_{ij} := P(C_{k+1} = j \mid C_k = i), \quad i, j = 0, 1, \ldots, n - 1 \]
Generalized carry process and riffle shuffle

Fumihiko NAKANO, Taizo SADAHIRO

Introduction

Amazing Matrix

(b, n, p)-process

Riffle Shuffle

(−b, n, p) - process

Miscellaneous

Application

Summary


\[ P_{ij} := P \left( C_{k+1} = j \mid C_k = i \right), \quad i, j = 0, 1, \ldots, n - 1 \]

E-values and left E-vectors of Amazing Matrix

E-values/ E-vectors depends only on \( b / n \).

\[ P = L^{-1} DL, \quad D = \text{diag} \left( 1, \frac{1}{b}, \frac{1}{b^2}, \ldots, \frac{1}{b^{n-1}} \right) \]

\[ L_{ij} = v_{ij}(n) = [x^j](A_m(x)) \cdot \]

\[ A_m(x) := (1 - x)^{n+1} \sum_{j \geq 0} (j + 1)^m x^j : \text{Eulerian polynomial.} \]
Amazing Matrix : Holte(1997)

\[ P_{ij} := \mathbf{P} \left( C_{k+1} = j \mid C_k = i \right), \quad i, j = 0, 1, \ldots , n - 1 \]

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\[ P = L^{-1} DL, \quad D = \text{diag} \left( 1, \frac{1}{b}, \frac{1}{b^2}, \ldots , \frac{1}{b^{n-1}} \right) \]

\[ L_{ij} = v_{ij}(n) = [x^j] \left( A_m(x) \right). \]

\[ A_m(x) := (1 - x)^{n+1} \sum_{j \geq 0} (j + 1)^{m} x^j : \text{Eulerian polynomial.} \]

Remark. \( P(b_1)P(b_2) = P(b_1 \cdot b_2). \)
Property of Left Eigenvectors

\[ L = \begin{pmatrix}
\text{(n-th Eulerian num.)} \\
\vdots \\
(-1)^j((n - 1)\text{-th Pascal num.})
\end{pmatrix} \]

[1]
Property of Left Eigenvectors

\[ L = \begin{pmatrix}
\begin{array}{cccc}
\text{(n-th Eulerian num.)} \\
\vdots \\
(-1)^j((n-1)\text{-th Pascal num.)}
\end{array}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & -2 & 1 \\
1 & 11 & 11 & 1 \\
1 & 3 & -3 & -1 \\
1 & -1 & -1 & 1 \\
1 & -3 & 3 & -1
\end{pmatrix}
\]
Property of Left Eigenvectors

\[ L = \begin{pmatrix}
(n\text{-th Eulerian num.}) \\
\vdots \\
(-1)^j((n-1)\text{-th Pascal num.})
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 4 & 1 \\
1 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 11 & 11 & 1 \\
1 & 3 & -3 & -1 \\
1 & -1 & -1 & 1 \\
1 & -3 & 3 & -1
\end{pmatrix}
\]

\[ E(n, k) := \#\{ \sigma \in S_n \text{ with } k\text{-descents} \} : n\text{-th Eulerian num.} \]
Property of Left Eigenvectors

\[ L = \begin{pmatrix} (n\text{-th Eulerian num.}) \\ \vdots \\ (-1)^j((n-1)\text{-th Pascal num.}) \end{pmatrix} \]

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\[
E(3, 0) = \#\{(123)\} = 1, \\
E(3, 1) = \#\{(132), (312), (231), (213)\} = 4, \\
E(3, 2) = \#\{(321)\} = 1.
\]
Property of Left Eigenvectors

\[ L = \begin{pmatrix} \begin{pmatrix} (n\text{-th Eulerian num.}) \\ \vdots \\ (-1)^j((n-1)\text{-th Pascal num.}) \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 11 & 11 & 1 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \end{pmatrix} \]

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\[ E(3, 0) = \#\{(123)\} = 1, \]
\[ E(3, 1) = \#\{(132), (312), (231), (213)\} = 4, \]
\[ E(3, 2) = \#\{(321)\} = 1. \]

[2] \( L \) is equal to the Foulkes character table of \( S_n \) (Diaconis-Fulman, 2012).
Foulkes character

Example

\[ \#\{ \sigma \in S_4 \mid \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) \} \]
\[ = \{(1324), (1423), (2314), (2413), (3412)\} = 5 \]
Foulkes character

Example

\[ \#\{ \sigma \in S_4 \mid \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) \} \]
\[ = \{(1324), (1423), (2314), (2413), (3412)\} = 5 \]

\[ \begin{array}{ccc}
+ & - & + \\
\Rightarrow & + & \times \\
\Rightarrow & + & -
\end{array} \]

\[ \begin{array}{ccc}
+ & + & \times \\
| & + & -
\end{array} \]
Foulkes character

Example

\[\#\{\sigma \in S_4 | \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4)\} = \{(1324), (1423), (2314), (2413), (3412)\} = 5\]

\[
\begin{array}{c}
+ \quad - \\
+ \times \\
+ \quad -
\end{array}
\]

\[
\begin{array}{c}
+ \quad + \\
\times \quad -
\end{array}
\]

\[
LR \simeq \begin{array}{c}
\text{dim} = 3 \\
\text{dim} = 2
\end{array}
\]
Property of Right Eigenvectors

Right Eigenvector of \( P \)

\[
P = RDR^{-1}
\]

\[
R_{ij} = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \binom{r}{n-j} (n - 1 - i)^{r-(n-j)}
\]
Property of Right Eigenvectors

Right Eigenvector of $P$

$$P = RDR^{-1}$$

$$R_{ij} = \sum_{r=n-j}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} \binom{r}{n-j} (n - 1 - i)^{r-(n-j)}$$

$$(1 \ 1) \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 6 & 11 & 6 \\ 1 & 2 & -1 & -2 \\ 1 & -2 & -1 & 2 \\ 1 & -6 & 11 & -6 \end{pmatrix}$$
Property of Right Eigenvectors

Right Eigenvector of $P$

$$P = RDR^{-1}$$

$$R_{ij} = \sum_{r=n-j}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} \binom{r}{n-j} (n - 1 - i)^{r-(n-j)}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 6 & 11 & 6 \\ 1 & 2 & -1 & -2 \\ 1 & -2 & -1 & 2 \\ 1 & -6 & 11 & -6 \end{pmatrix}$$

$$R(0, j) = S(n, n - j)$$

$$S(n, j) := \#\{\sigma \in S_n \text{ with } j\text{-cycles} \} \text{ Stirling num. of 1st kind}$$
Let $\{\sigma_1, \sigma_2, \cdots\}$ ($\sigma_0 = id$), be the Markov chain on $S_n$ induced by the repeated $b$-riffle shuffles on $n$-cards.
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**Relation to Riffle Shuffles (Diaconis-Fulman, 2009)**

\[
\{C_k\}_{k=1}^{\infty} \cong \{d(\sigma_k)\}_{k=1}^{\infty}, \quad d(\sigma): \text{the descent of } \sigma \in S_n.
\]
Riffle Shuffle

Let \( \{\sigma_1, \sigma_2, \cdots \} \) \((\sigma_0 = id)\), be the Markov chain on \( S_n \) induced by the repeated \( b \)-riffle shuffles on \( n \)-cards.

**Relation to Riffle Shuffles (Diaconis-Fulman, 2009)**

\[
\{C_k\}_{k=1}^{\infty} \overset{d}{=} \{d(\sigma_k)\}_{k=1}^{\infty}, \quad d(\sigma) : \text{the descent of } \sigma \in S_n.
\]

Since the stationary dist. of \( \{\sigma_k\} \) is uniform on \( S_n \),

\[
L_{0j} = \lim_{k \to \infty} P(C_k = j) = \lim_{k \to \infty} P(d(\sigma_k) = j) = P_{unif}(d(\sigma) = j) = E(n, j)/n!
\]

explaining why Eulerian num. appears.
Summary on Known Results

Amazing Matrix (the transition probability matrix $P$ of the carries process) has the following properties.

1. E-values depend only on $b$.
2. Eulerian number appears in the stationary distribution.
3. Left eigenvector matrix $L$ equals to the Foulkes character table of $S_n$.
4. Stirling number of the first kind appears in the right eigenvector matrix $R$.
5. Carries process has the same distribution to the descent process of the riffle shuffle.
Summary on Known Results

Amazing Matrix (the transition probability matrix $P$ of the carries process) has the following properties.

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(1) Eulerian num. appears in the stationary distribution.

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(3) Stirling num. of 1st kind appears in the right eigenvector matrix $R$. 
Summary on Known Results

Amazing Matrix (the transition probability matrix $P$ of the carries process) has the following properties.

(0) E-values depend only on $b$, and E-vectors depend only on $n$

(1) Eulerian num. appears in the stationary distribution.

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(3) Stirling num. of 1st kind appears in the right eigenvector matrix $R$.

(4) carries process has the same distribution to the descent process of the riffle shuffle.
(b, n, p)-Carries Process
(\(b, n, p\))-Carries Process

Add \(n\) base- \(b\) numbers. Let \(\frac{b-1}{p} \in \mathbb{N}, \frac{1}{p} + \frac{1}{p^*} = 1\).
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Generalized carry process and riffle shuffle

Fumihiko NAKANO, Taizo SADAHIRO

Introduction

Amazing Matrix \((b; n; p)\)-process

Riffle Shuffle

\((-b; n; p)\)-process

Miscellaneous

Application

Summary

\((b, n, p)\)-Carries Process

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($p = 1$ : usual carries process)
Remarks

(1) If we generalize the usual carries process by changing the digit set such as

\[ D_b = \{0, 1, \cdots, b - 1\} \quad \Rightarrow \quad D_b = \{d, d + 1, \cdots, d + b - 1\}, \]

we get \((b, n, p)\)-carries process, after some change of variables.
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we get \((b, n, p)\)-carries process, after some change of variables.

(2) \(C_k \in C_p(n)\) where \(C_p(n)\) is the carries set given by

\[ C_p(n) := \begin{cases} 
\{0, 1, \ldots, n - 1\} & (p = 1) \\
\{0, 1, \ldots, n\} & (p \neq 1)
\end{cases} \]
Left Eigenvectors

\[ \tilde{P} = \{ \tilde{P}_{ij} \} : \text{Transition probability of } (b, n, p)-\text{process} : \]

\[ \tilde{P}_{ij} = P \left( C_{k+1} = j \mid C_k = i \right). \]
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**Theorem 1**

E-values/ E-vectors depend only on \( b / n \).

\[ \tilde{P} = L_p^{-1} D L_p, \quad D = \text{diag} \left( 1, \frac{1}{b}, \cdots, \frac{1}{b^\#C_p(n)-1} \right) \]
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\[ A_{m,p}(x) := (1 - x)^{n+1} \sum_{j \geq 0} (pj + 1)^m x^j \]
Combinatorial meaning of $L$

[1] The stationary distribution $L_{0j}^{(p)}(n)$ gives

(1) $p = 1$: Eulerian number
(descent statistics of the permutation group)
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$M(2, 0) = \#\{(1−, 2−)\} = 1,$
$M(2, 1) = \#\{(1+, 2+), (1+, 2−), (1−, 2+), (2+, 1−), (2−, 1+), (2−, 1−)\} = 6,$
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(3) general $p \in \mathbb{N}$: descent statistics of the colored permutation group $G_{p,n}(\sim \mathbb{Z}_p \wr S_n)$
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**Combinatorial meaning of \( L \)**

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M(2, 0) = \#\{(1-, 2-)\} = 1, \\
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(3) general \( p \in \mathbb{N} \): descent statistics of the colored permutation group \( G_{p,n}(\simeq \mathbb{Z}_p \wr S_n) \)

[2] The left eigenvector matrix \( L \) equals to the Foulkes character table of \( G_{p,n} \).

[3] For \( p \notin \mathbb{N} \), we do not know...
Examples of $L(n = 3)$

$$p = 1 : \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \quad p = 2 : \begin{pmatrix} 1 & 23 & 23 & 1 \\ 1 & 5 & -5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$
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$p = 3$: \[
\begin{pmatrix}
1 & 60 & 93 & 8 \\
1 & 23 & -9 & -4 \\
1 & 0 & -3 & 2 \\
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\end{pmatrix}
\]

$p = 3/2$: \[
\begin{pmatrix}
1 & \frac{93}{8} & \frac{15}{2} & \frac{1}{8} \\
1 & \frac{9}{4} & -3 & -\frac{1}{4} \\
1 & -\frac{3}{2} & 0 & \frac{1}{2} \\
1 & -3 & 3 & -1
\end{pmatrix}
\]
Examples of $L(n = 3)$

$p = 1 : \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

$p = 2 : \begin{pmatrix} 1 & 23 & 23 & 1 \\ 1 & 5 & -5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$

$p = 3 : \begin{pmatrix} 1 & 60 & 93 & 8 \\ 1 & 23 & -9 & -4 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{pmatrix}$

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$p = 5/2 : \begin{pmatrix} 1 & \frac{311}{8} & \frac{101}{2} & \frac{27}{8} \\ 1 & \frac{33}{4} & -7 & -\frac{9}{4} \\ 1 & -\frac{1}{2} & -2 & \frac{3}{2} \\ 1 & -3 & 3 & -1 \end{pmatrix}$

No hits on OEIS...
Theorem 2

\[ R_p := L_p^{-1} = \{ R_{ij}^{(p)}(n) \}_{i,j=0,\ldots;\#C_p(n)-1} \]
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If \( p \in \mathbb{N} \),

(1) \( n!p^n R_{0,n-j}^{(p)} \) is equal to the Stirling-Frobenius cycle number.
Right Eigenvector

Theorem 2

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If \( p \in \mathbb{N} \),
(1) \( n!p^n R_{0,n-j}^{(p)} \) is equal to the Stirling-Frobenius cycle number.
(2) \( R_{ij}^{(p)}(n) = \left[ x^{n-j} \right] \# \left\{ \sigma \in G_{p,n} \mid \sigma : (x, n, p)\text{-shuffle with } d(\sigma^{-1}) = i \right\} \)
Colored Permutation Group

\[ \Sigma := [n] \times \mathbb{Z}_p \ (\ [n] := \{1, 2, \ldots, n\}), \ p \in \mathbb{N} \]
Colored Permutation Group

\[ \Sigma := [n] \times \mathbb{Z}_p \ (\ [n] := \{1, 2, \cdots, n\}, \ p \in \mathbb{N} \]

\[ T_q : (i, r) \mapsto (i, r + q), \ (i, r) \in \Sigma \ : \ q\text{-shift on colors} \]
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Example \( (n = 4, p = 3) \):

\[(1, 0) \ (2, 0) \ (3, 0) \ (4, 0) \]

\[ \downarrow \ \ \downarrow \ \ \downarrow \ \ \downarrow \]

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\[
(1, 0) \ (2, 0) \ (3, 0) \ (4, 0) \quad \quad (1, 1) \ (3, 1) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \Longrightarrow \quad \downarrow \quad \downarrow \\
(4, 1) \ (1, 0) \ (2, 2) \ (3, 2) \quad \quad (4, 2) \ (2, 0)
\]
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\[
\begin{array}{cccc}
(1, 0) & (2, 0) & (3, 0) & (4, 0) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
(4, 1) & (1, 0) & (2, 2) & (3, 2) \\
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
(1, 1) & (3, 1) & (4, 2) & (2, 0) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

This \(\sigma\) is determined by \((4, 1) (1, 0) (2, 2) (3, 2)\). so we abuse to write \(\sigma = ((4, 1), (1, 0), (2, 2), (3, 2))\).
Colored Permutation Group

\[ \Sigma := [n] \times \mathbb{Z}_p ([n] := \{1, 2, \cdots, n\}), \quad p \in \mathbb{N} \]

\[ T_q : (i, r) \mapsto (i, r + q), \quad (i, r) \in \Sigma : \quad q\text{-shift on colors} \]

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\[
\begin{align*}
(1, 0) & \quad (2, 0) & \quad (3, 0) & \quad (4, 0) & \quad (1, 1) & \quad (3, 1) \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \Downarrow & \quad \downarrow & \quad \downarrow \\
(4, 1) & \quad (1, 0) & \quad (2, 2) & \quad (3, 2) & \quad (4, 2) & \quad (2, 0)
\end{align*}
\]

This \(\sigma\) is determined by \((4, 1) \quad (1, 0) \quad (2, 2) \quad (3, 2)\).
so we abuse to write \(\sigma = ((4, 1), (1, 0), (2, 2), (3, 2))\).

In general, setting \((\sigma(i), \sigma^c(i)) := \sigma(i, 0) \in \Sigma, \quad i = 1, 2, \cdots, n, \)
Colored Permutation Group

\[ \Sigma := [n] \times \mathbb{Z}_p \left( [n] := \{1, 2, \cdots, n\}\right), \quad p \in \mathbb{N} \]

\[ T_q : (i, r) \mapsto (i, r + q), \quad (i, r) \in \Sigma : q\text{-shift on colors} \]

\[ G_{p,n} := \{ \sigma : \text{bijection on } \Sigma \mid \sigma \circ T_q = T_q \circ \sigma \}. \]

**Example** \((n = 4, p = 3)\):

\[
\begin{array}{cccc}
(1, 0) & (2, 0) & (3, 0) & (4, 0) \\
↓ & ↓ & ↓ & ↓ \\
(4, 1) & (1, 0) & (2, 2) & (3, 2) \\
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
(1, 1) & (3, 1) & & \\
↓ & ↓ & ↓ & \\
(4, 2) & (2, 0) & & \\
\end{array}
\]

This \(\sigma\) is determined by \((4, 1) (1, 0) (2, 2) (3, 2)\).

so we abuse to write \(\sigma = ((4, 1), (1, 0), (2, 2), (3, 2))\).

In general, setting \((\sigma(i), \sigma^c(i)) := \sigma(i, 0) \in \Sigma, \quad i = 1, 2, \cdots, n,\)

we write \(\sigma = ((\sigma(1), \sigma^c(1)), (\sigma(2), \sigma^c(2)), \cdots, (\sigma(n), \sigma^c(n)))\).
Descent on $G_{p,n}$

(1) Define a ordering on $\Sigma$

\[(1, 0) < (2, 0) < \cdots < (n, 0) \]
\[< (1, p - 1) < (2, p - 1) < \cdots < (n, p - 1) \]
\[< (1, p - 2) < (2, p - 2) < \cdots < (n, p - 2) \]
\[\cdots \]
\[< (1, 1) < \cdots < (n, 1). \]
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\cdots \\
< (1, 1) < \cdots < (n, 1).
\]

(2) “\( \sigma \in G_{p,n} \) has a descent at \( i \)”

\[
def \quad (i) \ (\sigma(i), \sigma^c(i)) > (\sigma(i + 1), \sigma^c(i + 1)) \ (\text{for } 1 \leq i \leq n - 1) \\
(ii) \ \sigma^c(n) \neq 0 \ (\text{for } i = n).
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\]

\[\quad \text{(ii) } \sigma^c(n) \neq 0 \quad (\text{for } i = n).
\]

(3) $d(\sigma)$ : the number of descents of $\sigma$.

Ex. ($p = 3$) : $d((5, 0) \downarrow (3, 0) \uparrow (2, 1) \downarrow (4, 2) \uparrow (1, 1) \downarrow) = 3$
Generalized Riffle Shuffle

$n$ cards
with $p$ colors

Introduction

Amazing Matrix

$(b; n; p)$-process

Riffle Shuffle

$(-b, n, p)$-process

Miscellaneous

Application

Summary
Generalized Riffle Shuffle

$n$ cards with $p$ colors

$b$-piles by multinomial

$\begin{align*}
n_0 & \quad n_1 & \quad \ldots \ldots & \quad n_{jp+r} & \quad \ldots \ldots & \quad n_{b-1}
\end{align*}$
Generalized Riffle Shuffle

Generalized carry process and riffle shuffle

Fumihiko NAKANO, Taizo SADAHIRO

Introduction

Amazing Matrix \( (b; n; p) \)-process

Riffle Shuffle \( (b; n; p) \)-process

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Application

Summary

This process defines a Markov chain \( f_{g1} = 0 \) on \( G_{p; n} \). (called the \((b; n; p)\)-shuffle)
Generalized Riffle Shuffle

$n$ cards with $p$ colors

$b$-piles by multinomial

$(b; n; p)$-process

$(n_0 \downarrow +0)$  
$(n_1 \downarrow +1)$  
$(n_{jp+r} \downarrow +r)$  
$(n_{b-1} \downarrow +0)$

shift colors
mix randomly

This process defines a Markov chain $f_{r_1^{r_p}}$ on $G_{p;n}$. (called the $(b; n; p)$-shuffle)
Generalized Riffle Shuffle

This process defines a Markov chain \( \{\sigma_r\}_{r=0}^{\infty} \) on \( G_{p,n} \).
(called the \((b, n, p)\)-shuffle)
Carries Process and Riffle Shuffle

\( p \in \mathbb{N}, \; b \equiv 1 \pmod{p} \)
\( \{C_r\}_{r=1}^{\infty} : (b, n, p) - \text{process} \)
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Remarks

(1) Theorem 3 explains why the descent statistics of \( G_{p,n} \) appears in the stationary distribution of \((b, n, p) - \text{process}\).
Carries Process and Riffle Shuffle

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Remarks
(1) Theorem 3 explains why the descent statistics of \( G_{p,n} \) appears in the stationary distribution of \((b, n, p)\) - process.

(2) By Theorem 3, \( \{d(\sigma_r)\}_r \) turns out to be a Markov chain.
What about \((-b)\)-case?

Any \(x \in \mathbb{Z}\) can be expanded uniquely as

\[
x = a_n(-b)^n + a_{n-1}(-b)^{n-1} + \cdots + a_0,
\]

\(a_k \in \{d, d + 1, \ldots, d + b - 1\}\).
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\(\implies\) E-vectors of \(\tilde{P}_{-b} : L_- = L_+, R_- = R_+\).
Dash - Descent on $\mathcal{G}_{p, n}$

(1) “Dash - order” $'$ on $\sum$:

$$(1, 0) <' (2, 0) <' \cdots <' (n, 0)$$

$$(1, 1) <' (2, 1) <' \cdots <' (n, 1)$$

$$<' \cdots$$

$$(1, p - 1) <' (2, p - 1) <' \cdots <' (n, p - 1)$$
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(2) “$\sigma \in G_{p,n}$ has a dash-descent at $i$”

\[ \defeq \]

(i) $(\sigma(i), \sigma^c(i)) >' (\sigma(i + 1), \sigma^c(i + 1))$ (1 \( \leq i \leq n - 1\))

(ii) $\sigma^c(n) = p - 1$ (\( i = n \)).
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(ii) $\sigma^c(n) = p - 1$ ($i = n$).

(3) $d'(\sigma)$: the number of dash-descents of $\sigma \in G_{p,n}$.
Dash - Descent on $G_{p,n}$

(1) "Dash - order" $\prec'$ on $\Sigma$:

$(1, 0) \prec' (2, 0) \prec' \cdots \prec' (n, 0)$
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$\prec' \cdots$
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(2) "$\sigma \in G_{p,n}$ has a dash-descent at $i$"

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$def$

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$d(\sigma) = d'(\sigma)$ for $p = 1$, $E'_p(n, k) = E_p(n, n - k)$.

Ex. ($p = 3$)

$\begin{align*}
&d' ((5, 0) \downarrow (3, 0) \uparrow (2, 1) \downarrow (4, 2) \uparrow (1, 1) \downarrow ) = 3, \\
&d' ((5, 0) \downarrow (3, 0) \uparrow (2, 1) \uparrow (4, 2) \downarrow (1, 1)) = 2.
\end{align*}$
Shuffles for \((-b, n, p)\) - process

\[ p \in \mathbb{N}, \ (-b) \equiv 1 \pmod{p} \]

\[ \{ C_r^{-} \}_{r=1}^{\infty} : (-b, n, p) - \text{process} \]
Shuffles for \((-b, n, p) - \text{process}\)

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\[ \{ C_r^- \}_{r=1}^\infty : \; (-b, n, p) - \text{process} \]
\[ \{ \sigma_r \}_{r=1}^\infty : \; (+b, n, p) - \text{shuffle} \]
\[ d_r^- := \begin{cases} 
  n - d'(\sigma_r) & (r: \text{odd}) \\
  d(\sigma_r) & (r: \text{even}) 
\end{cases} \]
Shuffles for \((-b, n, p)\) - process

\[ p \in \mathbb{N}, \quad (-b) \equiv 1 \pmod{p} \]

\[ \{ C_r^- \}_{r=1}^\infty : (-b, n, p) - \text{process} \]

\[ \{ \sigma_r \}_{r=1}^\infty : (+b, n, p) - \text{shuffle} \]

\[ d_{r}^- := \begin{cases} 
  n - d'(\sigma_r) & \text{(r : odd)} \\
  d(\sigma_r) & \text{(r : even)} 
\end{cases} \]

**Theorem 4**

\[ \{ C_r^- \}_r \overset{d}{=} \{ d_r^- \}_r \]
Shuffles for \((-b, n, p)\) - process

\[ p \in \mathbb{N}, \ (-b) \equiv 1 \pmod{p} \]
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**Theorem 4**

\[ \{C_r^-\}_r \overset{d}{=} \{d_r^-\}_r \]

**Remark.**
\[ \{d_r^-\}_r \text{ turns out to be a Markov chain.} \]
Description by the group algebra

\[ \Theta_i := \sum_{d(\sigma^{-1})=i} \sigma \in \mathbb{C}[G_{p,n}] \]
Description by the group algebra

\[ \Theta_i := \sum_{d(\sigma^{-1})=i} \sigma \in C[G_{p,n}] \]

\[ P_{shuffle} : \text{the trans. prob. matrix of the } (b, n, p) - \text{shuffle.} \]
Description by the group algebra

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\( P_{\text{shuffle}} \): the trans. prob. matrix of the \((b, n, p)\) - shuffle.

**Theorem 4**

(1) E-values of \( P_{\text{shuffle}} \) = \{1, \( \frac{1}{b} \), \ldots, \( \frac{1}{bn} \)\}.
Description by the group algebra

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**Theorem 4**

1. E-values of \( P_{shuffle} \) = \( \{1, \frac{1}{b}, \ldots, \frac{1}{bn}\} \).

2. E-space corr. to \( b^{-j} = RanL(E_j), E_j = \sum_i R_{ij} \Theta_i \)
Description by the group algebra

\[ \Theta_i \define \sum_{d(\sigma^{-1})=i} \sigma \in \mathbb{C}[G_{p,n}] \]

\( P_{\text{shuffle}} \) : the trans. prob. matrix of the \((b, n, p)\) - shuffle.

**Theorem 4**

(1) E-values of \( P_{\text{shuffle}} = \{1, \frac{1}{b}, \ldots, \frac{1}{bn}\} \).

(2) E-space corr. to \( b^{-j} = \text{Ran}L(E_j), E_j = \sum_i R_{ij} \Theta_i \) with multiplicity \( = \text{tr}L(E_j) = R_{0,j} p^n n! \) (=Stirling Frobenius cycle number)

\( L \) : the left regular representation of \( G_{p,n} \) on \( \mathbb{C}[G_{p,n}] \).
Define $e_0, \cdots, e_n$ such that

$$\sum_i \left( n + \frac{x-1}{p} - i \right) \Theta_i = \sum_{k=0}^n x^{n-k} e_{n-k}$$
Define $e_0, \cdots, e_n$ such that

$$
\sum_{i} \left( n + \frac{x-1}{n} - i \right) \Theta_i = \sum_{k=0}^{n} x^{n-k} e_{n-k}
$$

**Theorem 5**

$$
\Theta_i = \sum_{k} L_{ki} e_{n-k}, \quad e_{n-k} = \sum_{i} R_{ik} \Theta_i
$$
Let $Q$ : distribution of $(b, n, p)$-shuffle on $G_{p,n}$, and

$$m := \frac{3}{2} \log_b n + \log_b c,$$

$$\Phi(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} dx.$$
Let \( Q \) : distribution of \((b, n, p)\)-shuffle on \( G_{p,n} \), and

\[
m := \frac{3}{2} \log_b n + \log_b c,
\]

\[
\Phi(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} \, dx.
\]

**Theorem 6**

\[
\|Q^m - \text{Unif.}\|_{TV} = 1 - 2\Phi \left(-\frac{p}{4\sqrt{3}c}\right) + O(n^{-\frac{1}{2}})
\]

**Remark**

\[
1 - 2\Phi \left(-\frac{p}{4\sqrt{3}c}\right) \sim \begin{cases} 1 - \frac{p}{2c\sqrt{6\pi}} & (c \to \infty) \\ 1 - \frac{4c\sqrt{3}}{p\sqrt{2\pi}} \exp \left\{-\frac{1}{2} \left(\frac{-p}{4c\sqrt{3}}\right)^2\right\} & (c \to 0) \end{cases}
\]
Limit Theorem

For any $p \geq 1$, and for $n \geq 2$, $k = 0, 1, \cdots, n$, let

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p := [x^k](A_{p,n}(x))$$
Limit Theorem

For any $p \geq 1$, and for $n \geq 2$, $k = 0, 1, \cdots, n$, let

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p := [x^k] (A_{p,n}(x))$$

Let $Y_1, \cdots, Y_n$ be the independent, uniformly distributed r.v.'s on $[0, 1],$
Limit Theorem

For any $p \geq 1$, and for $n \geq 2$, $k = 0, 1, \cdots, n$, let

$$\begin{bmatrix} n \\ k \end{bmatrix} \equiv \left[ x^k \right] (A_{p,n}(x))$$

Let $Y_1, \cdots, Y_n$ be the independent, uniformly distributed r.v.’s on $[0, 1]$, and let $S_n := Y_1 + \cdots + Y_n$. 
Limit Theorem

For any $p \geq 1$, and for $n \geq 2$, $k = 0, 1, \ldots, n$, let

$$\langle \frac{n}{k} \rangle := \left[ x^k \right] \left( A_{p,n}(x) \right)$$

Let $Y_1, \ldots, Y_n$ be the independent, uniformly distributed r.v.'s on $[0, 1]$, and let $S_n := Y_1 + \cdots + Y_n$.

**Theorem 5**

$$P \left( S_n \in \frac{1}{p} + [k - 1, k] \right) = \langle \frac{n}{k} \rangle \left( p^n n! \right)^{-1}$$

for $k = 0, 1, \ldots, n$. 
Example

(1) $n = 3, \ p = 1$ : (Eulerian number)
Example

(1) \( n = 3, \ p = 1 \) : (Eulerian number)
Example

(1) \( n = 3, p = 1 \): (Eulerian number)

\[
\begin{array}{c}
1 & 1 & 4 & 2 \\
\end{array}
\times \frac{1}{3!}
\]
Example

(1) $n = 3, p = 1$ : (Eulerian number)

$\frac{1}{3!} \times 3! = 1$

(2) $n = 3, p = 2$ : (Macmahon number)

$\frac{1}{2!} \times 3! = 3$
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\frac{1}{3!}
\]

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Example

(1) $n = 3$, $p = 1$: (Eulerian number)

\[
\begin{align*}
&\times \frac{1}{3!} \\
&\times \frac{1}{2^3 \cdot 3!}
\end{align*}
\]

(2) $n = 3$, $p = 2$: (Macmahon number)
Example

(1) $n = 3, p = 1$ : (Eulerian number)

\[ \times \frac{1}{3!} \]

(2) $n = 3, p = 2$ : (Macmahon number)

\[ \times \frac{1}{2^3 \cdot 3!} \]
Example

(1) $n = 3, p = 1$: (Eulerian number)

(2) $n = 3, p = 2$: (Macmahon number)
Example

(1) $n = 3, p = 1$: (Eulerian number)

$1 \times \frac{1}{3!}$

(2) $n = 3, p = 2$: (Macmahon number)

$1 \times \frac{1}{2^3 \cdot 3!}$
Idea of Proof

Let $X'_1, \cdots, X'_m$ be independent, uniformly distributed r.v.'s on $[l, l + 1]$, and let $S'_m := X'_1 + \cdots + X'_m$. 


Idea of Proof

Let $X'_1, \ldots, X'_m$ be independent, uniformly distributed r.v.'s on $[l, l + 1]$, and let $S'_m := X'_1 + \cdots + X'_m$.

\[
\begin{array}{cccccc}
\text{Carry} & C_k & C_{k-1} & \cdots & C_1 & C_0 \\
\hline
\text{Addends} & X_{1,k} & \cdots & X_{1,2} & X_{1,1} & = X_1^{(k)} \\
& \vdots & \vdots & \vdots & \vdots & \\
& X_{m,k} & \cdots & X_{m,2} & X_{m,1} & = X_m^{(k)} \\
\hline
\text{Sum} & S_k & \cdots & S_2 & S_1
\end{array}
\]
### Idea of Proof

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<table>
<thead>
<tr>
<th>Carry</th>
<th>$C_k$</th>
<th>$C_{k-1}$</th>
<th>$\cdots$</th>
<th>$C_1$</th>
<th>$C_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addends</td>
<td>$X_{1,k}$</td>
<td>$\cdots$</td>
<td>$X_{1,2}$</td>
<td>$X_{1,1}$</td>
<td>$= X^{(k)}_1$</td>
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<td>$X_{m,1}$</td>
<td>$= X^{(k)}_m$</td>
</tr>
<tr>
<td>Sum</td>
<td>$S_k$</td>
<td>$\cdots$</td>
<td>$S_2$</td>
<td>$S_1$</td>
<td></td>
</tr>
</tbody>
</table>

Since $X^{(k)}_i \xrightarrow{k \to \infty} X'_i$, $X^{(k)}_1 + \cdots + X^{(k)}_m \xrightarrow{k \to \infty} S'_m$. 
Idea of Proof

Let $X'_1, \cdots, X'_m$ be independent, uniformly distributed r.v.'s on $[l, l + 1]$, and let $S'_m := X'_1 + \cdots + X'_m$.

<table>
<thead>
<tr>
<th>Carry</th>
<th>$C_k$</th>
<th>$C_{k-1}$</th>
<th>$\cdots$</th>
<th>$C_1$</th>
<th>$C_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addends</td>
<td>$X_{1,k}$</td>
<td>$\cdots$</td>
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<td>$= X_1^{(k)}$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
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</tr>
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$\mathbb{P}(C_k = j) = \mathbb{P}(X_1^{(k)} + \cdots + X_m^{(k)} \in [l, l + 1] + j)$
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\begin{array}{cccccc}
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\hline
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& \vdots & \vdots & \vdots & \vdots & \\
& X_{m,k} & \cdots & X_{m,2} & X_{m,1} & = X_m^{(k)} \\
\hline
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\end{array}
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\downarrow \\
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(1) Stationary distribution gives the descent statistics of \( G_{p,n} \)
Generalized carry process and riffle shuffle

Fumihiko NAKANO, Taizo SADAHIRO

Introduction

Amazing Matrix ($b; n; p$) - process

Riffle Shuffle

($-b; n; p$) - process

Miscellaneous

Application

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[3] for \( p \notin \mathbb{N} \), no combinatorial meaning is known so far...
References


