Generalized carry process and riffle shuffle

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Adding <u>3</u> numbers,

10111	10210	11102	11122	01011	11210	2112
43443	07082	04401	15299	64642	73497	38426
00171	55077	11440	95932	91116	17255	19649
49339	70267	68885	98147	70311	43856	37376
92954	32426	84728	09380	26070	34608	95451

then 1 seems to appear frequently. $(\sharp 0 : \sharp 1 : \sharp 2 = 7 : 20 : 7)$。

Transition Probability 1

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Example 1 $(b = 2, n = 2)$

$$\begin{array}{cc} 0 & 0 \\ \hline & 0 \\ & 0 \\ \hline & 0 \end{array} \quad \begin{array}{cc} 0 & 0 \\ \hline & 1 \\ & 0 \\ \hline & 1 \end{array} \quad \begin{array}{cc} 0 & 0 \\ \hline & 0 \\ & 1 \\ \hline & 1 \end{array} \quad \begin{array}{cc} 1 & 0 \\ \hline & 1 \\ & 1 \\ \hline & 0 \end{array} \quad \Longrightarrow \quad (P_{0,0}, P_{0,1}) = \frac{1}{2^2}(3, 1)$$

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For $b = 2, n = 2$

$$P = \frac{1}{2^2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad \Longrightarrow \quad \text{Stationary dist. } \pi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Transition Probability 2

Example 2 $(b = 2, n = 3)$

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$$
\begin{array}{cc}
0 & 0 \\
\hline
 & 0 \\
 & 0 \\
 & 0 \\
\hline
 & 0
\end{array}
\qquad
\begin{array}{cc}
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\hline
 & 1 \\
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$$\Longrightarrow \quad (P_{0,0}, P_{0,1}, P_{0,2}) = \frac{1}{2^3} \cdot (4, 4, 0)$$

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For $b = 2, n = 3$

$$P = \frac{1}{2^3} \begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix} \quad \Longrightarrow \quad \pi = \frac{1}{3!} \cdot (1, 4, 1)$$

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Add n base- b numbers $(b, n \in \mathbf{N}, b, n \geq 2)$

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$\mathbf{C_k}$: the carry coming out in the k-th digit.

Carry	C_{k+1}	C_k	$\cdots$	C_1	$C_0 = 0$
Addends		$X_{1,k}$	$\cdots$	$X_{1,1}$	$X_{1,0}$
		$\vdots$		$\vdots$	$\vdots$
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$\{C_k\}_{k=0}^{\infty}$ $(C_k \in \{0, \cdots, n-1\})$ is called the **carries process**.

Amazing Matrix : Holte(1997)

$$P_{ij} := \mathbf{P}\left(C_{k+1} = j \,|\, C_k = i\right), \quad i, j = \underline{0}, 1, \cdots, n-1$$

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E-values and left E-vectors of Amazing Matrix

E-values/ E-vectors depends only on b / n.

$$P = L^{-1} D L, \quad D = \text{ diag } \left(1, \frac{1}{b}, \frac{1}{b^2}, \cdots, \frac{1}{b^{n-1}}\right)$$

$$L_{ij} = v_{ij}(n) = [x^j]\left(A_m(x)\right).$$

$A_m(x) := (1-x)^{n+1} \sum_{j \geq 0} (j+1)^m x^j$: Eulerian polynomial.

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Remark. $\quad P(b_1)P(b_2) = P(b_1 \cdot b_2)$.

Property of Left Eigenvectors

$$[1] \quad L = \begin{pmatrix} (n\text{-th Eulerian num.}) \\ \vdots \\ (-1)^j((n-1)\text{-th Pascal num.}) \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 11 & 11 & 1 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

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$$E(3,0) = \sharp\{(123)\} = 1,$$
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$[2]$ L is equal to the <u>Foulkes character table</u> of S_n (Diaconis-Fulman, 2012).

Foulkes character

Example

$$\sharp\{\sigma \in S_4 \,|\, \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4)\}$$
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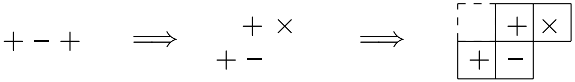

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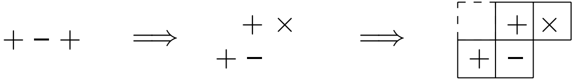

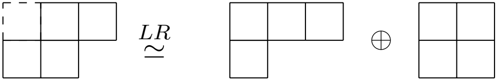

$\dim = 3$ $\qquad$ $\dim = 2$

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Right Eigenvector of P

$$P = RDR^{-1}$$

$$R_{ij} = \sum_{r=n-j}^{n} (-1)^{n-r} \left[\begin{array}{c} n \\ r \end{array} \right] \left(\begin{array}{c} r \\ n-j \end{array} \right) (n-1-i)^{r-(n-j)}$$

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$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 6 & 11 & 6 \\ 1 & 2 & -1 & -2 \\ 1 & -2 & -1 & 2 \\ 1 & -6 & 11 & -6 \end{pmatrix}$$

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$R(0,j) = S(n, n-j)$

$S(n,j) := \sharp\{\sigma \in S_n$ with j-cycles $\}$ Stirling num. of 1st kind

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Relation to Riffle Shuffles (Diaconis-Fulman, 2009)

$$\{C_k\}_{k=1}^{\infty} \overset{d}{=} \{d(\sigma_k)\}_{k=1}^{\infty}, \quad d(\sigma) : \text{ the descent of } \sigma \in S_n.$$

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Since the stationary dist. of $\{\sigma_k\}$ is uniform on S_n,

$$L_{0j} = \lim_{k\to\infty} \mathbf{P}(C_k = j) = \lim_{k\to\infty} \mathbf{P}(d(\sigma_k) = j) \\ = \mathbf{P}_{unif}(d(\sigma) = j) = E(n, j)/n!$$

explaining why Eulerian num. appears.

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(4) carries process has the same distribution to the descent process of the riffle shuffle.

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$$
\begin{array}{c|ccccc}
\text{Carry} & C_{k+1} & C_k & \cdots & C_1 & C_0 = 0 \\
\hline
\text{Addends} & & X_{1,k} & \cdots & X_{1,1} & X_{1,0} \\
 & & \vdots & & \vdots & \vdots \\
 & & X_{n,k} & \cdots & X_{n,1} & X_{n,0} \\
 & & \frac{b-1}{p^*} & \cdots & \frac{b-1}{p^*} & \frac{b-1}{p^*} \\
\hline
\text{Sum} & & S_k & \cdots & S_1 & S_0
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$\{C_k\}_{k=0}^{\infty}$ is called the (b, n, p)-**carries process**.
($p = 1$: usual carries process)

Remarks

(1) If we generalize the usual carries process by changing the digit set such as

$$D_b = \{0, 1, \cdots, b-1\} \quad \Longrightarrow \quad D_b = \{d, d+1, \cdots, d+b-1\},$$

we get (b, n, p)-carries process, after some change of variables.

Remarks

(1) If we generalize the usual carries process by changing the digit set such as

$$D_b = \{0, 1, \cdots, b-1\} \quad \Longrightarrow \quad D_b = \{d, d+1, \cdots, d+b-1\},$$

we get (b, n, p)-carries process, after some change of variables.

(2) $C_k \in \mathcal{C}_p(n)$ where $\mathcal{C}_p(n)$ is the carries set given by

$$\mathcal{C}_p(n) := \begin{cases} \{0, 1, \cdots, n-1\} & (p = 1) \\ \{0, 1, \cdots, n\} & (p \neq 1) \end{cases}$$

Left Eigenvectors

$\tilde{P} = \{\tilde{P}_{ij}\}$: Transition probability of (b, n, p)- process :

$$\tilde{P}_{ij} = \mathbf{P}\left(C_{k+1} = j \,|\, C_k = i\right).$$

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Theorem 1

E-values/ E-vectors depend only on b / n.

$$\tilde{P} = L_p^{-1} D L_p, \; D = \text{diag}\left(1, \frac{1}{b}, \cdots, \frac{1}{b^{\sharp \mathcal{C}_p(n) - 1}}\right)$$

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Combinatorial meaning of L

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[3] For $p \notin \mathbf{N}$, we do not know...

Examples of $L(n = 3)$

$$p = 1 : \begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \quad p = 2 : \begin{pmatrix} 1 & 23 & 23 & 1 \\ 1 & 5 & -5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

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$$p = 5/2 : \begin{pmatrix} 1 & \frac{311}{8} & \frac{101}{2} & \frac{27}{8} \\ 1 & \frac{33}{4} & -7 & -\frac{9}{4} \\ 1 & -\frac{1}{2} & -2 & \frac{3}{2} \\ 1 & -3 & 3 & -1 \end{pmatrix} \quad ?$$

No hits on OEIS...

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Theorem 2

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Descent on $G_{p,n}$

(1) Define a ordering on Σ

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$\overset{def}{\Longleftrightarrow}$ (i) $(\sigma(i), \sigma^c(i)) > (\sigma(i+1), \sigma^c(i+1))$ (for $1 \le i \le n-1$)

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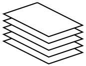
(3) $d(\sigma)$: the number of descents of σ.

Ex. $(p=3)$: $d\left((5,0)\searrow(3,0)\nearrow(2,1)\searrow(4,2)\nearrow(1,1)\searrow\right) = 3$

Generalized Riffle Shuffle



n cards
with p colors

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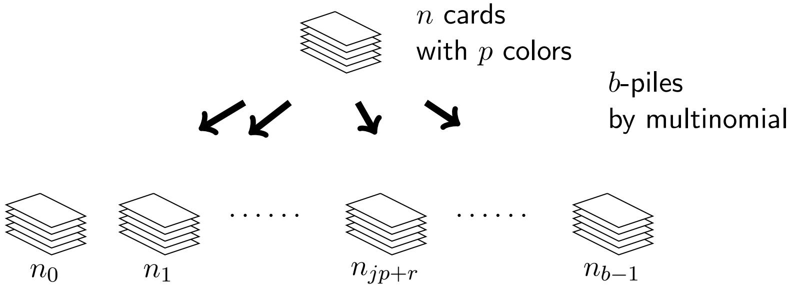

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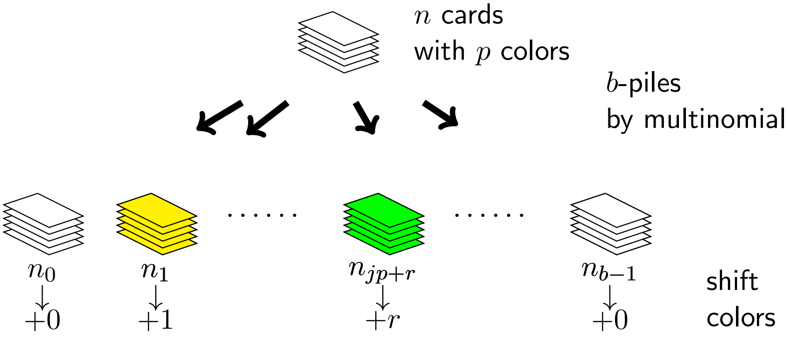

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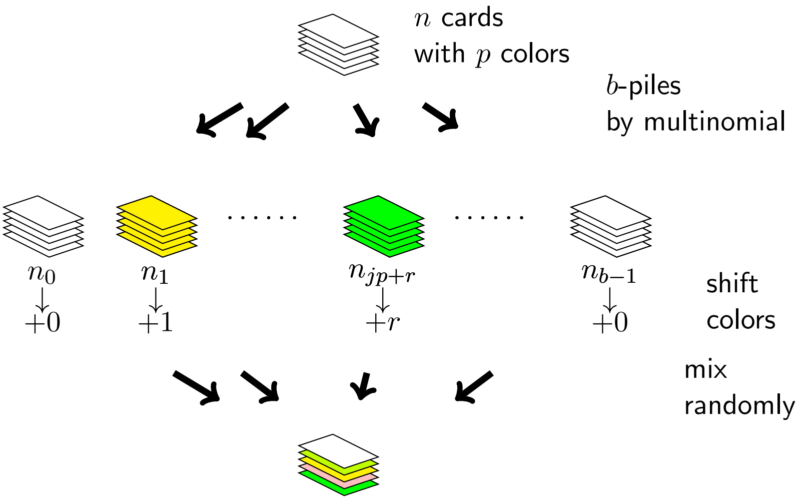

Fumihiko NAKANO, Taizo SADAHIRO

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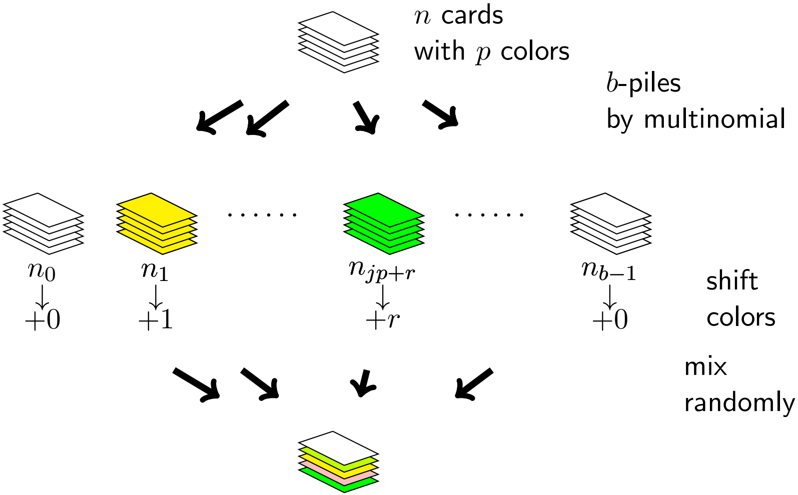

This process defines a Markov chain $\{\sigma_r\}_{r=0}^{\infty}$ on $G_{p,n}$.
(called the $\underline{(b, n, p)\text{-shuffle}}$)

Carries Process and Riffle Shuffle

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(2) By Theorem 3, $\{d(\sigma_r)\}_r$ turns out to be a Markov chain.

What about $(-b)$-case ?

Any $x \in \mathbf{Z}$ can be expanded uniquely as

$$x = a_n(-b)^n + a_{n-1}(-b)^{n-1} + \cdots + a_0,$$
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(1) " Dash - order " $<'$ on Σ:

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Dash - Descent on $G_{p,n}$

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Ex. $(p=3)$ $d\left((5,0)\searrow(3,0) \nearrow (2,1)\searrow(4,2) \nearrow (1,1)\searrow\right) = 3$,
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Remark.
$\{d_r^-\}_r$ turns out to be a Markov chain.

Description by the group algebra

$$\Theta_i := \sum_{d(\sigma^{-1})=i} \sigma \quad \in \mathbf{C}[G_{p,n}]$$

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with multiplicitiy $= trL(E_j) = R_{0,j}p^n n!$
(=Stirling Frobenius cycle number)

L : the left regular representation of $G_{p,n}$ on $\mathbf{C}[G_{p,n}]$.

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$$\Theta_i = \sum_k L_{ki} e_{n-k}, \quad e_{n-k} = \sum_i R_{ik} \Theta_i$$

Cut off

Let Q : distribution of (b, n, p)-shuffle on $G_{p,n}$, and

$$m := \frac{3}{2} \log_b n + \log_b c,$$

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Theorem 6

$$\|Q^m - \text{Unif.}\|_{TV} = 1 - 2\Phi\left(-\frac{p}{4\sqrt{3}c}\right) + O(n^{-\frac{1}{2}})$$

Remark

$$1 - 2\Phi\left(-\frac{p}{4\sqrt{3}c}\right) \sim \begin{cases} \dfrac{p}{2c\sqrt{6\pi}} & (c \to \infty) \\ 1 - \dfrac{4c\sqrt{3}}{p\sqrt{2\pi}}\exp\left\{-\dfrac{1}{2}\left(\dfrac{-p}{4c\sqrt{3}}\right)^2\right\} & (c \to 0) \end{cases}$$

Limit Theorem

For any $p \ge 1$, and for $n \ge 2$, $k = 0, 1, \cdots, n$, let

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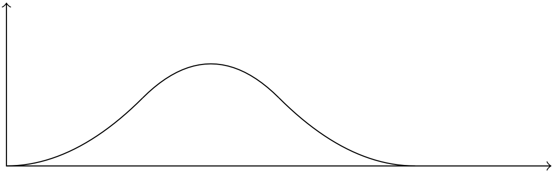
$$\mathbf{P} \left(S_n \in \frac{1}{p} + [k-1, k] \right) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_p (p^n n!)^{-1}$$

for $k = 0, 1, \cdots, n$.

Example

(1) $n = 3$, $p = 1$: (Eulerian number)



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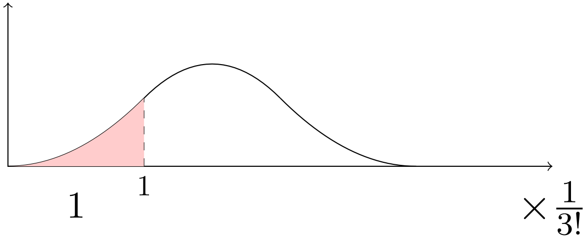

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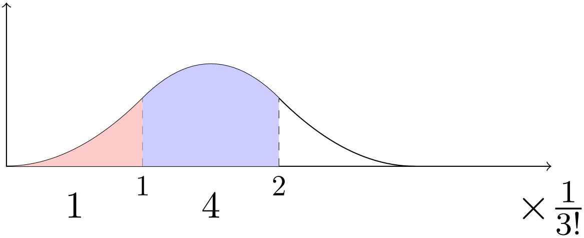

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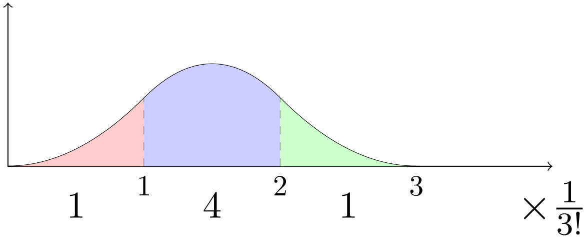

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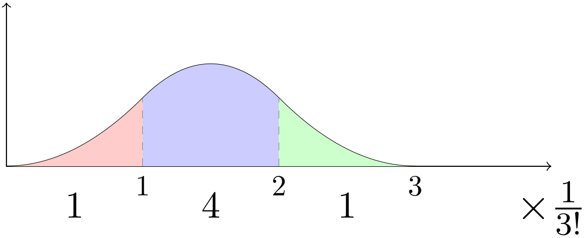

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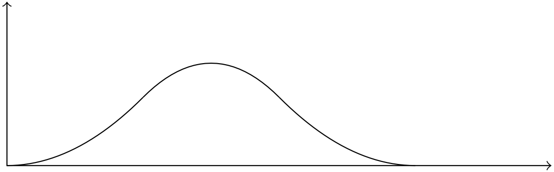

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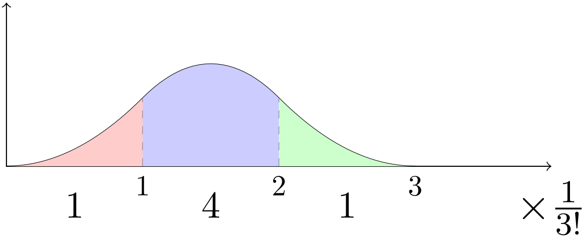

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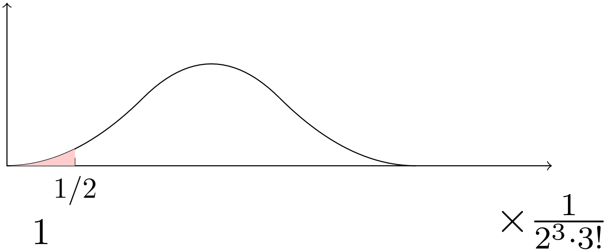

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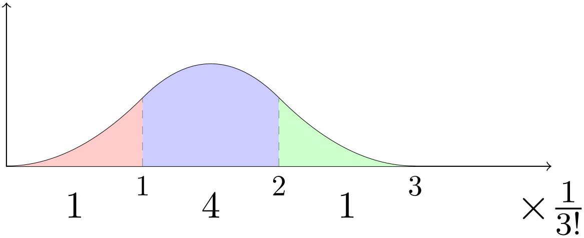

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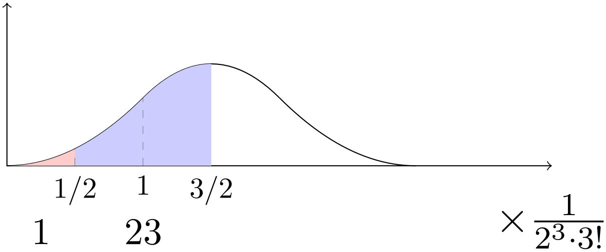

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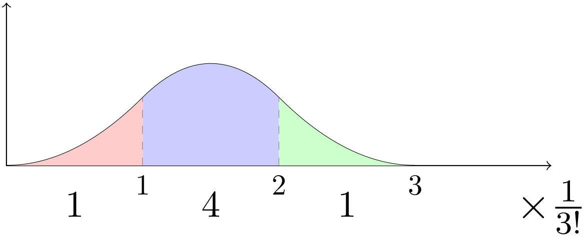

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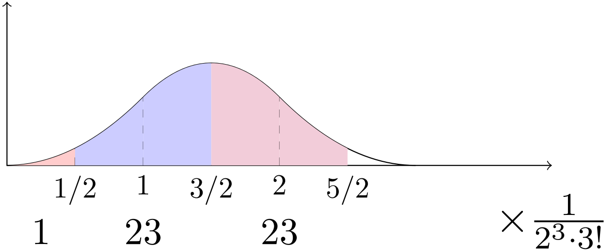

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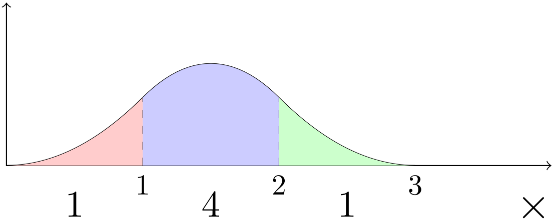

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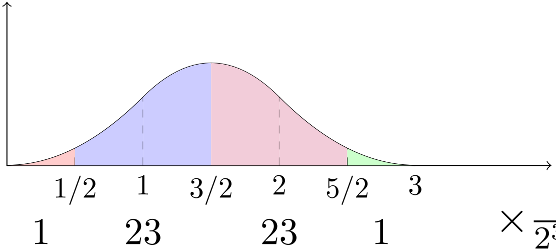

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Idea of Proof

Let $X'_1, \cdots, X'_m$ be independent, uniformly distributed r.v.'s on $[l, l+1]$, and let $S'_m := X'_1 + \cdots + X'_m$.

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[3] for $\underline{p \notin \mathbf{N}}$, no combinatorial meaning is known so far...

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