(q, t)-hook formula for Tailed Insets and a Macdonald polynomial identity

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Okada presented a conjecture on \((q, t)\)-hook formula for general \(d\)-complete posets in the paper, Soichi Okada, \((q, t)\)-Deformations of multivariate hook product formulae, *J. Algebr. Comb.* (2010) *32*, 399 – 416. We consider the Tailed Inset case, and reduce the conjectured identity to an identity of the Macdonald polynomials rephrasing Okada’s \((q,t)\)-weights via Pieri coefficients of the Macdonald polynomials. Joint work with Frederic Jouhet (University of Lyon I).
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Further references:


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Additional References

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Introduction
Definition

A *partially ordered set* (also called a *poset*) is a set $P$ with a binary relation “$\leq$” which is *antisymmetric*, *transitive*, and *reflexive*.

Definition (Stanley ’72)

Let $P$ be a poset. A *$P$-partition* is a map $\pi : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \implies \pi(x) \geq \pi(y) \text{ in } \mathbb{N},$$

where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{P}(P)$ be the set of $P$-partitions.

Example ($P$-partitions)
**Definition**

A partially ordered set (also called a poset) is a set $P$ with a binary relation “$\leq$” which is antisymmetric, transitive, and reflexive.

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$$x \leq y \text{ in } P \implies \pi(x) \geq \pi(y) \text{ in } \mathbb{N},$$

where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{A}(P)$ be the set of $P$-partitions.

**Example ($P$-partitions)**

Let $A$ be a partially ordered set (poset) with a binary relation $\leq$. A $P$-partition is a map $\pi : P \rightarrow \mathbb{N}$ satisfying

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where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{A}(P)$ be the set of $P$-partitions.
**Definition**

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where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{A}(P)$ be the set of $P$-partitions.

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A **partially ordered set** (also called a **poset**) is a set $P$ with a binary relation "$\leq$" which is **antisymmetric**, **transitive**, and **reflexive**.

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where $\mathbb{N}$ is the set of nonnegative integers. Let $A(P)$ be the set of $P$-partitions.

**Example ($P$-partitions)**

![Diagram of a partially ordered set](image)
**Definition**

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**Example ($P$-partitions)**

![Diagram of a partially ordered set]

Masao Ishikawa  (q, t)-hook formula for Tailed Insets
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**Example ($P$-partitions)**

1 0

1 2

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**Example ($P$-partitions)**

![Graph of a poset with elements 0, 1, and 2 connected by edges]

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*(q, t)-hook formula for Tailed Insets*
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where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathcal{A}(P)$ be the set of $P$-partitions.

**Example ($P$-partitions)**

![Diagram of a partial order with elements 1, 0, 2, and 1, 2 connected by arrows indicating the order]

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(q, t)-hook formula for Tailed Insets
(Shifted) diagrams

**Definition**

A *partition* is a nonincreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers with finitely many \( \lambda_i \) unequal to zero. The *length* and *weight* of \( \lambda \), denoted by \( \ell(\lambda) \) and \( |\lambda| \), are the number and sum of the non-zero \( \lambda_i \) respectively. A *strict partition* is a partition in which its parts are strictly decreasing. If \( \lambda \) is a partition (resp. strict partition), then its *diagram* \( D(\lambda) \) (resp. *shifted diagram* \( S(\lambda) \)) is defined by

\[
D(\lambda) = \{ (i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i \}
\]
\[
S(\lambda) = \{ (i, j) \in \mathbb{Z}^2 : i \leq j \leq \lambda_i + i - 1 \}.
\]

Example (The diagram and shifted diagram for \( \lambda = (4, 3, 1) \)).

\[
D(\lambda) = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
\[
S(\lambda) = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
A partition is a nonincreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers with finitely many \( \lambda_i \) unequal to zero. The length and weight of \( \lambda \), denoted by \( \ell(\lambda) \) and \( |\lambda| \), are the number and sum of the non-zero \( \lambda_i \) respectively. A strict partition is a partition in which its parts are strictly decreasing. If \( \lambda \) is a partition (resp. strict partition), then its diagram \( D(\lambda) \) (resp. shifted diagram \( S(\lambda) \)) is defined by

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Example (The diagram and shifted diagram for \( \lambda = (4, 3, 1) \))

\[D(\lambda) = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}\]
\[S(\lambda) = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}\]
A partition is a nonincreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers with finitely many \( \lambda_i \) unequal to zero. The length and weight of \( \lambda \), denoted by \( \ell(\lambda) \) and \( |\lambda| \), are the number and sum of the non-zero \( \lambda_i \) respectively. A strict partition is a partition in which its parts are strictly decreasing. If \( \lambda \) is a partition (resp. strict partition), then its diagram \( D(\lambda) \) (resp. shifted diagram \( S(\lambda) \)) is defined by

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Example (The diagram and shifted diagram for \( \lambda = (4, 3, 1) \))

\[
D(\lambda) = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{array} \quad \quad S(\lambda) = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\]
A partition is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with finitely many $\lambda_i$ unequal to zero. The length and weight of $\lambda$, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero $\lambda_i$ respectively. A strict partition is a partition in which its parts are strictly decreasing. If $\lambda$ is a partition (resp. strict partition), then its diagram $D(\lambda)$ (resp. shifted diagram $S(\lambda)$) is defined by

$$D(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\}$$

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Definition

A partition is a nonincreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers with finitely many \( \lambda_i \) unequal to zero. The length and weight of \( \lambda \), denoted by \( \ell(\lambda) \) and \( |\lambda| \), are the number and sum of the non-zero \( \lambda_i \) respectively. A strict partition is a partition in which its parts are strictly decreasing. If \( \lambda \) is a partition (resp. strict partition), then its diagram \( D(\lambda) \) (resp. shifted diagram \( S(\lambda) \)) is defined by

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\]

Example (The diagram and shifted diagram for \( \lambda = (4, 3, 1) \))

\[
D(\lambda) = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & & & \\
\end{array} \quad S(\lambda) = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & & & \\
\end{array}
\]
A diagram $D(\lambda)$ or a shifted diagram $S(\lambda)$ is regarded as a poset by defining its order structure by

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$ 

By this order the poset represented by a diagram $P = D(\lambda)$ is called a *shape*, and the posets $P = S(\lambda)$ is called *shifted shapes*.

Example (The shape and shifted shape for $\lambda = (4, 3, 1)$)
(Shifted) shapes

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Example (The shape and shifted shape for $\lambda = (4, 3, 1)$)

$D(\lambda) =$ 

$S(\lambda) =$
(Shifted) shapes

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Example (The shape and shifted shape for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \quad S(\lambda) =$$

\[\begin{array}{c}
\end{array}\]
For a partition (resp. strict partition) \( \lambda \) and a cell \((i, j) \in D(\lambda)\) (resp. \(S(\lambda)\)), the **hook at** \((i, j)\) in \(D(\lambda)\) (resp. \(S(\lambda)\)), is defined by

\[
H_{D(\lambda)}(i, j) = \{(i, j)\} \cup \{(i, l) \in D(\lambda) : l > j\} \cup \{(k, j) \in D(\lambda) : k > i\}
\]

(resp.

\[
H_{S(\lambda)}(i, j) = \{(i, j)\} \cup \{(i, l) \in S(\lambda) : l > j\}
\]

\[
\cup \{(k, j) \in D(\lambda) : k > i\} \cup \{(j + 1, l) \in S(\lambda) : l > j\}.
\]

Example (The hook at \((1, 2)\) in \(D(\lambda)\) and \(S(\lambda)\) for \(\lambda = (4, 3, 1)\))

\[D(\lambda) = \quad S(\lambda) =\]
For a partition (resp. strict partition) $\lambda$ and a cell $(i, j) \in D(\lambda)$ (resp. $S(\lambda)$), the hook at $(i, j)$ in $D(\lambda)$ (resp. $S(\lambda)$), is defined by

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(resp.

$$H_{S(\lambda)}(i, j) = \{(i, j)\} \cup \{(i, l) \in S(\lambda) : l > j\}$$

$$\cup \{(k, j) \in D(\lambda) : k > i\} \cup \{(j + 1, l) \in S(\lambda) : l > j\}.$$

Example (The hook at $(1, 2)$ in $D(\lambda)$ and $S(\lambda)$ for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \begin{array}{cccc}
4 & & & \\
&  & & \\
& & & \\
& & & \\
& & & \\
\end{array}$$

$$S(\lambda) = \begin{array}{cccc}
5 & & & \\
&  & & \\
& & & \\
& & & \\
& & & \\
\end{array}$$
Content and hook length

Definition

The hook length at \((i, j)\) is defined by \(h_{D(\lambda)}(i, j) = |H_{D(\lambda)}(i, j)|\) (resp. \(h_{S(\lambda)}(i, j) = |H_{S(\lambda)}(i, j)|\)). Further \(c(i, j) = j - i\) is called the content at \((i, j)\).

Example (The hook lengths in \(D(\lambda)\) and \(S(\lambda)\) for \(\lambda = (4, 3, 1)\))

\[
D(\lambda) = \begin{array}{ccc}
6 & 4 & 3 \\
4 & 2 & 1 \\
1 & & \\
\end{array}
\quad S(\lambda) = \begin{array}{ccc}
7 & 5 & 4 \\
4 & 3 & 1 \\
1 & & \\
\end{array}
\]

Example (The contents in \(D(\lambda)\) and \(S(\lambda)\) for \(\lambda = (4, 3, 1)\))

\[
D(\lambda) = \begin{array}{ccc}
0 & 1 & 2 \\
-1 & 0 & 1 \\
-2 & & \\
\end{array}
\quad S(\lambda) = \begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & & \\
\end{array}
\]
Content and hook length

Definition

The hook length at \((i, j)\) is defined by \(h_D(\lambda)(i, j) = |H_D(\lambda)(i, j)|\) (resp. \(h_S(\lambda)(i, j) = |H_S(\lambda)(i, j)|\)). Further \(c(i, j) = j - i\) is called the content at \((i, j)\).

Example (The hook lengths in \(D(\lambda)\) and \(S(\lambda)\) for \(\lambda = (4, 3, 1)\))

\[
D(\lambda) = \begin{array}{ccc}
6 & 4 & 3 \\
4 & 2 & 1 \\
1 & \end{array} \quad S(\lambda) = \begin{array}{ccc}
7 & 5 & 4 \\
4 & 3 & 1 \\
1 & \end{array}
\]

Example (The contents in \(D(\lambda)\) and \(S(\lambda)\) for \(\lambda = (4, 3, 1)\))

\[
D(\lambda) = \begin{array}{ccc}
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-1 & 0 & 1 \\
-2 & \end{array} \quad S(\lambda) = \begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & \end{array}
\]
Content and hook length

**Definition**

The **hook length at** \((i, j)\) is defined by \(h_D(\lambda)(i, j) = |H_D(\lambda)(i, j)|\) (resp. \(h_S(\lambda)(i, j) = |H_S(\lambda)(i, j)|\)). Further \(c(i, j) = j - i\) is called the **content** at \((i, j)\).

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\[
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6 & 4 & 3 & 1 \\
4 & 2 & 1 & \\
1 & & & \\
\end{array} \quad S(\lambda) = \begin{array}{cccc}
7 & 5 & 4 & 2 \\
4 & 3 & 1 & \\
1 & & & \\
\end{array}
\]

**Example (The contents in** \(D(\lambda)\) **and** \(S(\lambda)\) **for** \(\lambda = (4, 3, 1)\))

\[
D(\lambda) = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & 0 & 1 & \\
-2 & & & \\
\end{array} \quad S(\lambda) = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & \\
0 & & & \\
\end{array}
\]
Theorem (Frame-Robinson-Thrall ’54, Stanley ’72))

If $P = D(\lambda)$ or $S(\lambda)$, then we have

$$
\sum_{\pi \in \mathcal{A}(P)} z^{\lvert \pi \rvert} = \prod_{(i,j) \in P} \frac{1}{1 - z^{h_P(i,j)}},
$$

where the sum on the left-hand side runs over all $P$-partitions, and $|\pi| = \sum_{x \in P} \pi(x)$.

Example (An example of $P$-partition)

$$
\pi = \begin{array}{cccc}
0 & 0 & 1 & 2 \\
2 & 3 & 4 & \\
4 & & & \\
\end{array}
$$

$|\pi| = 16$

$z^{|\pi|} = z^{16}$
One Variable Hook Length Formula

Theorem (Frame-Robinson-Thrall '54, Stanley '72))

If \( P = D(\lambda) \) or \( S(\lambda) \), then we have

\[
\sum_{\pi \in \mathcal{A}(P)} z^{\left| \pi \right|} = \prod_{(i,j) \in P} \frac{1}{1 - z^{h_P(i,j)}},
\]

where the sum on the left-hand side runs over all \( P \)-partitions, and \( \left| \pi \right| = \sum_{x \in P} \pi(x) \).

Example (An example of \( P \)-partition)

\[
\pi = \begin{array}{cccc}
0 & 0 & 1 & 2 \\
2 & 3 & 4 & \\
4 & & & \\
\end{array}
\]

\( \left| \pi \right| = 16 \)

\( z^{\left| \pi \right|} = z^{16} \)
Example of One Variable Hook Length Formula

Example (The shape for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \pi_{11}\pi_{12}\pi_{13}\pi_{14}$$

$$\pi_{21}\pi_{22}\pi_{23}$$

$$\pi_{31}$$

$$D(\lambda) = \begin{array}{c}
6 \\ 4 \\ 2 \\ 1 \\
\end{array}$$

$$\begin{array}{c}
31 \\ 4 \\ 2 \\ 1 \\
\end{array}$$

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} z \sum_{(i, j) \in D(\lambda)} \pi_{ij} = \frac{1}{(1 - z)^3(1 - z^2)(1 - z^3)(1 - z^4)^2(1 - z^6)}.$$
Theorem (Gansner ’81, Sagan ’82)

Let \( \ldots, z_{-1}, z_0, z_1, z_2, \ldots \) be variables. If \( P = D(\lambda) \) or \( S(\lambda) \), then we have

\[
\sum_{\pi \in \mathcal{P}(P)} z^\pi = \prod_{(i,j) \in P} \frac{1}{1 - z[H_P(i,j)]},
\]

where the sum on the left-hand side runs over all \( P \)-partitions, \( z^\pi = \prod_{(i,j) \in P} z_{c(i,j)}^{\pi_{ij}} \) and \( z[H] = \prod_{(i,j) \in H} z_{c(i,j)} \) for any finite subset \( H \subset \mathbb{Z}^2 \). (Gansner used Hillman-Grassl ’76 algorithm.)

Example (An example of \( P \)-partition)

\[
\begin{align*}
\pi &= \begin{array}{cccc}
0 & 0 & 1 & 2 \\
2 & 3 & 4 \\
4 & & & \\
\end{array} \\

z^\pi &= z_2^4 z_0^2 z_1 z_2^3 z_3^4 z_2 z_2^2
\end{align*}
\]
Theorem (Gansner '81, Sagan '82)

Let \( \ldots, z_{-1}, z_0, z_1, z_2, \ldots \) be variables. If \( P = D(\lambda) \) or \( S(\lambda) \), then we have

\[
\sum_{\pi \in \mathcal{P}(P)} z^\pi = \prod_{(i,j) \in P} \frac{1}{1 - z[H_P(i, j)]},
\]

where the sum on the left-hand side runs over all \( P \)-partitions,

\[
z^\pi = \prod_{(i,j) \in P} z_c(i,j)^{\pi_{i,j}} \quad \text{and} \quad z[H] = \prod_{(i,j) \in H} z_c(i,j)
\]

for any finite subset \( H \subset \mathbb{Z}^2 \). (Gansner used Hillman-Grassl '76 algorithm.)

Example (An example of \( P \)-partition)

\[
\pi = \begin{array}{cccc}
2 & 3 & 4 \\
4 & & & \\
0 & 0 & 1 & 2 \\
\end{array}
\]

\[
z^\pi = z_2^4 z_{-1}^2 z_0^3 z_1^4 z_2 z_3^2
\]
Example of Multivariate Hook Length Formula

Example (The shape for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \sum_{\pi \in \mathcal{A}(P)} z_{-2}^{\pi_{31}} z_{-1}^{\pi_{21}} z_0^{\pi_{11}+\pi_{22}} z_1^{\pi_{12}+\pi_{23}} z_2^{\pi_{13}} z_3^{\pi_{14}}$$

$$= \frac{1}{(1 - z_{-2}z_{-1}z_0z_1z_2z_3)(1 - z_0z_1z_2z_3)(1 - z_1z_2z_3)(1 - z_3)} \times \frac{1}{(1 - z_{-2}z_{-1}z_0z_1)(1 - z_0z_1)(1 - z_1)(1 - z_{-2})}.$$
The Cauchy formula and the Littlewood formula

**Theorem (The Cauchy formula)**

Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) are \( n \)-tuples of variables. Then we have

\[
\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.
\]

**Theorem (The Littlewood formula)**

Let \( x = (x_1, \ldots, x_n) \) is an \( n \)-tuples of variables. Then we have

\[
\sum_{\lambda} s_{\lambda}(x) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}.
\]
The Cauchy formula and the Littlewood formula

**Theorem (The Cauchy formula)**

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are $n$-tuples of variables. Then we have

$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\lambda}(\mathbf{y}) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.$$ 

**Theorem (The Littlewood formula)**

Let $\mathbf{x} = (x_1, \ldots, x_n)$ is an $n$-tuples of variables. Then we have

$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}.$$
Conjecture (Okada ’10)

If \( P \) is a \( d \)-complete poset, then we have

\[
\sum_{\pi \in \mathcal{P}(P)} W_P(\pi; q, t) z^\pi = \prod_{(i,j) \in P} F(z[H_P(i,j)]; q, t),
\]

where the sum on the left-hand side runs over all \( P \)-partitions, and

\[
F(x; q, t) = \frac{(tx; q)_\infty}{(x; q)_\infty}.
\]

Example (The shape for \( \lambda = (4, 3, 1) \))

\[
D(\lambda) = \begin{array}{cccc}
\pi_{11} \pi_{12} \pi_{13} \pi_{14} \\
\pi_{21} \pi_{22} \pi_{23} \\
\pi_{31}
\end{array} \quad D(\lambda) = \begin{array}{cccc}
Z_0 & Z_1 & Z_2 & Z_3 \\
Z_{-1} & Z_0 & Z_1 & Z_2
\end{array}
\]
Example (The shape for $\lambda = (4, 3, 1)$)

$$D(\lambda) = D(\lambda) =$$

$$\sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) z_{-2}^{\pi_{31}} z_{-1}^{\pi_{21}} z_0^{\pi_{11} + \pi_{22}} z_1^{\pi_{12} + \pi_{23}} z_2^{\pi_{13}} z_3^{\pi_{14}}$$

$$= F(z_{-2}z_{-1}z_0z_1z_2z_3; q, t) F(z_0z_1z_2z_3; q, t) F(z_1z_2z_3; q, t)$$

$$\times F(z_3; q, t) F(z_{-2}z_{-1}z_0z_1; q, t) F(z_0z_1; q, t) F(z_1; q, t) F(z_{-2}; q, t).$$
Current situation

1. If $P$ is (1) Shape or (2) Shifted Shape, the $(q, t)$-hook formula is proven in the paper by Okada (2010).

2. If $P$ is (3) Bird or (6) Banner, the $(q, t)$-hook formula is proven by me (not yet published) 2013. We use Gasper’s identity.

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The Cauchy type identity for Macdonald polynomials

Theorem

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are $n$-tuples of variables. Then we have

$$
\sum_{\lambda} P_{\lambda}(x; q, t) Q_{\lambda}(y; q, t) = \prod_{i,j=1}^{n} F(x_iy_j; q, t).
$$
Warnaar’s formula

**Theorem (Warnaar ’06)**

\[ \sum_{\lambda} w^{r(\lambda)} \mathcal{H}_{\lambda}(q, t) \mathcal{P}_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + wx_i)(qtx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{i < j} \frac{(tx_ix_j; q)_\infty}{(x_ix_j; q)_\infty}, \]

where \( r(\lambda) \) is the number of rows of odd length.
Warnaar’s formula

**Theorem (Warnaar ’06)**

\[
\sum_{\lambda} w^{r(\lambda)} b_{\lambda}^{oa}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + wx_i)(qtx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty},
\]

where \( r(\lambda) \) is the number of rows of odd length.

**Corollary**

\[
\sum_{\lambda} w^{r(\lambda')} b_{\lambda}^{el}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(twx_i; q)_\infty}{(wx_i; q)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty}.
\]

**Proof.** Applying the \( F \)-algebra homomorphism \( w_{q,t} \) to the above identity.

**Further Corollary**

\[
\sum_{\lambda} w^{r(\lambda')} b_{\lambda}^{el}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(twx_i; q)_\infty}{(wx_i; q)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty}.
\]

Masao Ishikawa

\((q, t)\)-hook formula for Tailed Insets
Theorem (Warnaar ’06)

\[ \sum_{\lambda} w^{r(\lambda)} b^{oa}_{\lambda}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + wx_i)(qt x_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty}, \]

where \( r(\lambda) \) is the number of rows of odd length.

Corollary

\[ \sum_{\lambda} w^{r(\lambda')} b^{el}_{\lambda}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(tw x_i; q)_\infty}{(wx_i; q)_\infty} \prod_{i < j} \frac{(tx_i x_j; q)_\infty}{(x_i x_j; q)_\infty}. \]

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where \(r(\lambda)\) is the number of rows of odd length.

**Further Corollary**

\[
\sum_{\lambda} w^{\frac{|\lambda| + r(\lambda')}{2}} b^{el}_{\lambda}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(twx_i; q)_{\infty}}{(wx_i; q)_{\infty}} \prod_{i < j} \frac{(twx_ix_j; q)_{\infty}}{(wx_ix_j; q)_{\infty}},
\]

\[
\sum_{\lambda} w^{\frac{|\lambda| - r(\lambda')}{2}} b^{el}_{\lambda}(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(tx_i; q)_{\infty}}{(x_i; q)_{\infty}} \prod_{i < j} \frac{(twx_ix_j; q)_{\infty}}{(wx_ix_j; q)_{\infty}}.
\]
$d$-complete poset
The $d$-complete posets arise from the dominant minuscule heaps of the Weyl groups of simply-laced Kac-Moody Lie algebras.

Proctor gave completely combinatorial description of $d$-complete poset, which is a graded poset with $d$-complete coloring.

Proctor showed that any $d$-complete poset can be obtained from the 15 irreducible classes by slant-sum.

The $d$-complete coloring is important for the multivariate generating function. The content should be replaced by color for $d$-complete posets.

Okada’s $(q, t)$-weight $W_P(\pi; q, t)$

Hook monomials for $d$-complete posets
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Hook monomials for $d$-complete posets
**Contents of this section**

1. The \( d \)-complete posets arise from the dominant minuscule heaps of the Weyl groups of simply-laced Kac-Moody Lie algebras.

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6. Hook monomials for \( d \)-complete posets
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Okada’s $(q, t)$-weight $W_P(\pi; q, t)$

Hook monomials for $d$-complete posets
Definition

The double-tailed diamond poset $d_k(1)$ is the poset depicted below:

A $d_k$-interval is an interval isomorphic to $d_k(1)$.

A $d_k^-$-interval ($k \geq 4$) is an interval isomorphic to $d_k(1) - \{\text{top}\}$.

A $d_3^-$-interval consists of three elements $x$, $y$ and $w$ such that $w$ is covered by $x$ and $y$. 

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
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\[\text{top} \quad k - 2 \quad \text{side} \quad \text{side} \quad k - 2 \quad \text{bottom}\]
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Masao Ishikawa $(q, t)$-hook formula for Tailed Insets
Definition of $d$-complete poset

A poset $P$ is $d$-complete if it satisfies the following three conditions for every $k \geq 3$:

1. If $I$ is a $d_k^-$-interval, then there exists an element $v$ such that $v$ covers the maximal elements of $I$ and $I \cup \{v\}$ is a $d_k$-interval.

2. If $I = [w, v]$ is a $d_k$-interval and the top $v$ covers $u$ in $P$, then $u \in I$.

3. There are no $d_k^-$-intervals which differ only in the minimal elements.
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Properties of \( d \)-complete posets

Fact

If \( P \) is a connected \( d \)-complete poset, then

(a) \( P \) has a unique maximal element.

(b) \( P \) is graded, i.e., there exists a rank function \( r : P \to \mathbb{N} \) such that \( r(x) = r(y) + 1 \) if \( x \) covers \( y \).

Fact

(a) Any connected \( d \)-complete poset is uniquely decomposed into a slant sum of one-element posets and slant-irreducible \( d \)-complete posets.

## Properties of $d$-complete posets

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Examples

- rooted tree
- shape
- shifted shape
- swivel

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\((q, t)\)-hook formula for Tailed Insets
15 irreducible d-complete posets

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(q, t)-hook formula for Tailed Insets
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(9)

(10)

(11)
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(q, t)-hook formula for Tailed Insets
Top Tree and \(d\)-Complete Coloring

**Definition**

For a connected \(d\)-complete poset \(P\), we define its top tree by putting

\[
T = \{ x \in P : \text{ every } y \geq x \text{ is covered by at most one other element } \}
\]

**Fact**

Let \(I\) be a set of colors such that \(#I = \#T\). Then a bijection \(c : T \rightarrow I\) can be uniquely extended to a map \(c : P \rightarrow I\) satisfying the following three conditions:

1. If \(x\) and \(y\) are incomparable, then \(c(x) \neq c(y)\).
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Example

Top Tree and $d$-Complete Coloring of $d_5$-interval

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$(q, t)$-hook formula for Tailed Insets
Example

Top Tree and $d$-Complete Coloring of $d_5$-interval

Masao Ishikawa

$(q,t)$-hook formula for Tailed Insets
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Top Tree and $d$-Complete Coloring of $d_5$-interval

\[
\begin{array}{c}
\text{Top Tree and } d\text{-Complete Coloring of } d_5\text{-interval} \\
\end{array}
\]
Let $r$ be a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ be strict partitions such that

$$\alpha_1 > \cdots > \alpha_r \geq 0, \quad \beta_1 > \cdots > \beta_r \geq 0,$$

Let $P$ be the set $P = P_L \cup P_R$ of lattice points in $\mathbb{Z}^2$, where

$$P_R = \{(i, j) : 1 \leq i \leq j \leq \alpha_i + i - 1 \ (1 \leq i \leq r)\},$$

$$P_L = \{(i, j) : 1 \leq j \leq i \leq \beta_j + j - 1 \ (1 \leq j \leq r)\},$$

We regard $P$ as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$ 

We call this poset a \textit{shape} and denote it by $P = P_1(\alpha, \beta)$. 
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We call this poset a shape and denote it by $P = P_1(\alpha, \beta)$. 
Shapes

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
Shifted Shapes

Definition

Let $r$ be a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a strict partition such that

$$\alpha_1 > \cdots > \alpha_r \geq 0.$$ 

Define the \textit{shifted shape} $P = P_2(\alpha)$ by

$$P = \{(i, j) : i \leq j \leq \alpha_i + i - 1 \ (1 \leq i \leq r)\}.$$ 

We regard it as a poset by defining its order structure

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$
**Definition**

Let $r$ be a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a strict partition such that

$$\alpha_1 > \cdots > \alpha_r \geq 0.$$ 

Define the *shifted shape* $P = P_2(\alpha)$ by

$$P = \{ (i, j) : i \leq j \leq \alpha_i + i - 1 \ (1 \leq i \leq r) \}. $$

We regard it as a poset by defining its order structure

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$
Shifted Shapes

Definition

Let $r$ be a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a strict partition such that

$$\alpha_1 > \cdots > \alpha_r \geq 0.$$ 

Define the \textit{shifted shape} $P = P_2(\alpha)$ by

$$P = \{(i, j) : i \leq j \leq \alpha_i + i - 1 \ (1 \leq i \leq r)\}.$$ 

We regard it as a poset by defining its order structure

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$
Shifted Shape

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Define the \textit{bird} $P = P_3(\alpha, \beta; f)$ by

$$P = P_H \cup P_R \cup P_L \cup P_T$$

where

$$P_H = \{(1, j) : -f + 1 \leq j \leq 1\},$$

$$P_R = \{(i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2)\},$$

$$P_L = \{(i, j) : j \leq \beta_j + j - 1 \ (j = 1, 2)\},$$

$$P_T = \{(i, i) : 2 \leq i \leq f + 2\}$$

as a set and we regard it as a poset by defining its order structure

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$
**Definition**

Let \( \alpha = (\alpha_1, \alpha_2) \) and \( \beta = (\beta_1, \beta_2) \) be strict partitions such that \( \alpha_1 > \alpha_2 > 0 \) and \( \beta_1 > \beta_2 > 0 \). Define the *bird* \( P = P_3(\alpha, \beta; f) \) by

\[
P = P_H \cup P_R \cup P_L \cup P_T
\]

where

\[
P_H = \{(1, j) : -f + 1 \leq j \leq 1\},
\]
\[
P_R = \{(i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2)\},
\]
\[
P_L = \{(i, j) : j \leq i \leq \beta_j + j - 1 \ (j = 1, 2)\},
\]
\[
P_T = \{(i, i) : 2 \leq i \leq f + 2\}
\]

as a set and we regard it as a poset by defining its order structure

\[
(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.
\]

if and only if the both of \((i_1, j_1)\) and \((i_2, j_2)\) are in \(P_H \cup P_R \cup P_L\) or in \(P_T\).
Definition

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Define the bird $P = P_3(\alpha, \beta; f)$ by

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$$P_H = \{(1, j) : -f + 1 \leq j \leq 1\},$$
$$P_R = \{(i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2)\},$$
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$$P_T = \{(i, i) : 2 \leq i \leq f + 2\}$$

as a set and we regard it as a poset by defining its order structure

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$
Birds

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
Tailed Insets

Definition

Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2) \) be strict partitions such that

\[
\alpha_1 > \alpha_2 > \alpha_3 \geq 0, \quad \beta_1 > \beta_2 \geq 0.
\]

Let \( P \) be the set \( P = P_H \cup P_M \cup P_L \cup P_R \cup P_T \) of lattice points in \( \mathbb{Z}^2 \), where \( P_M = \{ (2, 1) \} \), \( P_T = \{ (4, 4) \} \) and

\[
\begin{align*}
P_H &= \{ (1, j) : -\beta_1 + 1 \leq j \leq 0 \}, \\
P_R &= \{ (i, j) : 1 \leq i \leq j \leq \alpha_i + i \ (i = 1, 2, 3) \}, \\
P_L &= \{ (i + 1, j + 1) : 1 \leq j \leq i \leq \beta_j + j \ (j = 1, 2) \}.
\end{align*}
\]

We regard \( P \) as a poset by defining the order relation

\[
(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.
\]

if neither of \((i_1, j_1)\) and \((i_2, j_2)\) is not in \( P_T \), whereas \((3, 3) < (4, 4)\). We call this poset a \textit{Tailed Inset}, denoted by \( P_5(\alpha, \beta) \).
Tailed Insets

Definition

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that

$$\alpha_1 > \alpha_2 > \alpha_3 \geq 0, \quad \beta_1 > \beta_2 \geq 0.$$ 

Let $P$ be the set $P = P_H \cup P_M \cup P_L \cup P_R \cup P_T$ of lattice points in $\mathbb{Z}^2$, where $P_M = \{(2, 1)\}$, $P_T = \{(4, 4)\}$ and

- $P_H = \{(1, j) : -\beta_1 + 1 \leq j \leq 0\}$,
- $P_R = \{(i, j) : 1 \leq i \leq j \leq \alpha_i + i \ (i = 1, 2, 3)\}$,
- $P_L = \{(i + 1, j + 1) : 1 \leq j \leq i \leq \beta_j + j \ (j = 1, 2)\}$.

We regard $P$ as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$ 

if neither of $(i_1, j_1)$ and $(i_2, j_2)$ is not in $P_T$, whereas $(3, 3) < (4, 4)$. We call this poset a *Tailed Inset*, denoted by $P_5(\alpha, \beta)$. 

(q, t)-hook formula for Tailed Insets
Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that

$$\alpha_1 > \alpha_2 > \alpha_3 \geq 0, \quad \beta_1 > \beta_2 \geq 0.$$  

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$$P_H = \{(1, j) : -\beta_1 + 1 \leq j \leq 0\},$$
$$P_R = \{(i, j) : 1 \leq i \leq j \leq \alpha_i + i (i = 1, 2, 3)\},$$
$$P_L = \{(i + 1, j + 1) : 1 \leq j \leq i \leq \beta_j + j (j = 1, 2)\}.$$  

We regard $P$ as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$  

if neither of $(i_1, j_1)$ and $(i_2, j_2)$ is not in $P_T$, whereas $(3, 3) < (4, 4)$. We call this poset a **Tailed Inset**, denoted by $P_5(\alpha, \beta)$. 
Tailed Insets

(1, 1)
Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) be a strict partition such that \( \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 0 \), and let \( f \geq 2 \) be a positive integer. Let \( P \) be the set \( P = P_H \cup P_W \cup P_T \) of lattice points in \( \mathbb{Z}^2 \), where

\[
\begin{align*}
P_H &= \{ (1, j) : -f + 2 \leq j \leq 1 \}, \\
P_W &= \{ (i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2, 3, 4) \}, \\
P_T &= \{ (i, 3) : 3 \leq i \leq f + 2 \}.
\end{align*}
\]

We regard \( P \) as a poset by defining the order relation \((i_1, j_1) \preceq (i_2, j_2) \iff i_1 \leq i_2 \) and \( j_1 \leq j_2 \). If both of \((i_1, j_1)\) and \((i_2, j_2)\) are in \( P_H \cup P_W \) or in \( P_T \), and call it a banner.
Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a strict partition such that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 0$, and let $f \geq 2$ be a positive integer. Let $P$ be the set $P = P_H \cup P_W \cup P_T$ of lattice points in $\mathbb{Z}^2$, where

$$
P_H = \{ (1, j) : -f + 2 \leq j \leq 1 \},$$

$$
P_W = \{ (i, j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2, 3, 4) \},$$

$$
P_T = \{ (i, 3) : 3 \leq i \leq f + 2 \}.$$

We regard $P$ as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$ 

if both of $(i_1, j_1)$ and $(i_2, j_2)$ are in $P_H \cup P_W$ or in $P_T$, and call it a banner.
Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a strict partition such that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 0$, and let $f \geq 2$ be a positive integer. Let $P$ be the set $P = P_H \cup P_W \cup P_T$ of lattice points in $\mathbb{Z}^2$, where

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P_T &= \{ (i,3) : 3 \leq i \leq f + 2 \}.
\end{align*}
\]

We regard $P$ as a poset by defining the order relation

\[(i_1,j_1) \geq (i_2,j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.\]

if both of $(i_1,j_1)$ and $(i_2,j_2)$ are in $P_H \cup P_W$ or in $P_T$, and call it a banner.
Masao Ishikawa \((q,t)\)-hook formula for Tailed Insets
Hook Monomials

**Definition**

Let $P$ be a connected $d$-complete poset and $T$ its top tree. Let $z_v$ ($v \in T$) be indeterminate. Let $c : P \rightarrow T$ be the $d$-complete coloring. For each $v \in P$, we define monomials $z[H_P(v)]$ by induction as follows:

(a) If $v$ is not the top of any $d_k$-interval, then we define

$$z[H_P(v)] = \prod_{w \leq v} z_{c(w)}.$$

(b) If $v$ is the top of a $d_k$-interval $[w, v]$, then we define

$$z[H_P(v)] = \frac{z[H_P(x)] z[H_P(y)]}{z[H_P(w)]}$$

where $x$ and $y$ are the sides of $[w, v]$. 
**Definition**

Let $P$ be a connected $d$-complete poset and $T$ its top tree. Let $z_v$ ($v \in T$) be indeterminate. Let $c : P \rightarrow T$ be the $d$-complete coloring. For each $v \in P$, we define monomials $z[H_P(v)]$ by induction as follows:

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$$z[H_P(v)] = \frac{z[H_P(x)] z[H_P(Y)]}{z[H_P(w)]}$$

where $x$ and $y$ are the sides of $[w, v]$. 

Masao Ishikawa 

(q, t)-hook formula for Tailed Insets
Hook Monomials

**Definition**

Let $P$ be a connected $d$-complete poset and $T$ its top tree. Let $z_v$ ($v \in T$) be indeterminate. Let $c : P \to T$ be the $d$-complete coloring. For each $v \in P$, we define monomials $z[H_P(v)]$ by induction as follows:

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$$z[H_P(v)] = \frac{z[H_P(x)] z[H_P(Y)]}{z[H_P(w)]}$$

where $x$ and $y$ are the sides of $[w, v]$. 
An example of hook monomials

Example

We consider the following poset $P = d_1(5)$. We give the following assignment of variables for $z_c(x)$, $x \in P$. 

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$(q, t)$-hook formula for Tailed Insets
Example

We consider the following poset $P = d_1(5)$. We give the following assignment of variables for $z_{c(x)}$, $x \in P$. 

Masao Ishikawa  
$(q, t)$-hook formula for Tailed Insets
Hook Monomials for $P = d_1(5)$

Example

We consider the following poset $P = d_1(5)$. The monomials associated to hooks of $P = d_1(5)$ are as follows:

$z_1 z_2^2 z_3^2 z_4 z_5$  $z_1 z_2 z_3 z_4 z_5$  $z_1 z_2 z_3 z_4$

$z_1 z_2 z_3^2 z_4 z_5$

$z_1 z_2 z_3 z_5$  $z_1 z_2 z_3$

$z_1 z_2$

$z_1$
Hook Monomials for $P = d_1(5)$

Example

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$$
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Hook Monomials for $P = d_1(5)$

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$z_1 z_2^2 z_3^2 z_4 z_5$  $z_1 z_2 z_3 z_4 z_5$

$z_1 z_2 z_3^2 z_4 z_5$

$z_1 z_2 z_3 z_5$

$z_1 z_2 z_3$

$z_1 z_2$

$z_1$
Hook Monomials for $P = d_1(5)$

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We consider the following poset $P = d_1(5)$. The monomials associated to hooks of $P = d_1(5)$ are as follows:

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$$Z_1 Z_2 Z_3 Z_4 Z_5$$

$$Z_1 Z_2 Z_3 Z_4$$

$$Z_1 Z_2 Z_3 Z_5$$

$$Z_1 Z_2 Z_3$$

$$Z_1 Z_2$$

$$Z_1$$
Hook Monomials for $P = d_1(5)$

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$$Z_1Z_2Z_3Z_4Z_5$$

$$Z_1Z_2Z_3^2Z_4Z_5$$

$$Z_1Z_2Z_3Z_4Z_5$$

$$Z_1Z_2Z_3$$

$$Z_1Z_2$$

$$Z_1$$
Hook Monomials for $P = d_1(5)$

Example

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$$z_1 z_2 z_3^2 z_4 z_5$$

$$z_1 z_2 z_3 z_5$$

$$z_1 z_2 z_3$$

$$z_1 z_2$$

$$z_1$$
(q, t)-Weight associated with $P$-partition $\pi$

**Definition**

Let $P$ be a connected $d$-complete poset with top tree $T$. Given a $P$-partition $\pi \in \mathcal{A}(P)$, we define $W_P(\pi; q, t)$ by

$$
\prod_{x, y \in \bar{P}, x < y \atop c(x) \text{ and } c(y) \text{ are adjacent in } \bar{T}} f\left(\widehat{\pi}(x) - \widehat{\pi}(y), \left\lfloor \frac{\widehat{r}(y) - \widehat{r}(x)}{2} \right\rfloor \right)
$$

$$
\prod_{x, y \in P, x < y \atop c(x) = c(y)} f\left(\frac{r(y) - r(x)}{2} \right) f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor - 1 \right).
$$

**Example**

Compute this weight $W_P(\pi; q, t)$ for $P = d_1(5)$.  

---

Masao Ishikawa

(q, t)-hook formula for Tailed Insets
Let $P$ be a connected $d$-complete poset with top tree $T$. Given a $P$-partition $\pi \in \mathcal{A}(P)$, we define $W_P(\pi; q, t)$ by

$$
\prod_{x, y \in P, x < y} f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor \right)
\prod_{x, y \in P, x < y, c(x) = c(y)} f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor \right) f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor - 1 \right).
$$

**Example**

Compute this weight $W_P(\pi; q, t)$ for $P = d_1(5)$. 

$$(q, t)$$-Weight associated with $P$-partition $\pi$
We consider the following poset $P = d_5(1)$. A $P$-partition $\pi$ must satisfy the following inequalities:

\[
\begin{align*}
\pi_{11} & \leq \pi_{12} \leq \pi_{13} \leq \pi_{14} \\
\land \\
\pi_{23} & \leq \pi_{24} \\
\land \\
\pi_{34} \\
\land \\
\pi_{44}
\end{align*}
\]
We consider the following poset $P = d_5(1)$. A $P$-partition $\pi$ must satisfy the following inequalities:

\[
\pi_{11} \leq \pi_{12} \leq \pi_{13} \leq \pi_{14} \\
\pi_{23} \leq \pi_{24} \\
\pi_{34} \\
\pi_{44}
\]
We consider the following poset $P = d_5(1)$. A $P$-partition $\pi$ must satisfy the following inequalities:

\[
\begin{align*}
\pi_{11} & \leq \pi_{12} \leq \pi_{13} \leq \pi_{14} \\
\pi_{23} & \leq \pi_{24} \\
\pi_{34} & \\
\pi_{44} & \end{align*}
\]
Numerator of $W_P(\pi; q, t)$ ($P = d_1(5)$)

Example

A $P$-partition $\pi$ extends to $\widehat{P}$-partition $\widehat{\pi}$.

\[
\text{num} \ W_P(\pi; q, t) = f(\pi_{11}; 0)f(\pi_{44}; 3) \times f(\pi_{12} - \pi_{11}; 0)f(\pi_{34} - \pi_{11}; 2)f(\pi_{44} - \pi_{12}; 2)f(\pi_{44} - \pi_{34}; 0) \\
\times f(\pi_{13} - \pi_{12}; 0)f(\pi_{24} - \pi_{12}; 1)f(\pi_{34} - \pi_{13}; 1)f(\pi_{34} - \pi_{13}; 0) \\
\times f(\pi_{14} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 0)f(\pi_{23} - \pi_{13}; 0)f(\pi_{24} - \pi_{23}; 0)
\]
A $P$-partition $\pi$ extends to $\widehat{P}$-partition $\widehat{\pi}$.

\[
\text{numer } W_P(\pi; q, t) = f(\pi_{11}; 0)f(\pi_{44}; 3) \\
\times f(\pi_{12} - \pi_{11}; 0)f(\pi_{34} - \pi_{11}; 2)f(\pi_{44} - \pi_{12}; 2)f(\pi_{44} - \pi_{34}; 0) \\
\times f(\pi_{13} - \pi_{12}; 0)f(\pi_{24} - \pi_{12}; 1)f(\pi_{34} - \pi_{13}; 1)f(\pi_{34} - \pi_{13}; 0) \\
\times f(\pi_{14} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 0)f(\pi_{23} - \pi_{13}; 0)f(\pi_{24} - \pi_{23}; 0)
\]
Numerator of $\mathcal{W}_P(\pi; q, t) (P = d_1(5))$

Example

A $P$-partition $\pi$ extends to $\widehat{P}$-partition $\widehat{\pi}$.

\[
\text{numer } \mathcal{W}_P(\pi; q, t) = f(\pi_{11}; 0)f(\pi_{44}; 3) \\
\times f(\pi_{12} - \pi_{11}; 0)f(\pi_{34} - \pi_{11}; 2)f(\pi_{44} - \pi_{12}; 2)f(\pi_{44} - \pi_{34}; 0) \\
\times f(\pi_{13} - \pi_{12}; 0)f(\pi_{24} - \pi_{12}; 1)f(\pi_{34} - \pi_{13}; 1)f(\pi_{34} - \pi_{13}; 0) \\
\times f(\pi_{14} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 0)f(\pi_{23} - \pi_{13}; 0)f(\pi_{24} - \pi_{23}; 0)
\]
Denominator of $W_P(\pi; q, t) \ (P = d_1(5))$

Example

A $P$-partition $\pi$

\[
\begin{align*}
\text{denom } W_P(\pi; q, t) &= f(\pi_{44} - \pi_{11}; 2)f(\pi_{44} - \pi_{11}; 3) \\
&\quad \times f(\pi_{34} - \pi_{12}; 1)f(\pi_{34} - \pi_{12}; 2)f(\pi_{24} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 1)
\end{align*}
\]
Denominator of $W_P(\pi; q, t) \ (P = d_1(5))$

Example

A $P$-partition $\pi$

\[
\text{denom } W_P(\pi; q, t) = f(\pi_{44} - \pi_{11}; 2)f(\pi_{44} - \pi_{11}; 3) \times f(\pi_{34} - \pi_{12}; 1)f(\pi_{34} - \pi_{12}; 2)f(\pi_{24} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 1)
\]
Denominator of $W_P(\pi; q, t) \quad (P = d_1(5))$

Example

A $P$-partition $\pi$

\[
\text{denom } W_P(\pi; q, t) = f(\pi_{44} - \pi_{11}; 2)f(\pi_{44} - \pi_{11}; 3) \times f(\pi_{34} - \pi_{12}; 1)f(\pi_{34} - \pi_{12}; 2)f(\pi_{24} - \pi_{13}; 0)f(\pi_{24} - \pi_{13}; 1)
\]
Okada’s (q, t)-hook formula conjecture

Then the following identity would hold for any d-complete posets $P$:

$$\sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t) z^\pi = \prod_{v \in P} F(z[H_P(v)])$$

where $z^\pi = \prod_{x \in P} z^{\pi(x)}_{c(x)}$. Here the sum on the left-hand side runs over all $P$-partitions $\pi \in \mathcal{A}(P)$, and the right-hand side is the product of all hook monomials for $v \in P$. 
Macdonald polynomials
Arm-length and leg-length

Definition

Let $\lambda$ be a partition. Let $s = (i, j)$ be a square in the diagram of $\lambda$, and let $a(s)$ and $l(s)$ be the arm-length and leg-length of $s$, i.e.,

$$a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i.$$ 

Definition

Define

$$b_{\lambda}(q, t) := \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

$$b_{\lambda}^{el}(q, t) := \prod_{s \in \lambda \atop l(s) \text{ even}} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

$$b_{\lambda}^{oa}(q, t) := \prod_{s \in \lambda \atop a(s) \text{ odd}} b_{\lambda}(s; q, t),$$

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$(q, t)$-hook formula for Tailed Insets
Arm-length and leg-length

**Definition**

Let $\lambda$ be a partition. Let $s = (i, j)$ be a square in the diagram of $\lambda$, and let $a(s)$ and $l(s)$ be the arm-length and leg-length of $s$, i.e.,

$$a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i.$$ 

**Definition**

Define

$$b_\lambda(q, t) := \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

$$b_{\lambda}^{el}(q, t) := \prod_{\begin{smallmatrix} s \in \lambda \\ l(s) \text{ even} \end{smallmatrix}} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

$$b_{\lambda}^{oa}(q, t) := \prod_{\begin{smallmatrix} s \in \lambda \\ a(s) \text{ odd} \end{smallmatrix}} b_\lambda(s; q, t).$$
Monomial symmetric function

**Definition**

If \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are two sequences of independent indeterminates, then we write

\[
\Pi(x; y; q, t) = \prod_{i,j} \frac{(tx_iy_j; q)\infty}{(x_iy_j; q)\infty} = \prod_{i,j} F(x_iy_j; q, t).
\]

**Definition**

Let \( \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \) and \( \Lambda \) denote the ring of symmetric polynomials in \( n \) independent variables and the ring of symmetric polynomials in countably many variables, respectively. For \( \lambda = (\lambda_1, \ldots, \lambda_n) \) a partition of at most \( n \) parts the monomial symmetric function \( m_\lambda \) is defined as

\[
m_\lambda(x) := \sum_{\alpha} x^\alpha
\]

where the sum is over all distinct permutations \( \alpha \) of \( \lambda \).
Monomial symmetric function

Definition

If \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are two sequences of independent indeterminates, then we write

\[
\Pi(x; y; q, t) = \prod_{i,j} \left( \frac{(tx_iy_j; q)_{\infty}}{(x_iy_j; q)_{\infty}} \right) = \prod_{i,j} F(x_iy_j; q, t).
\]

Definition

Let \( \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\subseteq_n} \) and \( \Lambda \) denote the ring of symmetric polynomials in \( n \) independent variables and the ring of symmetric polynomials in countably many variables, respectively. For \( \lambda = (\lambda_1, \ldots, \lambda_n) \) a partition of at most \( n \) parts the monomial symmetric function \( m_\lambda \) is defined as

\[
m_\lambda(x) := \sum_{\alpha} x^\alpha
\]

where the sum is over all distinct permutations \( \alpha \) of \( \lambda \).
Macdonald scalar product

Definition

For $r$ a nonnegative integer the power sums $p_r$ are given by $p_0 = 1$ and $p_r = m(r)$ for $r > 1$. More generally the power-sum products are defined as $p_\lambda(x) := p_{\lambda_1}(x)p_{\lambda_2}(x)\cdots$ for an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. Define the Macdonald scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu}z_\lambda \prod_i \prod_{i=1}^n \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

with $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ and $m_i = m_i(\lambda)$. 
Macdonald’s $P$-function

**Definition**

If we denote the ring of symmetric functions in $\Lambda_n$ variables over the field $\mathbb{F} = \mathbb{Q}(q, t)$ of rational functions in $q$ and $t$ by $\Lambda_{n, \mathbb{F}}$, then the Macdonald polynomial $P_\lambda(x) = P_\lambda(x; q, t)$ is the unique symmetric polynomial in $\Lambda_{n, \mathbb{F}}$ such that:

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu}(q, t)m_\mu(x)$$

with $u_{\lambda\lambda} = 1$ and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu.$$ 

The Macdonald polynomials $P_\lambda(x; q, t)$ with $\ell(\lambda) \leq n$ form an $\mathbb{F}$-basis of $\Lambda_{n, \mathbb{F}}$. If $\ell(\lambda) > n$ then $P_\lambda(x; q, t) = 0$. $P_\lambda(x; q, t)$ is called Macdonald’s $P$-function. Since $P_\lambda(x_1, \ldots, x_n, 0; q, t) = P_\lambda(x_1, \ldots, x_n; q, t)$ one can extend the Macdonald polynomials to symmetric functions containing an infinite number of independent variables $x = (x_1, x_2, \ldots)$, to obtain a basis of $\mathbb{F} = \Lambda \otimes \mathbb{F}$. 

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$(q, t)$-hook formula for Tailed Insets
Macdonald’s $Q$-function

Definition

A second Macdonald symmetric function, called *Macdonald’s $Q$-function*, is defined as

$$Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(x; q, t).$$

The normalization of the Macdonald inner product is then

$$\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu}$$

for all $\lambda, \mu$, which is equivalent to

$$\sum_\lambda P_\lambda(x; q, t) Q_\lambda(y; q, t) = \Pi(x; y; q, t).$$
Pieri coefficients $\varphi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$

**Definition**

Let $r$ be a positive integer, and let $\lambda, \mu$ be partitions such that $\lambda \supset \mu$ and $\lambda - \mu$ is a horizontal $r$-strip. The *Pieri coefficients* $\varphi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ are defined by

\[
P_{\mu} g_r = \sum_{\lambda} \varphi_{\lambda/\mu} P_{\lambda},
\]

\[
Q_{\mu} g_r = \sum_{\lambda} \psi_{\lambda/\mu} Q_{\lambda},
\]

where $g_r = Q_{(r)}$. 

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(q, t)-hook formula for Tailed Insets
Another direct expression for $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$

Proposition

From Macdonald’s book Chap.VI, §6, Ex.2(c), we have

$$
\varphi_{\lambda/\mu}(q, t) = \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{f(\lambda_i - \mu_j; j - i)f(\mu_i - \lambda_{j+1}; j - i)}{f(\lambda_i - \lambda_j; j - i)f(\mu_i - \mu_{j+1}; j - i)},
$$

$$
\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i < j \leq \ell(\mu)} \frac{f(\lambda_i - \mu_j; j - i)f(\mu_i - \lambda_{j+1}; j - i)}{f(\mu_i - \mu_j; j - i)f(\lambda_i - \lambda_{j+1}; j - i)}.
$$
Macdonald’s skew $Q$-function and skew $P$-function

**Definition**

For any three partitions $\lambda$, $\mu$, $\nu$ let $f_{\mu\nu}^{\lambda}$ be the coefficient $P_{\lambda}$ in the product $P_{\mu}P_{\nu}$:

$$P_{\mu}(x; q, t)P_{\nu}(x; q, t) = \sum_{\lambda} f_{\mu\nu}^{\lambda} P_{\lambda}(x; q, t)$$

Now let $\lambda$, $\mu$ be partitions and define $Q_{\lambda/\mu} \in \Lambda_F$ by

$$Q_{\lambda/\mu}(x; q, t) = \sum_{\nu} f_{\mu\nu}^{\lambda} Q_{\nu}(x; q, t).$$

Then $Q_{\lambda/\mu}(x; q, t) = 0$ unless $\lambda \supset \mu$, and $Q_{\lambda/\mu}$ is homogeneous of degree $|\lambda| - |\mu|$, which is called Macdonald’s skew $Q$-function.

**Definition**

We define Macdonald’s skew $P$-function $P_{\lambda/\mu}$ by

$$Q_{\lambda/\mu}(x; q, t) = \frac{b_{\lambda}(q, t)}{b_{\mu}(q, t)} P_{\lambda/\mu}(x; q, t).$$

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(q, t)-hook formula for Tailed Insets
Macdonald’s skew $Q$-function and skew $P$-function

Definition

For any three partitions $\lambda, \mu, \nu$ let $f_{\mu\nu}^{\lambda}$ be the coefficient $P_\lambda$ in the product $P_\mu P_\nu$:

$$P_\mu(x; q, t) P_\nu(x; q, t) = \sum_\lambda f_{\mu\nu}^{\lambda} P_\lambda(x; q, t)$$

Now let $\lambda, \mu$ be partitions and define $Q_{\lambda/\mu} \in \Lambda_F$ by

$$Q_{\lambda/\mu}(x; q, t) = \sum_\nu f_{\mu\nu}^{\lambda} Q_\nu(x; q, t).$$

Then $Q_{\lambda/\mu}(x; q, t) = 0$ unless $\lambda \supset \mu$, and $Q_{\lambda/\mu}$ is homogeneous of degree $|\lambda| - |\mu|$, which is called Macdonald’s skew $Q$-function.

Definition

We define Macdonald’s skew $P$-function $P_{\lambda/\mu}$ by

$$Q_{\lambda/\mu}(x; q, t) = \frac{b_{\lambda}(q, t)}{b_{\mu}(q, t)} P_{\lambda/\mu}(x; q, t).$$
Lemma

Let $\mu$ and $\nu$ be partitions, and $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are independent indeterminates.

$$\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t) = \prod(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)$$

Proof.

$$\sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w)$$

$$= \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w)$$

$$= \prod(x, z; y, w)$$

$$= \prod(x; y) \prod(x; w) \prod(z; y) \prod(z; w)$$
Lemma

Let \( \mu \) and \( \nu \) be partitions, and \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are independent indeterminates.

\[
\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t)
= \prod(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)
\]

Proof.

\[
\sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\nu}(z) P_{\nu}(w)
= \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w)
= \prod(x, z; y, w)
= \prod(x; y) \prod(x; w) \prod(z; y) \prod(z; w)
\]
Lemma

Let \( \mu \) and \( \nu \) be partitions, and \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are independent indeterminates.

\[
\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t) = \prod(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)
\]

Proof.

\[
\sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w) = \prod(x, z; y, w) = \prod(x; y) \prod(x; w) \prod(z; y) \prod(z; w)
\]
Lemma

Let $\mu$ and $\nu$ be partitions, and $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are independent indeterminates.

\[
\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t)
= \prod(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)
\]

Proof.

\[
\sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w)
= \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w)
= \prod(x, z; y, w)
= \prod(x; y) \prod(x; w) \prod(z; y) \prod(z; w)
\]
Lemma

Let $\mu$ and $\nu$ be partitions, and $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are independent indeterminates.

$$\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t)$$

$$= \prod(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)$$

Proof.

$$\sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w)$$

$$= \sum_{\mu, \nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w)$$

$$= \prod(x, z; y, w)$$

$$= \prod(x; y) \prod(x; w) \prod(z; y) \prod(z; w)$$
Proof

\[
= \prod(x; y) \sum_{\xi} Q_{\xi}(x) P_{\xi}(w) \sum_{\eta} Q_{\eta}(z) P_{\eta}(y) \sum_{\tau} Q_{\tau}(z) P_{\tau}(w)
\]

\[
= \prod(x; y) \sum_{\xi, \eta, \tau} Q_{\xi}(x) P_{\eta}(y) \sum_{\mu} \frac{b_\eta b_\tau}{b_\mu} f_{\eta \tau}^\mu Q_{\mu}(z) \sum_{\nu} f_{\xi \tau}^\nu P_{\nu}(w)
\]

\[
= \prod(x; y) \sum_{\mu, \nu, \tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_{\mu}(z) P_{\nu}(w)
\]

Hence, by comparing the coefficients of \( Q_{\mu}(z) P_{\nu}(w) \) in the both sides, we obtain the desired identity. This completes the proof.
Proof

\[ \Pi(x; y) \sum_{\xi} Q_{\xi}(x) P_{\xi}(w) \sum_{\eta} Q_{\eta}(z) P_{\eta}(y) \sum_{\tau} Q_{\tau}(z) P_{\tau}(w) \]

\[ = \Pi(x; y) \sum_{\xi,\eta,\tau} Q_{\xi}(x) P_{\eta}(y) \sum_{\mu} \frac{b_{\eta} b_{\tau}}{b_{\mu}} f_{\eta\tau}^{\mu} Q_{\mu}(z) \sum_{\nu} f_{\xi\tau}^{\nu} P_{\nu}(w) \]

\[ = \Pi(x; y) \sum_{\mu,\nu,\tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_{\mu}(z) P_{\nu}(w) \]

Hence, by comparing the coefficients of \( Q_{\mu}(z) P_{\nu}(w) \) in the both sides, we obtain the desired identity. This completes the proof.
Proof

\[ \prod(x; y) \sum_{\xi} Q_\xi(x) P_\xi(w) \sum_{\eta} Q_\eta(z) P_\eta(y) \sum_{\tau} Q_\tau(z) P_\tau(w) \]

\[ \prod(x; y) \sum_{\xi, \eta, \tau} Q_\xi(x) P_\eta(y) \sum_{\mu} \frac{b_\eta b_\tau}{b_\mu} f_{\eta \tau}^\mu Q_\mu(z) \sum_{\nu} f_{\xi \tau}^\nu P_\nu(w) \]

\[ \prod(x; y) \sum_{\mu, \nu, \tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_\mu(z) P_\nu(w) \]

Hence, by comparing the coefficients of \( Q_\mu(z) P_\nu(w) \) in the both sides, we obtain the desired identity. This completes the proof.
Proof

\[
= \Pi(x; y) \sum_{\xi} Q_\xi(x) P_\xi(w) \sum_{\eta} Q_\eta(z) P_\eta(y) \sum_{\tau} Q_\tau(z) P_\tau(w)
\]

\[
= \Pi(x; y) \sum_{\xi, \eta, \tau} Q_\xi(x) P_\eta(y) \sum_{\mu} \frac{b_\eta b_\tau}{b_\mu} f^\mu_{\eta \tau} Q_\mu(z) \sum_{\nu} f^\nu_{\xi \tau} P_\nu(w)
\]

\[
= \Pi(x; y) \sum_{\mu, \nu, \tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_\mu(z) P_\nu(w)
\]

Hence, by comparing the coefficients of \( Q_\mu(z) P_\nu(w) \) in the both sides, we obtain the desired identity. This completes the proof.
A generalization of Vuletić’s formula

**Theorem**

Fix a positive integer $T$ and two partitions $\mu^0$ and $\mu^T$. Let $x^0, \ldots, x^{T-1}, y^1, \ldots, y^T$ be sets of variables. Then we have

$$\sum_{(\lambda^1, \mu^1, \ldots, \lambda^T)} \prod_{i=1}^{T} Q_{\lambda^i/\mu^{i-1}}(x^{i-1}; q, t) P_{\lambda^i/\mu^i}(y^i; q, t)$$

$$= \prod_{0 \leq i < j \leq T} \prod(x^i, y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \ldots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \ldots, y^T; q, t)$$

where the sum runs over $(2T-1)$-tuples $(\lambda^1, \mu^1, \lambda^2, \ldots, \mu^{T-1}, \lambda^T)$ of partitions satisfying

$$\mu^0 \subset \lambda^1 \subset \mu^1 \subset \lambda^2 \subset \mu^2 \subset \ldots \subset \mu^{T-1} \subset \lambda^T \supset \mu^T.$$  

**Proof.** Use induction and the above lemma.
A generalization of Vuletić’s formula

Theorem

Fix a positive integer $T$ and two partitions $\mu^0$ and $\mu^T$. Let $x^0, \ldots, x^{T-1}$, $y^1, \ldots, y^T$ be sets of variables. Then we have

$$\sum_{(\lambda^1, \mu^1, \lambda^2, \ldots, \lambda^T)} \prod_{i=1}^{T} Q_{\lambda^i/\mu^{i-1}}(x^{i-1}; q, t) P_{\lambda^i/\mu^i}(y^i; q, t)$$

$$= \prod_{0 \leq i < j \leq T} \prod (x^i, y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \ldots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \ldots, y^T; q, t)$$

where the sum runs over $(2T-1)$-tuples $(\lambda^1, \mu^1, \lambda^2, \ldots, \mu^{T-1}, \lambda^T)$ of partitions satisfying

$$\mu^0 \subset \lambda^1 \subset \mu^1 \subset \lambda^2 \subset \mu^2 \subset \cdots \subset \mu^{T-1} \subset \lambda^T \subset \mu^T.$$ 

Proof. Use induction and the above lemma.
A corollary

Definition

We define $P_{[\lambda,\mu]}^\varepsilon(x; q, t)$ and $Q_{[\lambda,\mu]}^\varepsilon(x; q, t)$ for a pair $(\lambda, \mu)$ of partitions, a set $x = (x_1, x_2, \ldots)$ of independent variables and $\varepsilon = \pm$ by

\[
P_{[\lambda,\mu]}^\varepsilon(x; q, t) = \begin{cases} 
P_{\lambda/\mu}(x; q, t) & \text{if } \varepsilon = +, \\ 
Q_{\mu/\lambda}(x; q, t) & \text{if } \varepsilon = -, 
\end{cases}
\]

\[
Q_{[\lambda,\mu]}^\varepsilon(q, t) = \begin{cases} 
Q_{\lambda/\mu}(x; q, t) & \text{if } \varepsilon = +, \\ 
P_{\mu/\lambda}(x; q, t) & \text{if } \varepsilon = -.
\end{cases}
\]

Here we assume $\lambda \supset \mu$ if $\varepsilon = +$, and $\lambda \subset \mu$ if $\varepsilon = -$. 
A corollary

**Theorem**

Let $n$ be a positive integer, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ a sequence of $\pm$. Fix a positive integer $n$ and two partitions $\lambda^0$ and $\lambda^n$. Let $x^1, \ldots, x^n$ be sets of variables. Then we have

$$\sum_{(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})} \prod_{i=1}^{n} P_{\lambda^{i-1}, \lambda^i}(x^i; q, t) = \prod_{i<j} \prod_{(\epsilon_i, \epsilon_j) = (-, +)} \prod_{(\epsilon_i, \epsilon_j) = (+, -)} \prod_{X^i; x^j; q, t)}$$

$$\times \sum_{\nu} Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i = -}; q, t) P_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i = +}; q, t),$$

where the sum runs over $(n - 1)$-tuples $(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})$ of partitions satisfying

$$\begin{cases} 
\lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +, \\
\lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -. 
\end{cases}$$

**Proof.** Take $T = n$ and put $X^{i-1} = 0$ and $Y^i = x^i$ if $\epsilon_i = +1$, and $X^{i-1} = x^i$ and $Y^i = 0$ if $\epsilon_i = -1$. 

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$(q, t)$-hook formula for Tailed Insets
Theorem

Let $n$ be a positive integer, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ a sequence of $\pm$. Fix a positive integer $n$ and two partitions $\lambda^0$ and $\lambda^n$. Let $x^1, \ldots, x^n$ be sets of variables. Then we have

$$\sum_{(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})} \prod_{i=1}^{n} P_{[\lambda^{i-1}, \lambda^i]}^\epsilon (x^i; q, t) = \prod_{i<j}^{i<j} \Pi (x^i; x^j; q, t)$$

$$\times \sum_{\nu} Q_{\lambda^n/\nu} (\{x^i\}_{i=1}^{n}; q, t) P_{\lambda^0/\nu} (\{x^i\}_{i=1}^{n}; q, t),$$

where the sum runs over $(n-1)$-tuples $(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})$ of partitions satisfying

$$\begin{cases} \lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +, \\ \lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -. \end{cases}$$

Proof. Take $T = n$ and put $X^{i-1} = 0$ and $Y^i = x^i$ if $\epsilon_i = +1$, and $X^{i-1} = x^i$ and $Y^i = 0$ if $\epsilon_i = -1$. 

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$(q, t)$-hook formula for Tailed Insets
A corollary

Theorem

Let \( n \) be a positive integer, and \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) a sequence of \( \pm \). Fix a positive integer \( n \) and two partitions \( \lambda^0 \) and \( \lambda^n \). Let \( x^1, \ldots, x^n \) be sets of variables. Then we have

\[
\sum_{(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})} \prod_{i=1}^{n} P_{[\lambda^{i-1}, \lambda^i]}(x^i; q, t) = \prod_{i<j}^{n} \Pi(x^i; x^j; q, t)
\times \sum_{\nu} Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i=+}; q, t) P_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i=-}; q, t),
\]

where the sum runs over \((n-1)\)-tuples \((\lambda^1, \lambda^2, \ldots, \lambda^{n-1})\) of partitions satisfying

\[
\begin{cases} 
\lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +, \\
\lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -.
\end{cases}
\]

Proof. Take \( T = n \) and put \( X^{i-1} = 0 \) and \( Y^i = x^i \) if \( \epsilon_i = +1 \), and \( X^{i-1} = x^i \) and \( Y^i = 0 \) if \( \epsilon_i = -1 \).
Notation

Definition

Under the assumption that $\lambda \supseteq \mu$ if $\varepsilon = -$, or $\lambda \subseteq \mu$ if $\varepsilon = +$, we write

$$\Psi^\varepsilon_{\lambda/\mu} = \begin{cases} \psi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \varphi_{\mu/\lambda} & \text{if } \varepsilon = +, \end{cases} \quad \Phi^\varepsilon_{\lambda/\mu} = \begin{cases} \varphi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \psi_{\mu/\lambda} & \text{if } \varepsilon = +. \end{cases}$$

Definition

Here we assume $\lambda > \mu$ if $\delta = +1$, and $\lambda < \mu$ if $\delta = -1$. We also write

$$|\lambda - \mu|_\delta = \begin{cases} |\lambda - \mu| & \text{if } \delta = +1, \\ |\mu - \lambda| & \text{if } \delta = -1. \end{cases}$$
Notation

Definition

Under the assumption that $\lambda \supseteq \mu$ if $\varepsilon = -$, or $\lambda \subseteq \mu$ if $\varepsilon = +$, we write

$$
\Psi^{\varepsilon}_{\lambda/\mu} = \begin{cases} 
\psi_{\lambda/\mu} & \text{if } \varepsilon = -, \\
\phi_{\mu/\lambda} & \text{if } \varepsilon = +,
\end{cases}
\quad
\Phi^{\varepsilon}_{\lambda/\mu} = \begin{cases} 
\varphi_{\lambda/\mu} & \text{if } \varepsilon = -, \\
\psi_{\mu/\lambda} & \text{if } \varepsilon = +.
\end{cases}
$$

Definition

Here we assume $\lambda > \mu$ if $\delta = +1$, and $\lambda < \mu$ if $\delta = -1$. We also write

$$
|\lambda - \mu|_{\delta} = \begin{cases} 
|\lambda - \mu| & \text{if } \delta = +1, \\
|\mu - \lambda| & \text{if } \delta = -1.
\end{cases}
$$
Notation

Definition

Let $n$ be a positive integer. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be a sequence of $\pm 1$. Let $(\lambda^0, \lambda^1, \ldots, \lambda^n)$ be an $(n + 1)$-tuple of partitions such that $\lambda^{i-1} > \lambda^i$ if $\epsilon = +1$, and $\lambda^{i-1} < \lambda^i$ if $\epsilon = -1$. Then we write

$$\phi_\epsilon^\lambda(q, t) = \prod_{i=1}^n \phi_{\lambda^{i-1}, \lambda^i}(q, t),$$
$$\psi_\epsilon^\lambda(q, t) = \prod_{i=1}^n \psi_{\lambda^{i-1}, \lambda^i}(q, t).$$

Definition

Let $\alpha$ be a strict partition, and let $n$ be an integer such that $n \geq \alpha_1$. Define a sequence $\epsilon_n(\alpha) = (\epsilon_1(\alpha), \ldots, \epsilon_n(\alpha))$ of $\pm 1$ by putting

$$\epsilon_k(\alpha) = \begin{cases} +1 & \text{if } k \text{ is a part of } \alpha, \\ -1 & \text{if } k \text{ is not a part of } \alpha. \end{cases}$$
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$$
\phi_{\lambda^0, \lambda^1, \ldots, \lambda^n}^\epsilon(q, t) = \prod_{i=1}^n \phi_{\lambda^{i-1}, \lambda^i}^{\epsilon_i}(q, t), \quad \psi_{\lambda^0, \lambda^1, \ldots, \lambda^n}^\epsilon(q, t) = \prod_{i=1}^n \psi_{\lambda^{i-1}, \lambda^i}^{\epsilon_i}(q, t).
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Example of $\epsilon(\alpha)$ and $k$th trace $\pi[k]$

**Definition**

For each integer $k = 0, \ldots, n$ we define the $k$th trace $\pi[k]$ to be the sequence $(\ldots, \pi_{2,k+2}, \pi_{1,k+1})$ obtained by reading the $k$th diagonal from SE to NW. Here we use the convention that $\pi[k] = \emptyset$ if $k \geq \alpha_1$.

**Example**

For example, if $\alpha = (8, 5, 2, 1)$ and $n = 10$, then we have $\epsilon = (+ + - - + - - + - - + - - + - -)$.

We have $\pi[0] = (\pi_{44}, \pi_{33}, \pi_{22}, \pi_{11})$, $\pi[1] = (\pi_{34}, \pi_{23}, \pi_{12})$, $\ldots$, $\pi[10] = \emptyset$. 
Example of $\epsilon(\alpha)$ and $k$th trace $\pi[k]$

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For each integer $k = 0, \ldots, n$ we define the *$k$th trace $\pi[k]$* to be the sequence $(\ldots, \pi_{2,k+2}, \pi_{1,k+1})$ obtained by reading the $k$th diagonal from SE to NW. Here we use the convention that $\pi[k] = \emptyset$ if $k \geq \alpha_1$.

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Tailed Inset Case

(q, t)-hook formula for Tailed Insets
Tailed Insets

Definition

Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2) \) be strict partitions such that

\[
\alpha_1 > \alpha_2 > \alpha_3 \geq 0, \quad \beta_1 > \beta_2 \geq 0.
\]

Let \( P \) be the set \( P = P_H \cup P_M \cup P_L \cup P_R \cup P_T \) of lattice points in \( \mathbb{Z}^2 \), where \( P_M = \{ (2, 1) \} \), \( P_T = \{ (4, 4) \} \) and

\[
\begin{align*}
P_H &= \{ (1, j) : -\beta_1 + 1 \leq j \leq 0 \}, \\
P_R &= \{ (i, j) : 1 \leq i \leq j \leq \alpha_i + i (i = 1, 2, 3) \}, \\
P_L &= \{ (i + 1, j + 1) : 1 \leq j \leq i \leq \beta_j + j (j = 1, 2) \}.
\end{align*}
\]

We regard \( P \) as a poset by defining the order relation

\[
(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.
\]

if neither of \((i_1, j_1)\) and \((i_2, j_2)\) is not in \( P_T \), whereas \((3, 3) < (4, 4)\). We call this poset a Tailed Inset, denoted by \( P_5(\alpha, \beta) \).
Tailed Insets

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Tailed Insets

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
Tailed Insets

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
Definition

Let $\pi = (\sigma, \tau, \rho, \gamma, \delta) \in A(P)$ be a $P$-partition as in the following figure.
Let $p_i$ (resp. $q_i$) denote the number of vertices in the $i$th diagonal of $\lambda$ (resp. $\mu$) for $i \geq 1$, whereas we set $p_0 = 3$ and $q_0 = 2$. We define $\varepsilon = (\varepsilon_{c,c+1})_{c \in \mathbb{Z}}$ as follows. If $c \geq 1$,

$$\varepsilon_{c,c+1} = \begin{cases} + & \text{if } p_c = p_{c-1}, \\ - & \text{if } p_c = p_{c-1} - 1, \end{cases}$$

and if $c \leq 0$,

$$\varepsilon_{c,c+1} = \begin{cases} - & \text{if } q_{-c+1} = q_{-c}, \\ + & \text{if } q_{-c+1} = q_{-c} - 1. \end{cases}$$

The color of each vertex is shown in the figure above. In this example, we have $(p_i)_{i \geq 1} = (332211100 \ldots)$, $(q_i)_{i \geq 1} = (221100 \ldots)$ and $p_0 = 3$, $q_0 = 2$ by definition. Hence we have $\varepsilon_\lambda = (\cdots - - + - + - - + - + - - + - + - + + \cdots)$ as in the above figure.
If we set

\[ W^c_d(\pi; q, t) = \prod_{x, y \in \mathcal{P}, x < y} f_{q, t} \left( \pi(x) - \pi(y); \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor \right) \]

if \( c \) and \( d \) are adjacent colors in \( \hat{T} \), and

\[ W^c_{D+}(\pi; q, t) = \prod_{x, y \in \mathcal{P}, x < y} f_{q, t} \left( \pi(x) - \pi(y); \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor \right), \]

\[ W^c_{D-}(\pi; q, t) = \prod_{x, y \in \mathcal{P}, x < y} f \left( \pi(x) - \pi(y); \left\lfloor \frac{r(y) - r(x)}{2} \right\rfloor - 1 \right), \]

then we have

\[ W_P(\pi; q, t) = \frac{\prod_{c \text{ and } d \text{ are adjacent in } \hat{T}} W^c_d(\pi; q, t)}{\prod_{c \text{ all colors in } \hat{T}} W^c_D(\pi; q, t)}. \]

where

\[ W^c_D(\pi; q, t) = W^c_{D+}(\pi; q, t) W^c_{D-}(\pi; q, t). \]
If \( \lambda \) and \( \mu \) are partitions such that \( \lambda - \mu \) is a horizontal strip, then it is known that

\[
\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f_{q, t}(\lambda_i - \mu_j; j - i) f_{q, t}(\mu_i - \lambda_{j+1}; j - i)}{f_{q, t}(\mu_i - \mu_j; j - i) f_{q, t}(\lambda_i - \lambda_{j+1}; j - i)},
\]

\[
\varphi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{f_{q, t}(\lambda_i - \mu_j; j - i) f_{q, t}(\mu_i - \lambda_{j+1}; j - i)}{f_{q, t}(\lambda_i - \lambda_j; j - i) f_{q, t}(\mu_i - \mu_{j+1}; j - i)}.
\]

Under the assumption that \( \lambda \supseteq \mu \) if \( \varepsilon = - \), or \( \lambda \subseteq \mu \) if \( \varepsilon = + \), we write

\[
\psi_{<\lambda/\mu} = \begin{cases} 
\psi_{\lambda/\mu} & \text{if } \varepsilon = - , \\
\varphi_{\mu/\lambda} & \text{if } \varepsilon = + ,
\end{cases}
\]

\[
\Phi_{<\lambda/\mu} = \begin{cases} 
\varphi_{\lambda/\mu} & \text{if } \varepsilon = - , \\
\psi_{\mu/\lambda} & \text{if } \varepsilon = + .
\end{cases}
\]
Definition

I) For $0 \leq c \leq \lambda_1$, we define the partition $\Lambda^c$ of length $\leq p_c$ by

$$\Lambda^c = (\sigma_{p_c+p_c+c}, \ldots, \sigma_{1,1+c}) = (\sigma_{p+c+1-i, p+c+1-i+c})_{1\leq i \leq p_c}.$$ 

II) Now we set

$$\Lambda^{-1} = (\tau_{q_1+1,q_1}, \ldots, \tau_{2,1}, \gamma, \rho_1),$$

where $q_1 = 1$ or 2.

III) If $-\mu_1 \leq c \leq -2$, then we set

$$\Lambda^c = (\tau_{q_c-c,q_c}, \ldots, \tau_{1-c,1}, \gamma, \ldots, \gamma, \rho_c),$$

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(q, t)-weight by Pieri coefficient

**Theorem**

If \( P = P_5(\lambda, \mu) \) is the Tailed Insets corresponding to strict partitions \( \lambda \) and \( \mu \), then we have

\[
W_P(\pi; q, t) = \frac{f_{q,t}(\gamma; 0) \prod_{i=1}^{3} f_{q,t}(\delta - \sigma_{i,i}; 3 - i)}{f_{q,t}(\delta - \gamma; 2) f_{q,t}(\delta - \gamma; 1)} \prod_{c=-\mu_1-1}^{\lambda_1} \psi^{c,c+1}_{\Lambda_c/\Lambda_{c+1}}.
\]

**Proposition**

We set

\[
Z_c = \prod_{k=-\mu_1-1}^{c} z_k, \quad Z_{c,d} = \frac{Z_d}{Z_c} = \prod_{k=c+1}^{d} z_k,
\]

where \( z_{-\mu_1-1} \) is a dummy variable which does not appear in the original weight. Then we have

\[
z^{\pi} = \prod_{c=-\mu_1-1}^{\lambda_1} \frac{Z_c^{\gamma+\delta}}{Z_c^{\gamma}} \cdot \prod_{c=-\mu_1-1}^{\lambda_1} Z_c^{\Lambda_c - \Lambda_{c+1}}.
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z^{\pi} = \frac{Z_{\mu_1+1}^{\gamma+\delta}}{\prod_{c=-\mu_1-1}^{\lambda_1} Z_c^{c}} \cdot \prod_{c=-\mu_1-1}^{\lambda_1} Z_c^{c} Z_{\Lambda_c/\Lambda_{c+1}}.
\]
We use the convention that $\varepsilon_{-\mu_1-1,-\mu_1} = +$ and $\varepsilon_{c,c+1} = -$ for $c < -\mu_1 - 1$. Note that $\#\{c < 0 \mid \varepsilon_{c,c+1} = +\} = 2$. Because $\varepsilon_{-\mu_1-1,-\mu_1} = +$ and $\varepsilon_{c,c+1} = -$ for $c < -\mu_1 - 1$, we may set

$$\{c < 0 \mid \varepsilon_{c,c+1} = +\} = \{c_1^-, c_2^-\}.$$ 

where $-\mu_1 - 1 = c_2^- < c_1^- < 0$ Also note that $\#\{c \geq 0 \mid \varepsilon_{c,c+1} = -\} = 3$. Because $\varepsilon_{\lambda_1,\lambda_1+1} = -$ and $\varepsilon_{c,c+1} = +$ for $c > \lambda_1$, we may set

$$\{c \geq 0 \mid \varepsilon_{c,c+1} = -\} = \{c_1^+, c_2^+, c_3^+\}.$$ 

where $0 \leq c_1^+ < c_2^+ < c_3^+ = \lambda_1$. Hence we have

$$-\mu_1 - 1 = c_2^- < c_1^- < 0 \leq c_1^+ < c_2^+ < c_3^+ = \lambda_1.$$
Theorem

$$\sum W_P(\pi; q, t) z^\pi = \prod_{0 \leq i < j} \prod (Z_i^{-1}; Z_j; q, t) \prod_{i < j < 0} \prod (Z_i^{-1}; Z_j; q, t) \times$$

$$\times \sum_{\sigma_{1,1}, \sigma_{2,2}, \sigma_{3,3} \atop \gamma, \delta, \nu_3} f_{q,t}(\gamma; 0) \prod_{i=1}^{3} f_{q,t}(\delta - \sigma_{i,i}; 3 - i) \frac{f_{q,t}(\delta - \gamma; 2)f_{q,t}(\delta - \gamma; 1)}{f_{q,t}(\delta - \gamma; 2)} . \frac{Z_\gamma^{\gamma+\delta}}{\prod_{c=-\mu_1-1}^{-1} Z_c^\nu}$$

$$\times P_{\Lambda^0}(Z_{c_2}^+, Z_{c_3}^+, Z_{\lambda_1}; q, t) \times Q_{\Lambda^0/\nu}(Z_{-\mu_1-1}^{-1}, Z_{c_i}^{-1}; q, t) P_{\Lambda^{-\mu_1-1}/\nu}(Z_{-\mu_1}, \ldots, \widehat{Z_{c_i}}, \ldots Z_{-1}; q, t),$$

where the sum runs over

$$0 \leq \nu_3 \leq \sigma_{1,1} \leq \gamma \leq \sigma_{2,2} \leq \sigma_{3,3} \leq \delta$$

and $\Lambda^0 = (\sigma_{3,3}, \sigma_{2,2}, \sigma_{1,1})$, $\Lambda^{-\mu_1-1} = (\gamma, \gamma, \ldots, \gamma)$ and $\nu = (\gamma, \gamma, \nu_3)$. 

Masao Ishikawa

$(q, t)$-hook formula for Tailed Insets
We put $P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$, where

\begin{align*}
P_1 &= \{(i, i + c) \mid j > 3, \ 1 \leq i \leq p_i, \ 1 \leq c \leq \lambda_1, \}\, , \\
P_2 &= \{(j + 1 - c, j + 1) \mid i > 3, \ 1 \leq j \leq q_c, \ -\mu_1 \leq c \leq -1\} \\
P_3 &= \{(i, j) \mid 1 \leq i \leq 3, \ 2 \leq j \leq 3\} , \\
P_4 &= \{(2, 1), (1, 1), (1, 0)\} , \\
P_5 &= \{(1, c + 1) \mid -\mu_1 \leq c \leq -2\} , \\
P_6 &= \{(4, 4)\}
\end{align*}

and we write

\[ R_i = \prod_{v \in P_i} F(z[H_P(v)] ; q, t) , \]

for $i = 1, \ldots, 6$. 
Right-hand side

\[ \begin{align*}
& c_2^- = -5, \quad c_1^- = -3. \\
& c_1^+ = 2, \quad c_2^+ = 4, \quad c_3^+ = 7.
\end{align*} \]
Interpretation of RHS

Proposition

By direct computation, it is not hard to see

\[ R_1 = \prod_{0 \leq i < j \atop \varepsilon_{i,i+1} = + \atop \varepsilon_{i,j+1} = -} F(Z_{i,j}; q, t), \]

\[ R_2 = \prod_{i < j < 0 \atop \varepsilon_{i,i+1} = + \atop \varepsilon_{i,j+1} = -} F(Z_{i,j}; q, t). \]

\[ R_3 = \prod_{i < 0 \leq j \atop \varepsilon_{i,i+1} = + \atop \varepsilon_{i,j+1} = -} F(wZ_{i,j}; q, t) \]

\[ = F(wZ_{c_1^-c_1^+}; q, t) F(wZ_{c_1^-c_2^+}; q, t) F(wZ_{c_1^-c_3^+}; q, t) \]

\[ \times F(wZ_{c_2^-c_1^+}; q, t) F(wZ_{c_2^-c_2^+}; q, t) F(wZ_{c_2^-c_3^+}; q, t) \]
**Proposition**

\[ R_4 = F\left(wZ_{c_2^-}c_1^+Z_{c_1^-}c_2^+; q, t\right)F\left(wZ_{c_2^-}c_1^+Z_{c_1^-}c_3^+; q, t\right)F\left(wZ_{c_2^-}c_3^+Z_{c_1^-}c_2^+; q, t\right) \]

\[ P_5 = \prod_{c=-\mu_1}^{c_1^-} \prod_{c\neq c_1^-}^{c_1^-} F\left(w^2Z_{c_2^-}c_1^+Z_{c_1^-}c_2^+Z_{c,c_3^+}; q, t\right) \]
Thank you for your attention!