(q, t)-hook formula for Tailed Insets and a Macdonald polynomial identity

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Abstract

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Okada presented a conjecture on (q, t)-hook formula for general d-complete posets in the paper, Soichi Okada, (q, t)-Deformations of multivariate hook product formulae, *J. Algebr. Comb.* (2010) **32**, 399 – 416. We consider the Tailed Inset case, and reduce the conjectured identity to an indentity of the Macodonald polynomials rephrasing Okada's (q,t)-weights via Pieri coefficients of the Macodonald polynomials. Joint work with Frederic Jouhet (University of Lyon I).

References

In this talk

1. M. Ishikawa, "(q, t)-hook formula for Birds and Banners", `arXiv:1302.1968 [math.CO]`.
2. S. Okada, "(q, t)-deformations of multivariate hook product formulae", `arXiv:0909.0086 [math.CO]` *J. Algebraic Combin.* **32** (2010), 399-416.
3. R. Proctor, "Dynkin diagram classication of λ-minusule Bruhat lattices and of d-complete posets", *J. Algebraic Combin.* **9** (1999), 61 – 94.
4. M. Vuletić, "A generalization of Macmahon's formula", `arXiv:0707.0532 [math.CO] 4Jul 2007`, *Trans. Amer. Math. Soc.* **361** (2009), 2789-2804.
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Introduction

P-partitions

Definition

A *partially ordered set* (also called a *poset*) is a set P with a binary relation "$\leq$" which is *antisymmetric*, *transitive*, and *reflexive*.

Definition (Stanley '72)

Let P be a poset. A *P-partition* is a map $\pi : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \pi(x) \geq \pi(y) \text{ in } \mathbb{N},$$

where $\mathbb{N}$ is the set of nonnegative integers. Let $\mathscr{A}(P)$ be the set of P-partitions.

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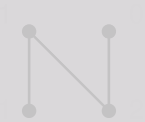

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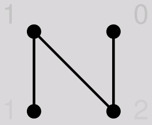

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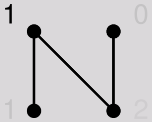

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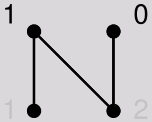

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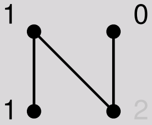

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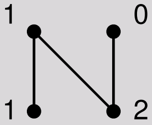

(Shifted) diagrams

Definition

A *partiton* is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers with finitely many λ_i unequal to zero. The *length* and *weight* of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero λ_i respectively. A *strict partition* is a partition in which its parts are strictly decreasing. If λ is a partition (resp. strict partition), then its *diagram* $D(\lambda)$ (resp. *shifted diagram* $S(\lambda)$) is defined by

$$D(\lambda) = \{\, (i,j) \in \mathbb{Z}^2 \;:\; 1 \le j \le \lambda_i \}$$

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Example (The diagram and shifted diagram for $\lambda = (4, 3, 1)$)

$$D(\lambda) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array} \qquad S(\lambda) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \end{array}$$

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(Shifted) shapes

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A diagram $D(\lambda)$ or a shifted diagram $S(\lambda)$ is regarded as a poset by defining its order structure by

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$

By this order the poset represented by a diagram $P = D(\lambda)$ is called a *shape*, and the posets $P = S(\lambda)$ is called *shifted shapes*.

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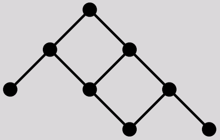
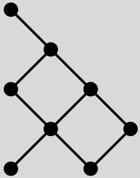
$D(\lambda) =$ 

$S(\lambda) =$ 

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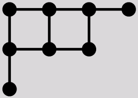
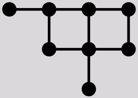
$D(\lambda) =$ 

$S(\lambda) =$ 

Definition

For a partition (resp. strict partition) λ and a cell $(i,j) \in D(\lambda)$ (resp. $S(\lambda)$), the *hook at (i,j)* in $D(\lambda)$ (resp. $S(\lambda)$), is defined by

$$H_{D(\lambda)}(i,j) = \{(i,j)\} \cup \{(i,l) \in D(\lambda) : l > j\} \cup \{(k,j) \in D(\lambda) : k > i\}$$

(resp.

$$\begin{aligned} H_{S(\lambda)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in S(\lambda) : l > j\} \\ &\cup \{(k,j) \in D(\lambda) : k > i\} \cup \{(j+1,l) \in S(\lambda) : l > j\}). \end{aligned}$$

Example (The hook at $(1,2)$ in $D(\lambda)$ and $S(\lambda)$ for $\lambda = (4,3,1)$)

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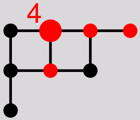
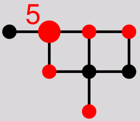
$D(\lambda) =$ 

$S(\lambda) =$ 

Content and hook length

Definition

The *hook length at* (i,j) is defined by $h_{D(\lambda)}(i,j) = |H_{D(\lambda)}(i,j)|$ (resp. $h_{S(\lambda)}(i,j) = |H_{S(\lambda)}(i,j)|$). Further $c(i,j) = j - i$ is called the *content at* (i,j).

Example (The hook lenghs in $D(\lambda)$ and $S(\lambda)$ for $\lambda = (4,3,1)$)

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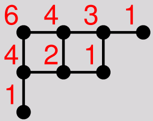
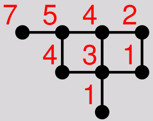
$D(\lambda) =$ 

$S(\lambda) =$ 

Example (The contents in $D(\lambda)$ and $S(\lambda)$ for $\lambda = (4,3,1)$)

$D(\lambda) =$

$S(\lambda) =$

Content and hook length

Definition

The *hook length at* (i,j) is defined by $h_{D(\lambda)}(i,j) = |H_{D(\lambda)}(i,j)|$ (resp. $h_{S(\lambda)}(i,j) = |H_{S(\lambda)}(i,j)|$). Further $c(i,j) = j - i$ is called the *content at* (i,j).

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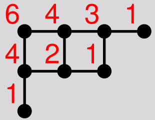
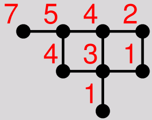
$D(\lambda) =$ 

$S(\lambda) =$ 

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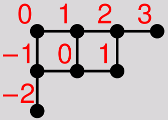
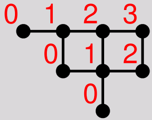
$D(\lambda) =$ 

$S(\lambda) =$ 

One Variable Hook Length Formula

Theorem (Frame-Robinson-Thrall '54, Stanley '72))

If $P = D(\lambda)$ or $S(\lambda)$, then we have

$$\sum_{\pi \in \mathscr{A}(P)} z^{|\pi|} = \prod_{(i,j)\in P} \frac{1}{1 - z^{h_P(i,j)}},$$

where the sum on the left-hand side runs over all P-partitions, and $|\pi| = \sum_{x\in P} \pi(x)$.

Example (An example of P-partition)

$\pi =$

$|\pi| = 16$

$z^{|\pi|} = z^{16}$

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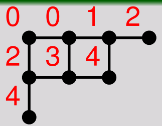

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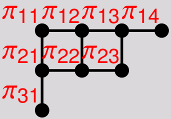
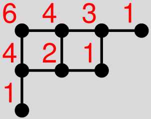
Example of One Variable Hook Length Formula

Example (The shape for $\lambda = (4,3,1)$)

$D(\lambda) =$ 

$D(\lambda) =$ 

$$\sum_{\pi \in \mathscr{A}(D(\lambda))} z^{\sum_{(i,j)\in D(\lambda)} \pi_{i,j}} = \frac{1}{(1-z)^3(1-z^2)(1-z^3)(1-z^4)^2(1-z^6)}.$$

Multivariate Hook Length Formula

Theorem (Gansner '81, Sagan '82)

Let $\dots, z_{-1}, z_0, z_1, z_2, \dots$ be variables. If $P = D(\lambda)$ or $S(\lambda)$, then we have

$$\sum_{\pi \in \mathscr{A}(P)} z^{\pi} = \prod_{(i,j)\in P} \frac{1}{1 - z[H_P(i,j)]},$$

where the sum on the left-hand side runs over all P-partitions, $z^{\pi} = \prod_{(i,j)\in P} {z_{c(i,j)}}^{\pi_{i,j}}$ and $z[H] = \prod_{(i,j)\in H} z_{c(i,j)}$ for any finite subset $H \subset \mathbb{Z}^2$. (Gansner used Hillman-Grassl '76 algorithm.)

Example (An example of P-partition)

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$$z^{\pi} = z_{-2}^4 z_{-1}^2 z_0^3 z_1^4 z_2 z_3^2$$

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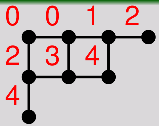
$\pi =$ 

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Example of Multivariate Hook Length Formula

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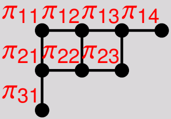
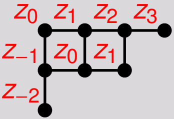
$D(\lambda) =$ 

$D(\lambda) =$ 

$$\sum_{\pi \in \mathscr{A}(P)} z_{-2}^{\pi_{31}} z_{-1}^{\pi_{21}} z_0^{\pi_{11}+\pi_{22}} z_1^{\pi_{12}+\pi_{23}} z_2^{\pi_{13}} z_3^{\pi_{14}}$$

$$= \frac{1}{(1 - z_{-2}z_{-1}z_0z_1z_2z_3)(1 - z_0z_1z_2z_3)(1 - z_1z_2z_3)(1 - z_3)}$$

$$\times \frac{1}{(1 - z_{-2}z_{-1}z_0z_1)(1 - z_0z_1)(1 - z_1)(1 - z_{-2})}.$$

The Cauchy formula and the Littlewood formula

Therem (The Cauchy formula)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are n-tuples of variables. Then we have

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.$$

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Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ is an n-tuples of variables. Then we have

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(q,t)-hook formula

Conjecture (Okada '10)

If P is a d-complete poset, then we have

$$\sum_{\pi \in \mathscr{A}(P)} W_P(\pi; q, t)\, z^{\pi} = \prod_{(i,j) \in P} F\left(z[H_P(i,j)]; q, t\right),$$

where the sum on the left-hand side runs over all P-partitions, and

$$F(x; q, t) = \frac{(tx; q)_\infty}{(x; q)_\infty}.$$

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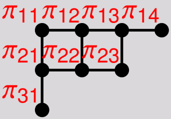
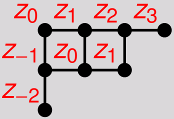
$D(\lambda) =$ 

$D(\lambda) =$ 

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$$= F(z_{-2}z_{-1}z_0z_1z_2z_3; q, t)\, F(z_0z_1z_2z_3; q, t)\, F(z_1z_2z_3; q, t)$$
$$\times F(z_3; q, t)\, F(z_{-2}z_{-1}z_0z_1; q, t)\, F(z_0z_1; q, t)\, F(z_1; q, t)\, F(z_{-2}; q, t).$$

Current situation

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1. If P is (1) Shape or (2) Shfted Shape, the (q, t)-hook formula is proven in the paper by Okada(2010).
2. If P is (3) Bird or (6) Banner, the (q, t)-hook formula is proven by me (not yet published) 2013. We use Gasper's identity.
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Theorem

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ are n-tuples of variables. Then we have

$$\sum_{\lambda} P_\lambda(\boldsymbol{x}; q, t) Q_\lambda(\boldsymbol{y}; q, t) = \prod_{i,j=1}^{n} F(x_i y_j; q, t).$$

Warnaar's formula

Theorem (Warnaar '06)

$$\sum_{\lambda} w^{r(\lambda)} b_{\lambda}^{\mathrm{oa}}(q,t) P_{\lambda}(x;q,t) = \prod_{i\geq 1} \frac{(1+wx_i)(qtx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} \prod_{i<j} \frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty},$$

where $r(\lambda)$ is the number of rows of odd length.

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Corollary

$$\sum_{\lambda} w^{r(\lambda')} b_{\lambda}^{\mathrm{el}}(q,t) P_{\lambda}(x;q,t) = \prod_{i\geq 1} \frac{(twx_i;q)_\infty}{(wx_i;q)_\infty} \prod_{i<j} \frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}.$$

Proof. Applying the $\mathbb{F}$-algebra homomorphism $w_{q,t}$ to the above identity.

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$$\sum_{\lambda} w^{\frac{|\lambda|+r(\lambda')}{2}} b_\lambda^{\mathrm{el}}(q,t) P_\lambda(x;q,t) = \prod_{i\geq 1} \frac{(twx_i;q)_\infty}{(wx_i;q)_\infty} \prod_{i<j} \frac{(twx_ix_j;q)_\infty}{(wx_ix_j;q)_\infty},$$

$$\sum_{\lambda} w^{\frac{|\lambda|-r(\lambda')}{2}} b_\lambda^{\mathrm{el}}(q,t) P_\lambda(x;q,t) = \prod_{i\geq 1} \frac{(tx_i;q)_\infty}{(x_i;q)_\infty} \prod_{i<j} \frac{(twx_ix_j;q)_\infty}{(wx_ix_j;q)_\infty}.$$

d-complete poset

d-complete poset

Contents of this section

1. The *d*-complete posets arise from the dominant minuscule heaps of the Weyl groups of simply-laced Kac-Moody Lie algebras.
2. Proctor gave completely combinatorial description of *d*-complete poset, which is a graded poset with *d*-complete coloring.
3. Proctor showed that any *d*-complete poset can be obtained from the 15 *irreducible* classes by *slant-sum*.
4. The *d-complete coloring* is important for the multivariate generating function. The content should be replaced by color for *d*-complete posets.
5. Okada's (q,t)-weight $W_P(\pi;q,t)$
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Double-tailed diamond poset

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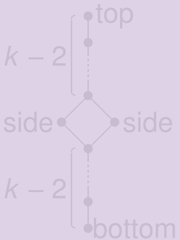
- The double-tailed diamond poset $d_k(1)$ is the poset depicted below:



- A d_k-interval is an interval isomorphic to $d_k(1)$.
- A d_k^--interval ($k \geq 4$) is an interval isomorphic to $d_k(1) - \{\mathrm{top}\}$.
- A d_3^--interval consists of three elements x, y and w such that w is covered by x and y.

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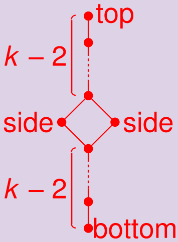

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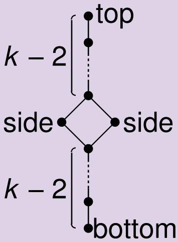

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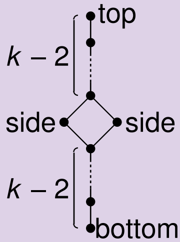

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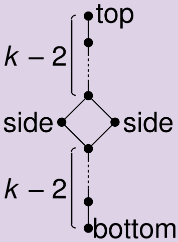

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Definition

A poset P is *d-complete* if it satisfies the following three conditions for every $k \geq 3$:

1. If I is a d_k^--interval, then there exists an element v such that v covers the maximal elements of I and $I \cup \{v\}$ is a d_k-interval.
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3. There are no d_k^--intervals which differ only in the minimal elements.

Definition of d-complete poset

Definition

A poset P is *d-complete* if it satisfies the following three conditions for every $k \geq 3$:

1. If I is a d_k^--interval, then there exists an element v such that v covers the maximal elements of I and $I \cup \{v\}$ is a d_k-interval.
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Properties of d-complete posets

Fact

If P is a connected d-complete poset, then

(a) P has a unique maximal element.

(b) P is *graded*, i.e., there exists a rank function $r : P \to \mathbb{N}$ such that $r(x) = r(y) + 1$ if x covers y.

Fact

Any connected d-complete poset is uniquely decomposed into a slant sum of one-element posets and slant-irreducible d-complete posets.

Slant irreducible d-complete posets are classified into 15 families : (1) Shapes, (2) Shifted shapes, (3) Birds, (4) Insets, (5) Tailed insets, (6) Banners, (7) Nooks, (8) Swivels, (9) Tailed swivels, (10) Tagged swivels, (11) Swivel shifts, (12) Pumps, (13) Tailed pumps, (14) Near bats, (15) Bat.

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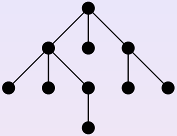
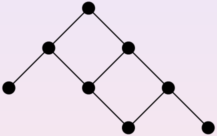
Examples

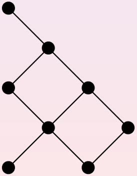
rooted tree



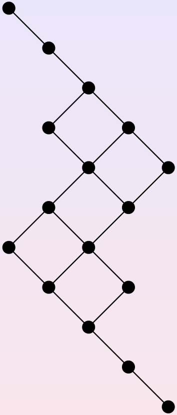
shape



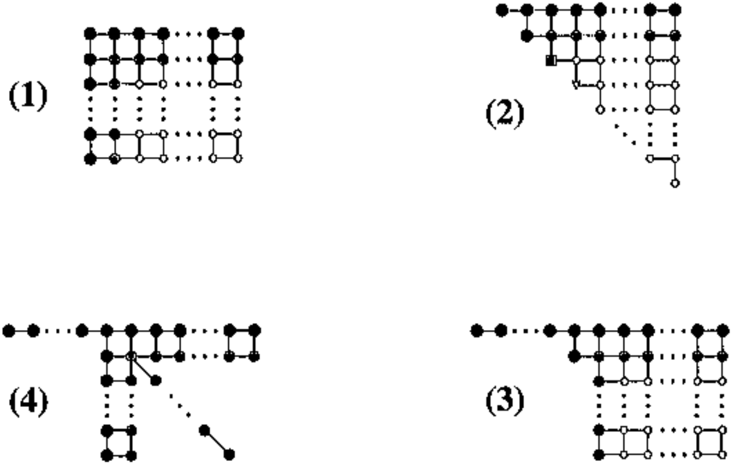
shifted shape



swivel



15 irreducible d-complete posets



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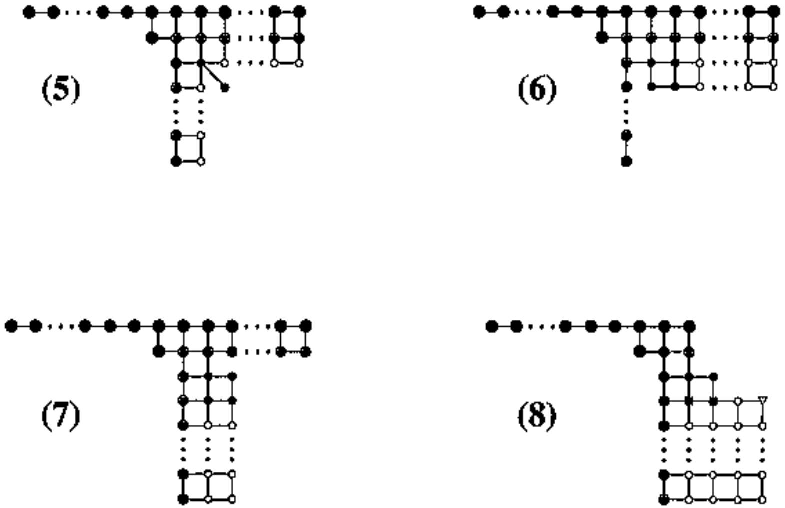

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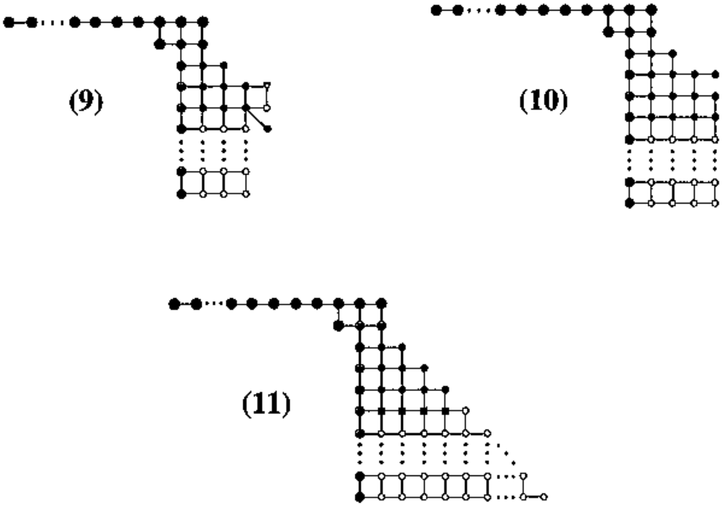

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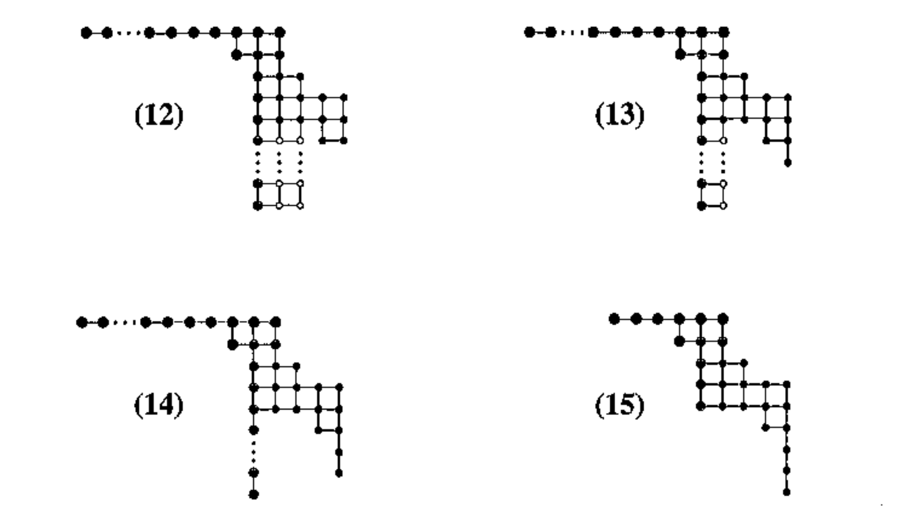

Top Tree and d-Complete Coloring

Definition

For a connected d-complete poset P, we define its top tree by putting

$$T = \{x \in P : \text{ every } y \geq x \text{ is covered by at most one other element } \}$$

Fact

Let I be a set of colors such that $\#I = \#T$. Then a bijection $c : T \to I$ can be uniquely extended to a map $c : P \to I$ satisfying the following three conditions:

- If x and y are incomparable, then $c(x) \neq c(y)$.
- If an interval $[w, v]$ is a chain, then the colors $c(x)$ ($x \in [w, v]$) are distinct.
- If $[w, v]$ is a d_k-interval then $c(w) = c(v)$.

Such a map $c : P \to I$ is called a d-complete coloring.

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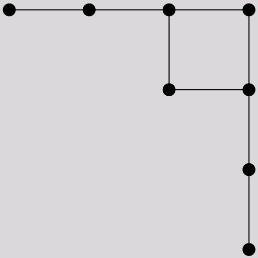
Example

Top Tree and d-Complete Coloring of d_5-interval



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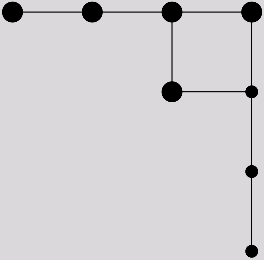

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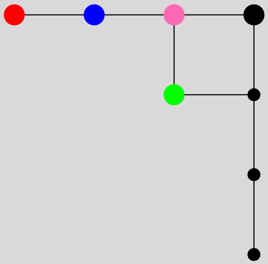

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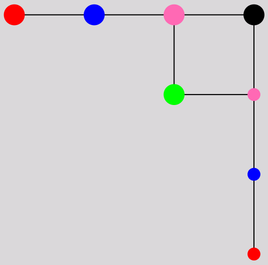

Shapes

Definition

Let r be a postive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ be strict partitions such that

$$\alpha_1 > \cdots > \alpha_r \geq 0, \qquad \beta_1 > \cdots > \beta_r \geq 0,$$

Let P be the set $P = P_{\mathrm{L}} \cup P_{\mathrm{R}}$ of lattice points in $\mathbb{Z}^2$, where

$$P_{\mathrm{R}} = \{\, (i,j) \,:\, 1 \leq i \leq j \leq \alpha_i + i - 1 \ (1 \leq i \leq r) \,\},$$
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We regard P as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$

We call this poset a *shape* and denote it by $P = P_1(\alpha, \beta)$.

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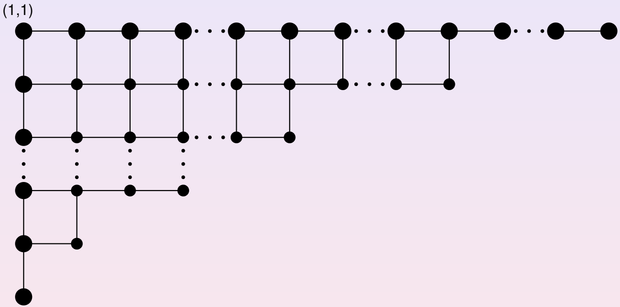

Shifted Shapes

Definition

Let r be a postive integer, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a strict partition such that

$$\alpha_1 > \cdots > \alpha_r \geq 0.$$

Define the *shifted shape* $P = P_2(\alpha)$ by

$$P = \{ (i, j) \; : \; i \leq j \leq \alpha_i + i - 1 \; (1 \leq i \leq r) \}.$$

We regard it as a poset by defining its order structure

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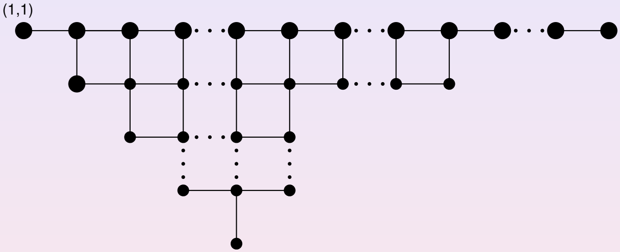

Birds

Definition

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Define the *bird* $P = P_3(\alpha, \beta; f)$ by

$$P = P_{\mathrm{H}} \cup P_{\mathrm{R}} \cup P_{\mathrm{L}} \cup P_{\mathrm{T}}$$

where

$$\begin{aligned}
P_{\mathrm{H}} &= \{\, (1, j) \;:\; -f + 1 \le j \le 1 \},\\
P_{\mathrm{R}} &= \{\, (i, j) \;:\; i \le j \le \alpha_i + i - 1 \ (i = 1, 2) \,\},\\
P_{\mathrm{L}} &= \{\, (i, j) \;:\; j \le i \le \beta_j + j - 1 \ (j = 1, 2) \,\},\\
P_{\mathrm{T}} &= \{\, (i, i) \;:\; 2 \le i \le f + 2 \}
\end{aligned}$$

as a set and we regard it as a poset by defining its order structure

$$(i_1, j_1) \ge (i_2, j_2) \iff i_1 \le i_2 \text{ and } j_1 \le j_2.$$

if and only if the both of (i_1, j_1) and (i_2, j_2) are in $P_{\mathrm{H}} \cup P_{\mathrm{R}} \cup P_{\mathrm{L}}$ or in P_{T}

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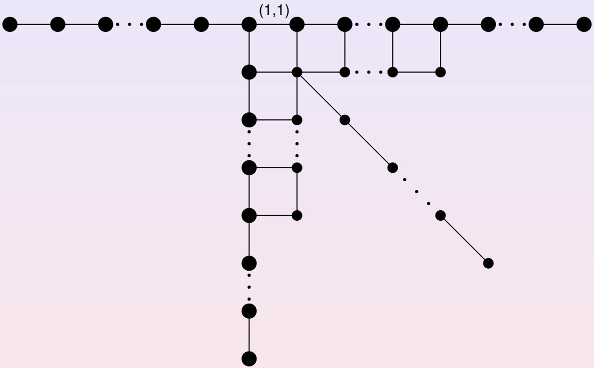

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if neither of (i_1, j_1) and (i_2, j_2) is not in P_{T}, whereas $(3,3) < (4,4)$. We call this poset a *Tailed Inset*, denoted by $P_5(\alpha, \beta)$.

Tailed Insets

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Let P be the set $P = P_{\mathrm{H}} \cup P_{\mathrm{M}} \cup P_{\mathrm{L}} \cup P_{\mathrm{R}} \cup P_{\mathrm{T}}$ of lattice points in $\mathbb{Z}^2$, where $P_{\mathrm{M}} = \{ (2,1) \}$, $P_{\mathrm{T}} = \{ (4,4) \}$ and

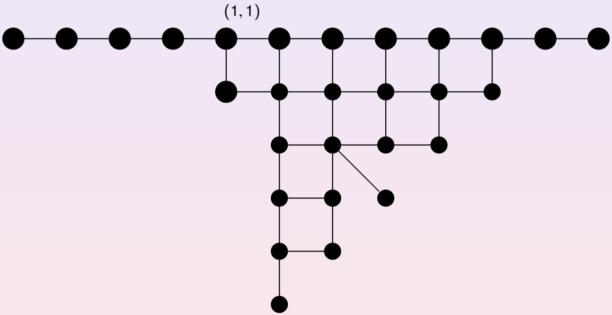
$$\begin{aligned} P_{\mathrm{H}} &= \{ (1,j) \, : \, -\beta_1 + 1 \leq j \leq 0 \}, \\ P_{\mathrm{R}} &= \{ (i,j) \, : \, 1 \leq i \leq j \leq \alpha_i + i \; (i = 1,2,3) \}, \\ P_{\mathrm{L}} &= \{ (i+1, j+1) \, : \, 1 \leq j \leq i \leq \beta_j + j \; (j = 1,2) \}. \end{aligned}$$

We regard P as a poset by defining the order relation

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$

if neither of (i_1, j_1) and (i_2, j_2) is not in P_{T}, whereas $(3,3) < (4,4)$. We call this poset a *Tailed Inset*, denoted by $P_5(\alpha, \beta)$.

Tailed Insets



Banners

Definition

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a strict partition such that $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 0$, and let $f \geq 2$ be a positive integer. Let P be the set $P = P_{\mathrm{H}} \cup P_{\mathrm{W}} \cup P_{\mathrm{T}}$ of lattice points in $\mathbb{Z}^2$, where

$$
\begin{aligned}
P_{\mathrm{H}} &= \{\, (1, j) \;:\; -f + 2 \leq j \leq 1 \},\\
P_{\mathrm{W}} &= \{\, (i, j) \;:\; i \leq j \leq \alpha_i + i - 1 \;(i = 1, 2, 3, 4)\,\},\\
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if both of (i_1, j_1) and (i_2, j_2) are in $P_{\mathrm{H}} \cup P_{\mathrm{W}}$ or in P_{T}, and call it a *banner*

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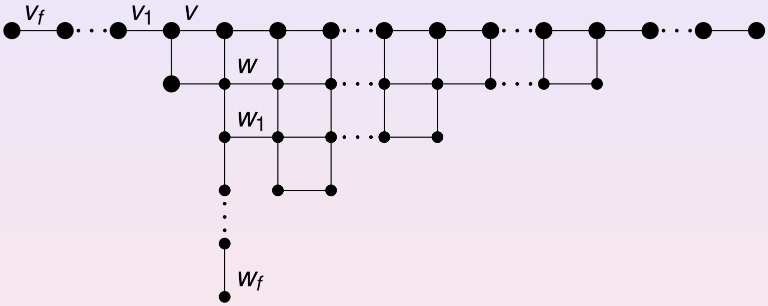
Banners



Definition

Let P be a connected d-complete poset and T its top tree. Let z_v ($v \in T$) be indeterminate. Let $c : P \to T$ be the d-complete coloring. For each $v \in P$, we define monomials $z\left[H_P(v)\right]$ by induction as follows:

(a) If v is not the top of any d_k-interval, then we deine

$$z\left[H_P(v)\right] = \prod_{w \le v} z_{c(w)}.$$

(b) If v is the top of a d_k-interval $[w, v]$, then we de?ne

$$z\left[H_P(v)\right] = \frac{z\left[H_P(x)\right] z\left[H_P(Y)\right]}{z\left[H_P(w)\right]}$$

where x and y are the sides of $[w, v]$.

Hook Monomials

Definition

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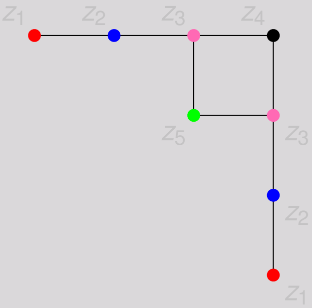
An example of hook monomials

Example

We consider the following poset $P = d_1(5)$. We give the following assignment of variavles for $z_{c(x)}$, $x \in P$.



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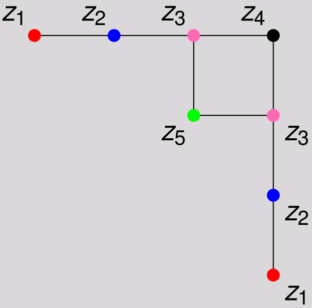

Hook Monomials for $P = d_1(5)$

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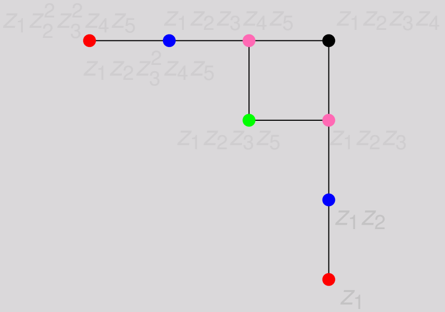

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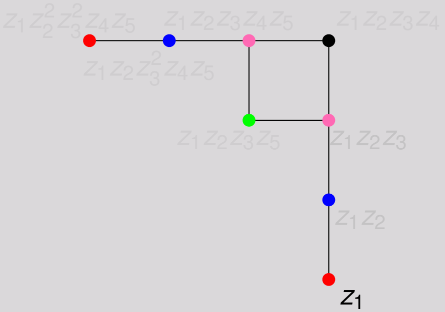

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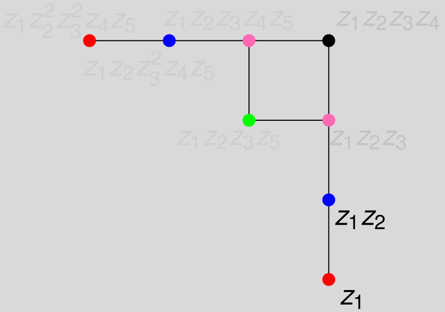

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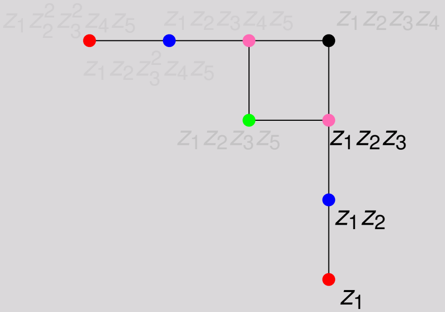

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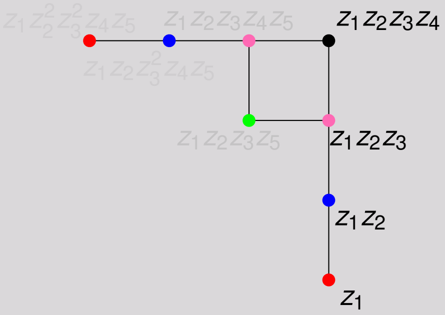

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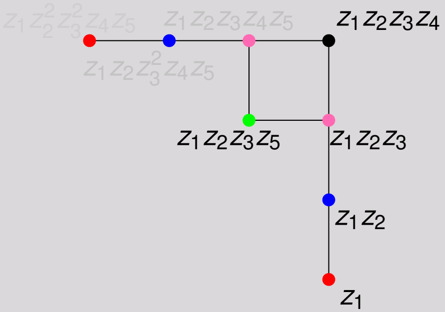

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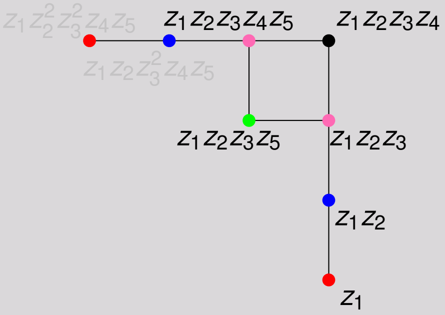

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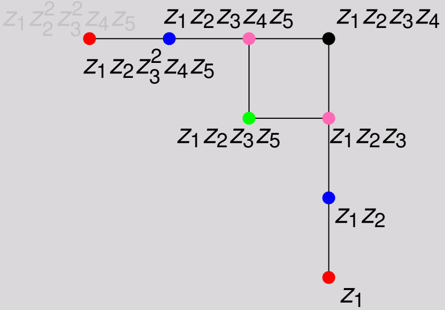

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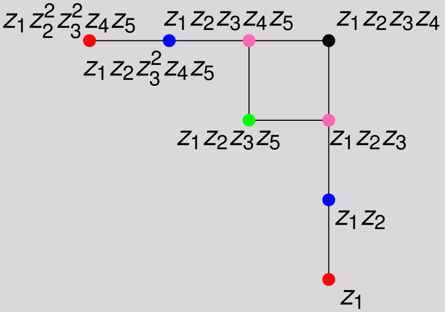

(q,t)-Weight associated with P-partition π

Definition

Let P be a connected d-complete poset with top tree T. Given a P-partition $\pi \in \mathscr{A}(P)$, we define $W_P(\pi; q, t)$ by

$$\frac{\displaystyle\prod_{\substack{x,y \in \widehat{P},\, x<y \\ c(x) \text{ and } c(y) \text{ are adjacent in } \widehat{T}}} f\left(\widehat{\pi}(x) - \widehat{\pi}(y), \left\lfloor \frac{\widehat{r}(y) - \widehat{r}(x)}{2} \right\rfloor\right)}{\displaystyle\prod_{\substack{x,y \in P,\, x<y \\ c(x)=c(y)}} f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y)-r(x)}{2} \right\rfloor\right) f\left(\pi(x) - \pi(y), \left\lfloor \frac{r(y)-r(x)}{2} \right\rfloor - 1\right)}.$$

Example

Compute this weight $W_P(\pi; q, t)$ for $P = d_1(5)$.

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Example

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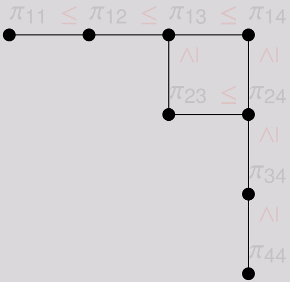
We consider the following poset $P = d_5(1)$. A P-partition π must satisfy the following inequalities



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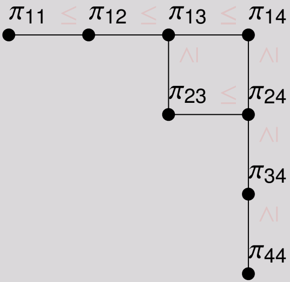

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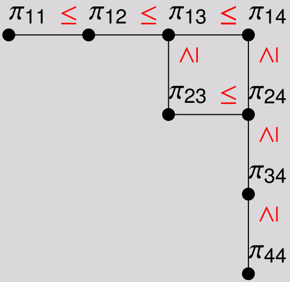
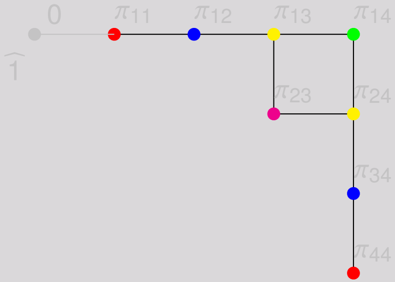


Example

A P-partition π extends to $\widehat{P}$-partition $\widehat{\pi}$.



$$
\begin{aligned}
\mathrm{numer}\, W_P(\pi; q, t) = {} & f(\pi_{11}; 0) f(\pi_{44}; 3) \\
& \times f(\pi_{12} - \pi_{11}; 0) f(\pi_{34} - \pi_{11}; 2) f(\pi_{44} - \pi_{12}; 2) f(\pi_{44} - \pi_{34}; 0) \\
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Numerator of $W_P(\pi; q, t)$ ($P = d_1(5)$)

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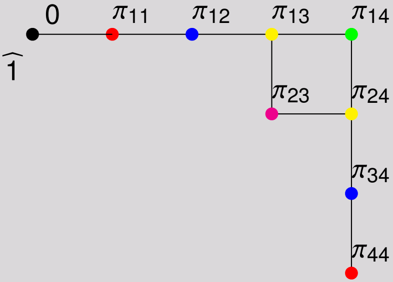

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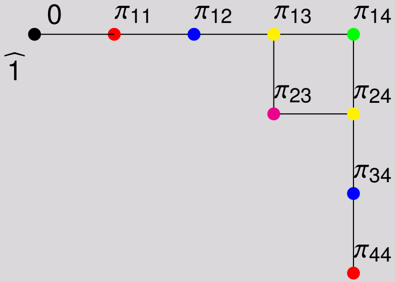

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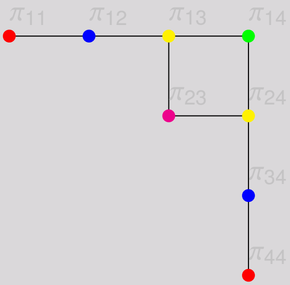
Denominator of $W_P(\pi; q, t)$ ($P = d_1(5)$)

Example

A P-partition π



$$\begin{aligned}\text{denom}\, W_P(\pi; q, t) &= f(\pi_{44} - \pi_{11}; 2) f(\pi_{44} - \pi_{11}; 3) \\ &\times f(\pi_{34} - \pi_{12}; 1) f(\pi_{34} - \pi_{12}; 2) f(\pi_{24} - \pi_{13}; 0) f(\pi_{24} - \pi_{13}; 1)\end{aligned}$$

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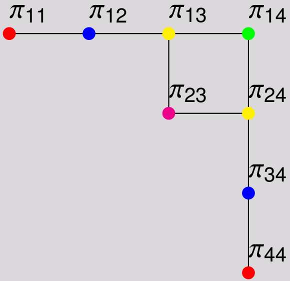

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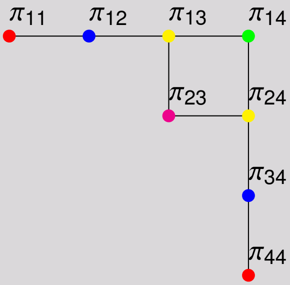

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Okada's (q,t)-hook formula conjecture

Okada's Conjecture

Then the following identity would hold for any d-complete posets P:

$$\sum_{\pi \in \mathscr{A}(P)} W_P(\pi; q, t) z^{\pi} = \prod_{v \in P} F(z[H_P(v)])$$

where $z^{\pi} = \prod_{x \in P} z_{c(x)}^{\pi(x)}$. Here the sum on the left-hand side runs over all P-partitions $\pi \in \mathscr{A}(P)$, and the right-hand side is the product of all hook monomials for $v \in P$.

Macdonald polynomials

Arm-length and leg-length

Definition

Let λ be a partition. Let $s = (i, j)$ be a square in the diagram of λ, and let $a(s)$ and $l(s)$ be the arm-length and leg-length of s, i.e.,

$$a(s) = \lambda_i - j, \qquad l(s) = \lambda'_j - i.$$

Definition

Define

$$b_\lambda(q,t) := \prod_{s\in\lambda} \frac{1 - q^{a(s)}t^{l(s)+1}}{1 - q^{a(s)+1}t^{l(s)}} = \prod_{\substack{i\geq 1\\ m\geq 0}} \frac{f(\lambda_i - \lambda_{i+m+1}, m)}{f(\lambda_i - \lambda_{i+m}, m)},$$

$$b^{\mathrm{el}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ l(s)\text{ even}}} \frac{1 - q^{a(s)}t^{l(s)+1}}{1 - q^{a(s)+1}t^{l(s)}} = \prod_{\substack{i\geq 1\\ m\geq 0\text{ even}}} \frac{f(\lambda_i - \lambda_{i+m+1}, m)}{f(\lambda_i - \lambda_{i+m}, m)}$$

$$b^{\mathrm{oa}}_\lambda(q,t) := \prod_{\substack{s\in\lambda\\ a(s)\text{ odd}}} b_\lambda(s; q, t).$$

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$$b_\lambda^{\mathrm{oa}}(q,t) := \prod_{\substack{s\in\lambda\\ a(s)\text{ odd}}} b_\lambda(s;q,t).$$

Monomial symmeytric function

Definition

If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two sequences of independent indeterminates, then we write

$$\Pi(x; y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \prod_{i,j} F(x_i y_j; q, t).$$

Definition

Let $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ and Λ denote the ring of symmetric polynomials in n independent variables and the ring of symmetric polynomials in countably many variables, respectively. For $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition of at most n parts the *monomial symmetric function* m_λ is defined as

$$m_\lambda(x) := \sum_{\alpha} x^\alpha$$

where the sum is over all distinct permutations α of λ.

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Definition

If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are two sequences of independent indeterminates, then we write

$$\Pi(x; y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \prod_{i,j} F(x_i y_j; q, t).$$

Definition

Let $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ and Λ denote the ring of symmetric polynomials in n independent variables and the ring of symmetric polynomials in countably many variables, respectively. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of at most n parts the *monomial symmetric function* m_λ is defined as

$$m_\lambda(x) := \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations α of λ.

Macdonald scalar product

Definition

For r a nonnegative integer the *power sums* p_r are given by $p_0 = 1$ and $p_r = m_{(r)}$ for $r > 1$. More generally the power-sum products are defined as $p_\lambda(x) := p_{\lambda_1}(x)p_{\lambda_2}(x)\cdots$ for an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \dots)$. Define the *Macdonald scalar product* $\langle\cdot,\cdot\rangle_{q,t}$ on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu\rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_i \prod_{i=1}^{n} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

with $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ and $m_i = m_i(\lambda)$.

Macdonald's P-function

Definition

If we denote the ring of symmetric functions in Λ_n variables over the field $\mathbb{F} = \mathbb{Q}(q,t)$ of rational functions in q and t by $\Lambda_{n,\mathbb{F}}$, then the *Macdonald polynomial $P_\lambda(x) = P_\lambda(x;q,t)$* is the unique symmetric polynomial in $\Lambda_{n,\mathbb{F}}$ such that :

$$P_\lambda = \sum_{\mu \le \lambda} u_{\lambda\mu}(q,t) m_\mu(x)$$

with $u_{\lambda\lambda} = 1$ and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu.$$

The Macdonald polynomials $P_\lambda(x;q,t)$ with $\ell(\lambda) \le n$ form an $\mathbb{F}$-basis of $\Lambda_{n,\mathbb{F}}$. If $\ell(\lambda) > n$ then $P_\lambda(x;q,t) = 0$. $P_\lambda(x;q,t)$ is called *Macdonald's P-function*. Since $P_\lambda(x_1,\dots,x_n,0;q,t) = P_\lambda(x_1,\dots,x_n;q,t)$ one can extend the Macdonald polynomials to symmetric functions containing an infinite number of independent variables $x = (x_1, x_2, \dots)$, to obtain a basis of $\mathbb{F} = \Lambda \otimes \mathbb{F}$.

Definition

A second Macdonald symmetric function, called *Macdonald's Q-function*, is defined as

$$Q_\lambda(x;q,t) = b_\lambda(q,t)P_\lambda(x;q,t).$$

The normalization of the Macdonald inner product is then $\langle P_\lambda, Q_\mu\rangle_{q,t} = \delta_{\lambda\mu}$ for all λ,μ, which is equivalent to

$$\sum_\lambda P_\lambda(x;q,t)Q_\lambda(y;q,t) = \Pi(x;y;q,t).$$

Pieri coefficients $\varphi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$

Definition

Let r be a positive integer, and let λ, μ be partitions such that $\lambda \supset \mu$ and $\lambda - \mu$ is a horizontal r-strip. The *Pieri coefficients* $\varphi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ are defined by

$$P_\mu g_r = \sum_\lambda \varphi_{\lambda/\mu} P_\lambda,$$
$$Q_\mu g_r = \sum_\lambda \psi_{\lambda/\mu} Q_\lambda,$$

where $g_r = Q_{(r)}$.

Another direct expression for $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$

Proposition

From Macdonald's book Chap.VI, §6, Ex.2(c), we have

$$\varphi_{\lambda/\mu}(q,t) = \prod_{1\le i<j\le \ell(\lambda)} \frac{f(\lambda_i-\mu_j;j-i)f(\mu_i-\lambda_{j+1};j-i)}{f(\lambda_i-\lambda_j;j-i)f(\mu_i-\mu_{j+1};j-i)},$$

$$\psi_{\lambda/\mu}(q,t) = \prod_{1\le i<j\le \ell(\mu)} \frac{f(\lambda_i-\mu_j;j-i)f(\mu_i-\lambda_{j+1};j-i)}{f(\mu_i-\mu_j;j-i)f(\lambda_i-\lambda_{j+1};j-i)}.$$

Macdonald's skew Q-function and skew P-function

Definition

For any three partitions λ, μ, ν let $f^{\lambda}_{\mu\nu}$ be the coefficient P_λ in the product $P_\mu P_\nu$:

$$P_\mu(x;q,t)P_\nu(x;q,t) = \sum_{\lambda} f^{\lambda}_{\mu\nu} P_\lambda(x;q,t)$$

Now let λ, μ be partitions and define $Q_{\lambda/\mu} \in \Lambda_{\mathbb{F}}$ by

$$Q_{\lambda/\mu}(x;q,t) = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_\nu(x;q,t).$$

Then $Q_{\lambda/\mu}(x;q,t) = 0$ unless $\lambda \supset \mu$, and $Q_{\lambda/\mu}$ is homogeneous of degree $|\lambda| - |\mu|$, which is called *Macdonald's skew Q-function*.

Definition

We define *Macdonald's skew P-function* $P_{\lambda/\mu}$ by

$$Q_{\lambda/\mu}(x;q,t) = \frac{b_\lambda(q,t)}{b_\mu(q,t)} P_{\lambda/\mu}(x;q,t).$$

Macdonald's skew Q-function and skew P-function

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$$Q_{\lambda/\mu}(x; q, t) = \frac{b_\lambda(q, t)}{b_\mu(q, t)} P_{\lambda/\mu}(x; q, t).$$

Lemma

Lemma

Let μ and ν be partitions, and $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are independent indeterminates.

$$\sum_{\lambda} Q_{\lambda/\mu}(x; q, t) P_{\lambda/\nu}(y; q, t) = \Pi(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)$$

Proof.

$$\begin{aligned} &\sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_\mu(z) P_\nu(w) \\ &= \sum_{\mu,\nu} \sum_{\lambda} Q_\lambda(x, z) P_\lambda(y, w) \\ &= \Pi(x, z; y, w) \\ &= \Pi(x; y)\Pi(x; w)\Pi(z; y)\Pi(z; w) \end{aligned}$$

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$$= \Pi(x; y; q, t) \sum_{\tau} Q_{\nu/\tau}(x; q, t) P_{\mu/\tau}(y; q, t)$$

Proof.

$$\sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w)$$

$$= \sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w)$$

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$$= \Pi(x; y) \Pi(x; w) \Pi(z; y) \Pi(z; w)$$

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Proof.

$$\sum_{\mu,\nu}\sum_{\lambda} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y)Q_{\mu}(z)P_{\nu}(w)$$
$$=\sum_{\mu,\nu}\sum_{\lambda} Q_{\lambda}(x,z)P_{\lambda}(y,w)$$
$$=\Pi(x,z;y,w)$$
$$=\Pi(x;y)\Pi(x;w)\Pi(z;y)\Pi(z;w)$$

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Proof.

$$\begin{aligned}
&\sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w) \\
&= \sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda}(x, z) P_{\lambda}(y, w) \\
&= \Pi(x, z; y, w) \\
&= \Pi(x; y) \Pi(x; w) \Pi(z; y) \Pi(z; w)
\end{aligned}$$

Lemma

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Let μ and ν be partitions, and $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are independent indeterminates.

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Proof.

$$\begin{aligned}
&\sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) Q_{\mu}(z) P_{\nu}(w) \\
&= \sum_{\mu,\nu} \sum_{\lambda} Q_{\lambda}(x,z) P_{\lambda}(y,w) \\
&= \Pi(x,z;y,w) \\
&= \Pi(x;y)\Pi(x;w)\Pi(z;y)\Pi(z;w)
\end{aligned}$$

Proof

$$= \Pi(x;y)\sum_{\xi} Q_{\xi}(x)P_{\xi}(w)\sum_{\eta} Q_{\eta}(z)P_{\eta}(y)\sum_{\tau} Q_{\tau}(z)P_{\tau}(w)$$

$$= \Pi(x;y)\sum_{\xi,\eta,\tau} Q_{\xi}(x)P_{\eta}(y)\sum_{\mu}\frac{b_{\eta}b_{\tau}}{b_{\mu}} f^{\mu}_{\eta\tau}Q_{\mu}(z)\sum_{\nu} f^{\nu}_{\xi\tau}P_{\nu}(w)$$

$$= \Pi(x;y)\sum_{\mu,\nu,\tau} Q_{\nu/\tau}(x)P_{\mu/\tau}(y)Q_{\mu}(z)P_{\nu}(w)$$

Hence, by comparing the coefficients of $Q_{\mu}(z)P_{\nu}(w)$ in the both sides, we obtain the desired identity. This completes the proof.

Proof

$$= \Pi(x;y)\sum_{\xi} Q_\xi(x)P_\xi(w)\sum_{\eta} Q_\eta(z)P_\eta(y)\sum_{\tau} Q_\tau(z)P_\tau(w)$$

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Proof

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&= \Pi(x;y) \sum_{\xi} Q_\xi(x) P_\xi(w) \sum_{\eta} Q_\eta(z) P_\eta(y) \sum_{\tau} Q_\tau(z) P_\tau(w) \\
&= \Pi(x;y) \sum_{\xi,\eta,\tau} Q_\xi(x) P_\eta(y) \sum_{\mu} \frac{b_\eta b_\tau}{b_\mu} f^{\mu}_{\eta\tau} Q_\mu(z) \sum_{\nu} f^{\nu}_{\xi\tau} P_\nu(w) \\
&= \Pi(x;y) \sum_{\mu,\nu,\tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y) Q_\mu(z) P_\nu(w)
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Hence, by comparing the coefficients of $Q_{\mu}(z)P_{\nu}(w)$ in the both sides, we obtain the desired identity. This completes the proof.

A generalization of Vuletić's formula

Theorem

Fix a positive integer T and two partitions μ^0 and μ^T. Let $x^0, \dots, x^{T-1}$, $y^1, \dots, y^T$ be sets of variables. Then we have

$$\sum_{(\lambda^1,\mu^1,\lambda^2,\dots,\lambda^T)} \prod_{i=1}^{T} Q_{\lambda^i/\mu^{i-1}}(x^{i-1}; q, t) P_{\lambda^i/\mu^i}(y^i; q, t)$$
$$= \prod_{0 \le i < j \le T} \Pi(x^i, y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \dots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \dots, y^T; q, t)$$

where the sum runs over $(2T-1)$-tuples $(\lambda^1, \mu^1, \lambda^2, \dots, \mu^{T-1}, \lambda^T)$ of partitions satisfying

$$\mu^0 \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \cdots \supset \mu^{T-1} \subset \lambda^T \supset \mu^T.$$

Proof. Use induction and the above lemma.

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$$= \prod_{0 \le i < j \le T} \Pi(x^i, y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \dots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \dots, y^T; q, t)$$

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Proof. Use induction and the above lemma.

A corollary

Definition

We define $P^{\varepsilon}_{[\lambda,\mu]}(x;q,t)$ and $Q^{\varepsilon}_{[\lambda,\mu]}(x;q,t)$ for a pair (λ,μ) of partitions, a set $x=(x_1,x_2,\dots)$ of independent variables and $\varepsilon=\pm$ by

$$P^{\varepsilon}_{[\lambda,\mu]}(x;q,t)=\begin{cases}P_{\lambda/\mu}(x;q,t) & \text{if } \varepsilon=+,\\ Q_{\mu/\lambda}(x;q,t) & \text{if } \varepsilon=-,\end{cases}$$

$$Q^{\varepsilon}_{[\lambda,\mu]}(q,t)=\begin{cases}Q_{\lambda/\mu}(x;q,t) & \text{if } \varepsilon=+,\\ P_{\mu/\lambda}(x;q,t) & \text{if } \varepsilon=-.\end{cases}$$

Here we assume $\lambda\supset\mu$ if $\varepsilon=+$, and $\lambda\subset\mu$ if $\varepsilon=-$.

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Theorem

Let n be a positive integer, and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ a sequence of $\pm$, Fix a positive integer n and two partitions λ^0 and λ^n. Let $x^1, \dots, x^n$ be sets of variables. Then we have

$$\sum_{(\lambda^1, \lambda^2, \dots, \lambda^{n-1})} \prod_{i=1}^{n} P^{\epsilon_i}_{[\lambda^{i-1}, \lambda^i]}(x^i; q, t) = \prod_{\substack{i<j \\ (\epsilon_i, \epsilon_j) = (-,+)}} \Pi(x^i; x^j; q, t)$$

$$\times \sum_{\nu} Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i=-}; q, t) P_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i=+}; q, t),$$

where the sum runs over $(n-1)$-tuples $(\lambda^1, \lambda^2, \dots, \lambda^{n-1})$ of partitions satisfying

$$\begin{cases} \lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +, \\ \lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -. \end{cases}$$

Proof. Take $T = n$ and put $X^{i-1} = 0$ and $Y^i = x^i$ if $\epsilon_i = +1$, and $X^{i-1} = x^i$ and $Y^i = 0$ if $\epsilon_i = -1$.

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Notation

Definition

Under the assumption that $\lambda \supseteq \mu$ if $\varepsilon = -$, or $\lambda \subseteq \mu$ if $\varepsilon = +$, we write

$$\Psi^{\varepsilon}_{\lambda/\mu} = \begin{cases} \psi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \varphi_{\mu/\lambda} & \text{if } \varepsilon = +, \end{cases} \qquad \Phi^{\varepsilon}_{\lambda/\mu} = \begin{cases} \varphi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \psi_{\mu/\lambda} & \text{if } \varepsilon = +. \end{cases}$$

Definition

Here we assume $\lambda > \mu$ if $\delta = +1$, and $\lambda < \mu$ if $\delta = -1$. We also write

$$|\lambda - \mu|_{\delta} = \begin{cases} |\lambda - \mu| & \text{if } \delta = +1, \\ |\mu - \lambda| & \text{if } \delta = -1. \end{cases}$$

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Notation

Definition

Let n be a positive integer. Let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be a sequence of ± 1. Let $(\lambda^0, \lambda^1, \dots, \lambda^n)$ be an $(n+1)$-tuple of partitions such that $\lambda^{i-1} \succ \lambda^i$ if $\epsilon = +1$, and $\lambda^{i-1} \prec \lambda^i$ if $\epsilon = -1$. Then we write

$$\phi^{\epsilon}_{[\lambda^0,\lambda^1,\dots,\lambda^n]}(q,t) = \prod_{i=1}^{n} \phi^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(q,t), \qquad \psi^{\epsilon}_{[\lambda^0,\lambda^1,\dots,\lambda^n]}(q,t) = \prod_{i=1}^{n} \psi^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(q,t).$$

Definition

Let α be a strict partition, and let n be an integer such that $n \geq \alpha_1$. Define a sequence $\epsilon_n(\alpha) = (\epsilon_1(\alpha), \dots, \epsilon_n(\alpha))$ of ± 1 by putting

$$\epsilon_k(\alpha) = \begin{cases} +1 & \text{if } k \text{ is a part of } \alpha, \\ -1 & \text{if } k \text{ is not a part of } \alpha. \end{cases}$$

Notation

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Example of $\epsilon(\alpha)$ and kth trace $\pi[k]$

Definition

For each integer $k = 0, \ldots, n$ we define the *kth trace $\pi[k]$* to be the sequence $(\ldots, \pi_{2,k+2}, \pi_{1,k+1})$ obtained by reading the kth diagonal from SE to NW. Here we use the convention that $\pi[k] = \emptyset$ if $k \geq \alpha_1$.

Example

For example, if $\alpha = (8, 5, 2, 1)$ and $n = 10$, then we have $\epsilon = (++--+--+--)$.

We have $\pi[0] = (\pi_{44}, \pi_{33}, \pi_{22}, \pi_{11})$, $\pi[1] = (\pi_{34}, \pi_{23}, \pi_{12}), \ldots, \pi[10] = \emptyset$,

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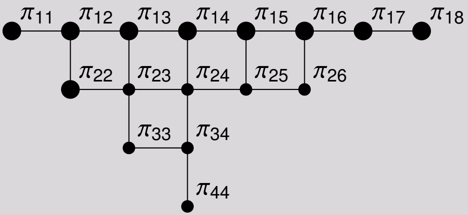

We have $\pi[0] = (\pi_{44}, \pi_{33}, \pi_{22}, \pi_{11})$, $\pi[1] = (\pi_{34}, \pi_{23}, \pi_{12})$,..., $\pi[10] = \emptyset$,

Tailed Inset Case

Definition

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that

$$\alpha_1 > \alpha_2 > \alpha_3 \geq 0, \qquad \beta_1 > \beta_2 \geq 0.$$

Let P be the set $P = P_{\mathrm{H}} \cup P_{\mathrm{M}} \cup P_{\mathrm{L}} \cup P_{\mathrm{R}} \cup P_{\mathrm{T}}$ of lattice points in $\mathbb{Z}^2$, where $P_{\mathrm{M}} = \{(2,1)\}$, $P_{\mathrm{T}} = \{(4,4)\}$ and

$$P_{\mathrm{H}} = \{(1,j) \,:\, -\beta_1 + 1 \leq j \leq 0\},$$
$$P_{\mathrm{R}} = \{(i,j) \,:\, 1 \leq i \leq j \leq \alpha_i + i \; (i = 1,2,3)\},$$
$$P_{\mathrm{L}} = \{(i+1,j+1) \,:\, 1 \leq j \leq i \leq \beta_j + j \; (j = 1,2)\}.$$

We regard P as a poset by defining the order relation

$$(i_1,j_1) \geq (i_2,j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$

if neither of (i_1,j_1) and (i_2,j_2) is not in P_{T}, whereas $(3,3) < (4,4)$. We call this poset a *Tailed Inset*, denoted by $P_5(\alpha,\beta)$.

Tailed Insets

Definition

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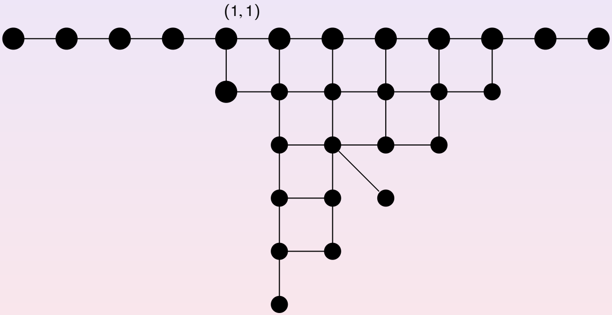
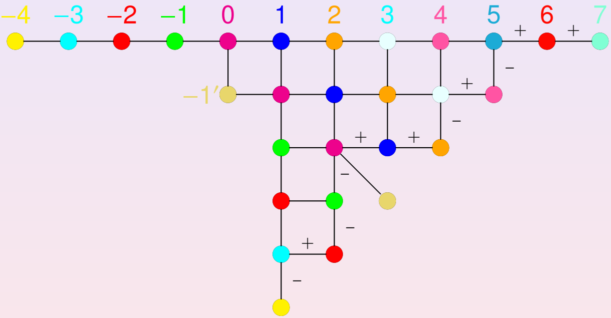
Tailed Insets

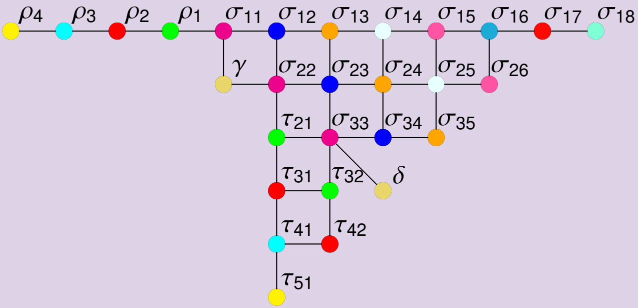


Tailed Insets



Definition

Let $\pi = (\sigma, \tau, \rho, \gamma, \delta) \in \mathscr{A}(P)$ be a P-partition as in the following figure.



Banner partition

Definition

Let p_i (resp. q_i) denote the number of vertices in the ith diagonal of λ (resp. μ) for $i \geq 1$, whereas we set $p_0 = 3$ and $q_0 = 2$. We define $\varepsilon = (\varepsilon_{c,c+1})_{c \in \mathbb{Z}}$ as follows. If $c \geq 1$,

$$\varepsilon_{c,c+1} = \begin{cases} + & \text{if } p_c = p_{c-1}, \\ - & \text{if } p_c = p_{c-1} - 1, \end{cases}$$

and if $c \leq 0$,

$$\varepsilon_{c,c+1} = \begin{cases} - & \text{if } q_{-c+1} = q_{-c}, \\ + & \text{if } q_{-c+1} = q_{-c} - 1. \end{cases}$$

The color of each vetex is shown in the figure above. In this example, we have $(p_i)_{i \geq 1} = (332211100\ldots)$, $(q_i)_{i \geq 1} = (221100\ldots)$ and $p_0 = 3$, $q_0 = 2$ by definition. Hence we have $\varepsilon_\lambda = (\cdots - - + - + - - + + - + - + + - + + \cdots)$ as in the above figure.

(q,t)-weight

Definition

If we set

$$W_N^{c,d}(\pi;q,t)=\prod_{\substack{x,y\in\widehat{P},\,x<y\\ c(x)=c\text{ and }c(y)=d}} f_{q,t}\left(\widehat{\pi}(x)-\widehat{\pi}(y);\left\lfloor\frac{\widehat{r}(y)-\widehat{r}(x)}{2}\right\rfloor\right)$$

if c and d are adjacent colors in $\widehat{T}$, and

$$W_D^{c,+}(\pi;q,t)=\prod_{\substack{x,y\in P,\,x<y\\ c(x)=c(y)=c}} f_{q,t}\left(\pi(x)-\pi(y),\left\lfloor\frac{r(y)-r(x)}{2}\right\rfloor\right),$$

$$W_D^{c,-}(\pi;q,t)=\prod_{\substack{x,y\in P,\,x<y\\ c(x)=c(y)=c}} f\left(\pi(x)-\pi(y);\left\lfloor\frac{r(y)-r(x)}{2}\right\rfloor-1\right),$$

then we have $W_P(\pi;q,t)=\dfrac{\prod_{c\text{ and }d\text{ are adjacent in }\widehat{T}} W_N^{c,d}(\pi;q,t)}{\prod_{c\text{ all colors in }T} W_D^{c}(\pi;q,t)}$. where $W_D^c(\pi;q,t)=W_D^{c,+}(\pi;q,t)W_D^{c,-}(\pi;q,t)$.

$W_P(\pi; q, t)$ for Banners

Definition

If λ and μ are partitions such that $\lambda - \mu$ is a horizontal strip, then it is known that

$$\psi_{\lambda/\mu}(q,t) = \prod_{1 \le i \le j \le \ell(\mu)} \frac{f_{q,t}(\lambda_i - \mu_j; j-i) f_{q,t}(\mu_i - \lambda_{j+1}; j-i)}{f_{q,t}(\mu_i - \mu_j; j-i) f_{q,t}(\lambda_i - \lambda_{j+1}; j-i)},$$

$$\varphi_{\lambda/\mu}(q,t) = \prod_{1 \le i \le j \le \ell(\lambda)} \frac{f_{q,t}(\lambda_i - \mu_j; j-i) f_{q,t}(\mu_i - \lambda_{j+1}; j-i)}{f_{q,t}(\lambda_i - \lambda_j; j-i) f_{q,t}(\mu_i - \mu_{j+1}; j-i)}.$$

Under the assumption that $\lambda \supseteq \mu$ if $\varepsilon = -$, or $\lambda \subseteq \mu$ if $\varepsilon = +$, we write

$$\Psi^{\varepsilon}_{\lambda/\mu} = \begin{cases} \psi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \varphi_{\mu/\lambda} & \text{if } \varepsilon = +, \end{cases} \qquad \Phi^{\varepsilon}_{\lambda/\mu} = \begin{cases} \varphi_{\lambda/\mu} & \text{if } \varepsilon = -, \\ \psi_{\mu/\lambda} & \text{if } \varepsilon = +. \end{cases}$$

Definition

I) For $0 \le c \le \lambda_1$, we define the partition Λ^c of length $\le p_c$ by

$$\Lambda^c = (\sigma_{p_c,p_c+c},\dots,\sigma_{1,1+c}) = (\sigma_{p_c+1-i,p_c+1-i+c})_{1\le i\le p_c}.$$

II) Now we set

$$\Lambda^{-1} = (\underbrace{\tau_{q_1+1,q_1},\dots,\tau_{2,1}}_{q_1},\gamma,\rho_1),$$

where $q_1 = 1$ or 2.

III) If $-\mu_1 \le c \le -2$, then we set

$$\Lambda^c = (\underbrace{\tau_{q_{-c}-c,q_{-c}},\dots,\tau_{1-c,1}}_{q_{-c}},\underbrace{\gamma,\dots,\gamma}_{-c},\rho_{-c}),$$

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(q, t)-weight by Pieri coefficient

Theorem

If $P = P_5(\lambda, \mu)$ is the Tailed Insets corresponding to strict partitions λ and μ, then we have

$$W_P(\pi; q, t) = \frac{f_{q,t}(\gamma; 0) \prod_{i=1}^{3} f_{q,t}(\delta - \sigma_{i,i}; 3 - i)}{f_{q,t}(\delta - \gamma; 2) f_{q,t}(\delta - \gamma; 1)} \prod_{c=-\mu_1-1}^{\lambda_1} \Psi_{\Lambda_c/\Lambda_{c+1}}^{\varepsilon_{c,c+1}}.$$

Proposition

We set

$$Z_c = \prod_{k=-\mu_1-1}^{c} z_k, \qquad Z_{c.d} = \frac{Z_d}{Z_c} = \prod_{k=c+1}^{d} z_k,$$

where $z_{-\mu_1-1}$ is a dummy variable which does not appear in the original weight. Then we have $z^\pi = \dfrac{z_{-1'}^{\gamma+\delta}}{\prod_{c=-\mu_1-1}^{-1} Z_c^{\gamma}} \cdot \prod_{c=-\mu_1-1}^{\lambda_1} Z_c^{|\Lambda^c|-|\Lambda^{c+1}|}$.

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Definition

We use the convention that $\varepsilon_{-\mu_1-1,-\mu_1} = +$ and $\varepsilon_{c,c+1} = -$ for $c < -\mu_1 - 1$. Note that $\sharp\{c < 0 \,|\, \varepsilon_{c,c+1} = +\} = 2$. Because $\varepsilon_{-\mu_1-1,-\mu_1} = +$ and $\varepsilon_{c,c+1} = -$ for $c < -\mu_1 - 1$, we may set

$$\{c < 0 \,|\, \varepsilon_{c,c+1} = +\} = \{c_1^-, c_2^-\}.$$

where $-\mu_1 - 1 = c_2^- < c_1^- < 0$ Also note that $\sharp\{c \geq 0 \,|\, \varepsilon_{c,c+1} = -\} = 3$. Because $\varepsilon_{\lambda_1,\lambda_1+1} = -$ and $\varepsilon_{c,c+1} = +$ for $c > \lambda_1$, we may set

$$\{c \geq 0 \,|\, \varepsilon_{c,c+1} = -\} = \{c_1^+, c_2^+, c_3^+\}.$$

where $0 \leq c_1^+ < c_2^+ < c_3^+ = \lambda_1$. Hence we have

$$-\mu_1 - 1 = c_2^- < c_1^- < 0 \leq c_1^+ < c_2^+ < c_3^+ = \lambda_1.$$

Theorem

$$\sum W_P(\pi; q,t) z^\pi = \prod_{\substack{0\le i<j \\ \varepsilon_{i,i+1}=+ \\ \varepsilon_{j,j+1}=-}} \Pi(Z_i^{-1}; Z_j; q,t) \prod_{\substack{i<j<0 \\ \varepsilon_{i,i+1}=+ \\ \varepsilon_{j,j+1}=-}} \Pi(Z_i^{-1}; Z_j; q,t)$$

$$\times \sum_{\substack{\sigma_{1,1},\sigma_{2,2},\sigma_{3,3} \\ \gamma,\delta,\nu_3}} \frac{f_{q,t}(\gamma; 0) \prod_{i=1}^{3} f_{q,t}(\delta - \sigma_{i,i}; 3-i)}{f_{q,t}(\delta-\gamma; 2) f_{q,t}(\delta-\gamma; 1)} \cdot \frac{z_{-1'}^{\gamma+\delta}}{\prod_{c=-\mu_1-1}^{-1} Z_c^{\gamma}}$$

$$\times P_{\Lambda^0}(Z_{c_2^+}, Z_{c_3^+}, Z_{\lambda_1}; q,t)$$

$$\times Q_{\Lambda^0/\nu}(Z_{-\mu_1-1}^{-1}, Z_{c_1^-}^{-1}; q,t) P_{\Lambda^{-\mu_1-1}/\nu}(Z_{-\mu_1}, \dots, \widehat{Z_{c_1^-}}, \cdots Z_{-1}; q,t),$$

where the sum runs over

$$0 \le \nu_3 \le \sigma_{1,1} \le \gamma \le \sigma_{2,2} \le \sigma_{3,3} \le \delta$$

and $\Lambda^0 = (\sigma_{3,3}, \sigma_{2,2}, \sigma_{1,1})$, $\Lambda^{-\mu_1-1} = (\underbrace{\gamma, \gamma, \dots, \gamma}_{\mu_1+1})$ and $\nu = (\gamma, \gamma, \nu_3)$.

Right-hand side

Definition

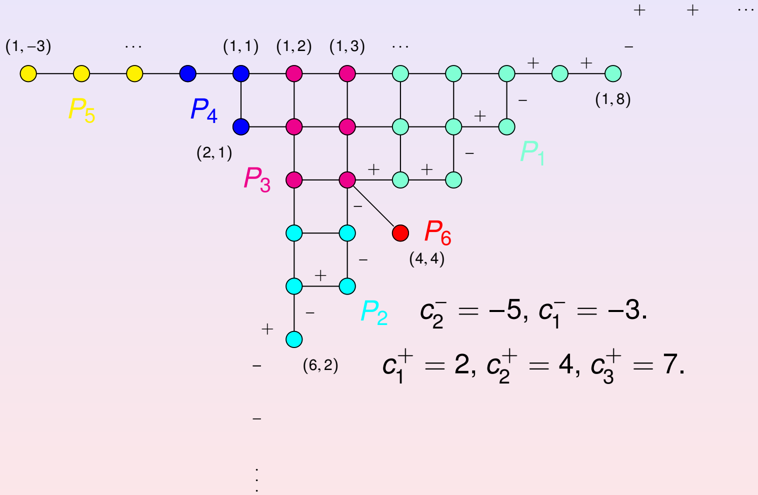
We put $P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$, where

$$
\begin{aligned}
P_1 &= \{(i, i + c) \,|\, j > 3,\ 1 \le i \le p_i,\ 1 \le c \le \lambda_1, \} \\
P_2 &= \{(j + 1 - c, j + 1) \,|\, i > 3,\ 1 \le j \le q_c,\ -\mu_1 \le c \le -1\}, \\
P_3 &= \{(i, j) \,|\, 1 \le i \le 3,\ 2 \le j \le 3\}, \\
P_4 &= \{(2, 1), (1, 1), (1, 0)\}, \\
P_5 &= \{(1, c + 1) \,|\, -\mu_1 \le c \le -2\}, \\
P_6 &= \{(4, 4)\}
\end{aligned}
$$

and we write

$$R_i = \prod_{v \in P_i} F\left(z\left[H_P(v)\right]; q, t\right),$$

for $i = 1, \ldots, 6$.



$$c_2^- = -5,\ c_1^- = -3.$$

$$c_1^+ = 2,\ c_2^+ = 4,\ c_3^+ = 7.$$

Interpretation of RHS

Proposition

By direct computation, is is not hard to see

$$R_1 = \prod_{\substack{0\le i<j \\ \varepsilon_{i,i+1}=+ \\ \varepsilon_{j,j+1}=-}} F\left(Z_{i,j};q,t\right),$$

$$R_2 = \prod_{\substack{i<j<0 \\ \varepsilon_{i,i+1}=+ \\ \varepsilon_{j,j+1}=-}} F\left(Z_{i,j};q,t\right).$$

$$\begin{aligned} R_3 &= \prod_{\substack{i<0\le j \\ \varepsilon_{i,i+1}=+ \\ \varepsilon_{j,j+1}=-}} F\left(wZ_{i,j};q,t\right) \\ &= F\left(wZ_{c_1^-,c_1^+};q,t\right) F\left(wZ_{c_1^-,c_2^+};q,t\right) F\left(wZ_{c_1^-,c_3^+};q,t\right) \\ &\quad \times F\left(wZ_{c_2^-,c_1^+};q,t\right) F\left(wZ_{c_2^-,c_2^+};q,t\right) F\left(wZ_{c_2^-,c_3^+};q,t\right) \end{aligned}$$

Interpretation of RHS

Proposition

$$R_4 = F\left(wZ_{c_2^-,c_1^+}Z_{c_1^-,c_2^+};q,t\right)F\left(wZ_{c_2^-,c_1^+}Z_{c_1^-,c_3^+};q,t\right)F\left(wZ_{c_2^-,c_3^+}Z_{c_1^-,c_2^+};q,t\right)$$

$$P_5 = \prod_{\substack{c=-\mu_1\\ c\neq c_1^-}}^{-1} F\left(w^2 Z_{c_2^-,c_1^+}Z_{c_1^-,c_2^+}Z_{c,c_3^+};q,t\right)$$

The end

Thank you for your attention!