A Pieri-type formula and a factorization formula for K-k-Schur functions

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Introduction

K-k-Schur functions $g_\lambda^{(k)}$
:symmetric functions parametrized by k-bounded partitions λ

Roughly speaking ... they appear in "type affine A combinatorics"

(A)
partitions $\mathcal{P}$, Schur functions s_λ, symmetric groups S_n, Grassmannian, ...

(affine A)
k-bounded partitions $\mathcal{P}_k$, $k+1$-core partitions $\mathcal{C}_{k+1}$, affine symmetric groups $\tilde{S}_{k+1}$, affine Grassmannian, k-Schur functions $s_\lambda^{(k)}$, K-k-Schur functions $g_\lambda^{(k)}$,...

Introduction

History

(conjecturally) equivalent definitions of the k-Schur function $s_\lambda^{(k)}$:

- Using *tableaux atoms* (Lascoux, Lapointe and Morse, 2003)
- Using a certain symmetric function operator (Lapointe and Morse, 2003)
- Using a Pieri-type formula (Lapointe and Morse, 2007)
  - shown to be the Schubert basis of the homology of the affine Grassmannian (Lam, 2008)
  - K-theoretic version: K-k-Schur function $g_\lambda^{(k)}$
    - as the Schubert basis of the K-homology of the affine Grassmannian (Lam, Schilling and Shimozono, 2010)
    - Pieri-type formula (Morse, 2012) $\leftarrow$ **we start here**
- As a generating function of *strong marked tableaux* (Lam, Lapointe, Morse and Shimozono, 2010)

Introduction

Outline

* Start with the Pieri formula for K-k-Schur functions $g_\lambda^{(k)}$
* Consider $\widetilde{g}_\lambda^{(k)} := \sum_{\mu \leq \lambda} g_\mu^{(k)}$
* Prove a Pieri-type formula

$$\widetilde{g}_\lambda^{(k)} \widetilde{h}_r = \sum_{\mu} \pm \widetilde{g}_\mu^{(k)}$$

  where $\widetilde{h}_r = h_0 + h_1 + \cdots + h_r$.
* Prove a k-rectangle factorization formula

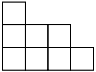
$$\widetilde{g}_{R_t \cup \lambda}^{(k)} = \widetilde{g}_{R_t}^{(k)} \widetilde{g}_\lambda^{(k)}.$$

  $(R_t = (t^{k+1-t}) = (t, \ldots, t))$
* Main tool: Bruhat orderings of affine symmetric groups

1. Background Objects
2. Weak strips and (K-)k-Schur functions
3. Results

Preliminaries: Type A

$\mathcal{P} = \{\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \mid \lambda_i \in \mathbb{Z}_{\geq 0},\ \sum_i \lambda_i < \infty\}$.

e.g. $(4,3,1) \longleftrightarrow$  its Young diagram (French notation)

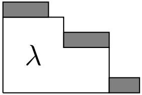
$\Lambda = \mathbb{Z}[h_1, h_2, \dots]$: the ring of symmectic functions,

where $h_r = \displaystyle\sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \dots x_{i_r}$.

Schur function s_λ $(\lambda \in \mathcal{P})$ is characterized by the Pieri rule:

$$h_r s_\lambda = \sum_{\mu/\lambda:\text{horizontal strip of size } r} s_\mu \qquad \text{for } \lambda \in \mathcal{P},\ r \geq 1$$

$(\mu/\lambda$: horizontal strip $\iff \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots)$



Preliminaries

In the theory of (K-)k-Schur functions,

Underlying objects: $\mathcal{P} \rightsquigarrow \mathcal{P}_k (\simeq \mathcal{C}_{k+1} \simeq \tilde{S}^{\circ}_{k+1})$

Pieri rule : horizontal strips $\rightsquigarrow$ weak strips

Fix $k \in \mathbb{Z}_{>0}$.

$\mathcal{P}_k = \{k\text{-bounded partitions}\} := \{\lambda \in \mathcal{P} \mid \lambda_1 \leq k\}$

$\mathcal{C}_{k+1} = \{k+1\text{-core partitions}\}$

$\quad := \{\kappa \in \mathcal{P} \mid \text{No hook with length} = k+1\}$

$$\tilde{S}_{k+1} = \langle s_0, s_1, \ldots, s_k \rangle / \begin{pmatrix} s_i^2 = 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \ (i - j \not\equiv 0, \pm 1) \end{pmatrix}$$

(all indices are mod $k+1$)

$S_{k+1} = \langle s_1, \ldots, s_k \rangle \subset \tilde{S}_{k+1}$.

$\tilde{S}^\circ_{k+1} = \{\text{affine Grassmannian elements}\}$

$\quad :=$ minimal length representatives of $\tilde{S}_{k+1}/S_{k+1}$

$\quad = \{w \in \tilde{S}_{k+1} \mid l(wx) = l(w) + l(x) \ (\forall x \in S_{k+1})\}$

$\quad = \{w \in \tilde{S}_{k+1} \mid \text{any reduced expression for } w \text{ ends in } s_0\}$.

Theorem

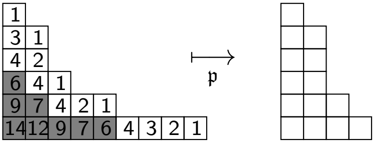
$\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \tilde{S}^\circ_{k+1}$.

Bijection $\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \tilde{S}^{\circ}_{k+1}$

$\mathfrak{p}\colon \mathcal{C}_{k+1} \longrightarrow \mathcal{P}_k; \kappa \mapsto \lambda$

given by $\lambda_i = \#\{j \mid (i,j) \in \kappa,\ \mathrm{hook}_{(i,j)}(\kappa) \leq k\}$.

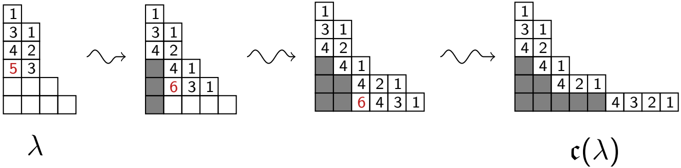
Example. $k = 4$ ($\mathcal{C}_5 \longrightarrow \mathcal{P}_4$)



"count the cells with hook length $\leq k$"

$\mathfrak{c}\colon \mathcal{P}_k \longrightarrow \mathcal{C}_{k+1}; \lambda \mapsto \kappa$

Example. $k = 4$ ($\mathcal{P}_3 \longrightarrow \mathcal{C}_4$)



for $i = l(\lambda), \ldots, 1$:
    while there is a cell with hook length $> k$ in i-th row :
        slide i-th row to the right

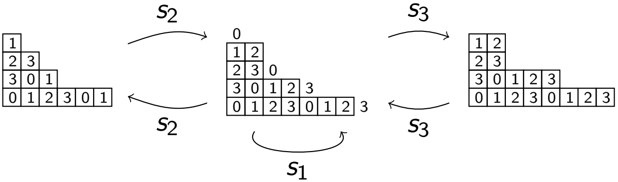
$\mathfrak{s} : \tilde{S}^{\circ}_{k+1} \longrightarrow \mathcal{C}_{k+1}$

Def. residue of a cell $(i, j) := j - i \mod k + 1 \in \mathbb{Z}_{k+1}$.

Action of s_i on $\mathcal{C}_{k+1}$: (exactly one of the following happens)

- ∗ if there exist addable corners of residue i, add them all
- ∗ if there exist removable corners of residue i, remove them all
- ∗ otherwise, do nothing

Example. $k = 3$

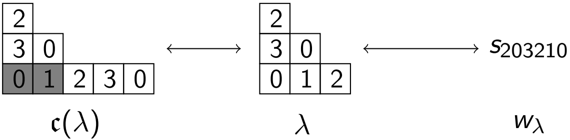


Fact This gives a well-defined action of $\tilde{S}_{k+1}$ on $\mathcal{C}_{k+1}$, and induces a bijection $\mathfrak{s} : \tilde{S}^{\circ}_{k+1} \longrightarrow \mathcal{C}_{k+1}; w \mapsto w \cdot \varnothing$.

$\mathcal{P}_k \longrightarrow \tilde{S}^{\circ}_{k+1}; \lambda \mapsto w_\lambda$

"reading the residues" from the shortest row to the largest, and within each row from right to left.

Example. $k = 3$ ($\mathcal{C}_4 \simeq \mathcal{P}_3 \simeq \tilde{S}^{\circ}_4$)



The strong and weak orders

(W, S): a Coxeter group
The (left) weak order $\leq_L$ is given by the covering relation

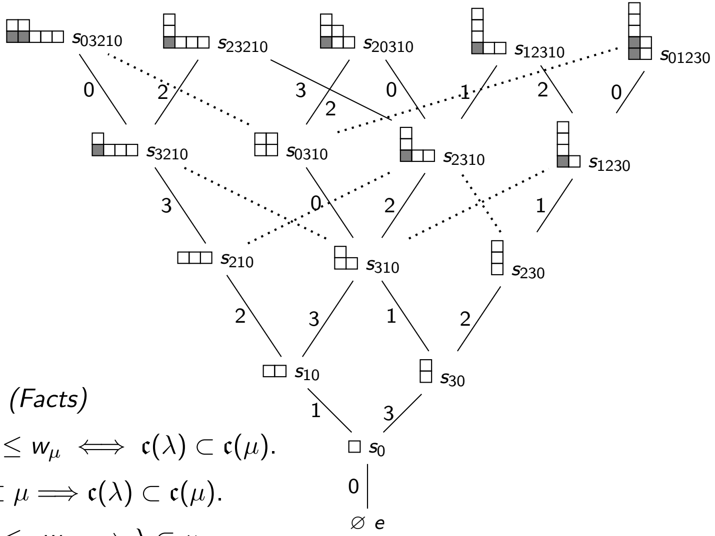
$$u \lessdot_L v \iff l(u) + 1 = l(v),\ su = v\ (\exists s \in S).$$

The strong order (Bruhat order) $\leq$ is given by the covering relation

$$\begin{aligned} u \lessdot v \iff & l(u) + 1 = l(v), \\ & u = s_1 \dots \widehat{s_i} \dots s_m\ (\exists i) \\ & \text{for } \exists \text{ reduced expression } v = s_1 \dots s_m. \end{aligned}$$

$\rightsquigarrow$ We induce $\leq, \leq_L$ on $\tilde{S}^{\circ}_{k+1} (\simeq \mathcal{C}_{k+1} \simeq \mathcal{P}_k)$.

Example. $\mathcal{P}_3 \simeq \mathcal{C}_4 \simeq \tilde{S}_4^\circ$. ($\mathfrak{c}(\lambda)$ and w_λ are displayed)



Remark. (Facts)

- $*$ $w_\lambda \leq w_\mu \iff \mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu)$.
- $*$ $\lambda \subset \mu \Longrightarrow \mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu)$.
- $*$ $w_\lambda \leq_L w_\mu \Longrightarrow \lambda \subset \mu$.

Definition (Cyclically decreasing sequence)

A sequence $(i_1, \ldots, i_m)$ *in* $\mathbb{Z}_{k+1} = \{0, 1, \ldots, k\}$ *is cyclically decreasing if*

$$\nexists a < b \text{ s.t. } i_b \equiv i_a \text{ or } i_a + 1 \mod k+1.$$

i.e. there is no appearance of

$$\ldots, j, \ldots, j, \ldots \quad \textit{nor}$$

$$\ldots, j, \ldots, j+1, \ldots$$

in $(i_1, \ldots, i_m)$.

Definition (Cyclically decreasing element)

For $A = \{i_1, \ldots, i_m\} \subsetneq \mathbb{Z}_{k+1}$ *where* $(i_1, \ldots, i_m)$ *is cyclically decreasing,*

$$d_A := s_{i_1} \ldots s_{i_m} \; (\in \tilde{S}_{k+1}).$$

Example. $k = 5$. $A = \{0, 1, 3, 5\} \subsetneq \mathbb{Z}_6$.
$\rightsquigarrow d_A = s_1 s_0 s_5 s_3 = s_1 s_0 s_3 s_5 = s_1 s_3 s_0 s_5 = s_3 s_1 s_0 s_5$.

Weak Strips and k-Schur functions

Definition (weak strip)

For $\lambda, \mu \in \mathcal{P}_k$, μ/λ: a weak strip of size r

$$:\iff \exists A \subsetneq \mathbb{Z}_{k+1} \text{ s.t. } |A| = r,\ w_\mu = d_A w_\lambda \geq_L w_\lambda$$

$$(\iff \exists A \subsetneq \mathbb{Z}_{k+1} \text{ s.t. } |A| = r,\ w_\mu = d_A w_\lambda,\ l(w_\mu) = l(d_A) + l(w_\lambda)\ )$$

Remark. In fact,

$$\mu/\lambda\text{: a weak strip} \iff \begin{cases} \mathfrak{c}(\mu)/\mathfrak{c}(\lambda)\text{: horizontal strip} \\ w_\mu \geq_L w_\lambda \end{cases}$$

$$\iff \begin{cases} \mu/\lambda\text{: horizontal strip} \\ \mathfrak{p}(\mathfrak{c}(\mu)')/\mathfrak{p}(\mathfrak{c}(\lambda)')\text{: vertical strip} \end{cases}.$$

Weak Strips and k-Schur functions

Definition (k-Schur function)

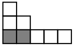
k-Schur functions $s_\lambda^{(k)}$ is characterized by $s_\varnothing^{(k)} = 1$ and

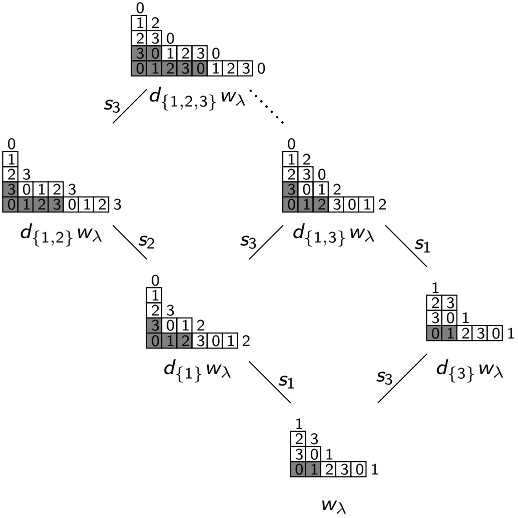
$$h_r s_\lambda^{(k)} = \sum_{\mu/\lambda:\ \text{weak strip of size } r} s_\mu^{(k)}. \quad (\text{ for } \lambda \in \mathcal{P}_k,\ r \leq k)$$

Remark.

$\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ is $\mathbb{Z}$-bases of $\Lambda^{(k)} = \mathbb{Z}[h_1, \ldots, h_k]$.

$s_\lambda^{(k)}$: homogeneous

Example. $k = 3$. $\lambda =$  $\in \mathcal{P}_3$, $\mathfrak{c}(\lambda) =$  $\in \mathcal{C}_4$, $w_\lambda = s_{203210} \in \tilde{S}_4^\circ$.



e.g.
$\{$weak strips$/w_\lambda$ of size 2$\}$
$= \{d_{1,2}w_\lambda, d_{1,3}w_\lambda\}$.

$$h_2 s_\lambda^{(3)} = s^{(3)}_{d_{\{1,3\}}\lambda} + s^{(3)}_{d_{\{1,2\}}\lambda}.$$

Weak set-valued strips and K-k-Schur functions

For $i \in \mathbb{Z}_{k+1}$, let $\phi_i \colon \tilde{S}_{k+1} \longrightarrow \tilde{S}_{k+1}$; $\phi_i(x) = \begin{cases} x & (\text{if } x > s_i x) \\ s_i x & (\text{if } x < s_i x) \end{cases}$.

For $x = s_{i_1} \dots s_{i_m}$ (red. exp.), let $\phi_x = \phi_{i_1} \dots \phi_{i_m}$.

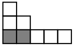
Definition (weak set-valued strip)

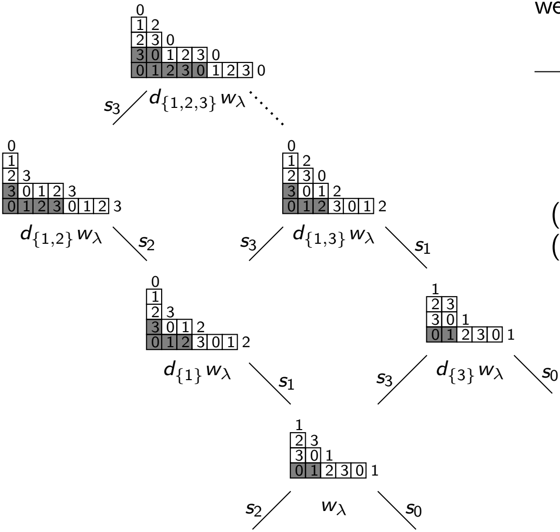
$(\mu/\lambda, A)$: a weak set-valued strip of size r
$:\iff \exists A \subsetneq \mathbb{Z}_{k+1}$ s.t. $|A| = r$, $w_\mu = \phi_{d_A}(w_\lambda)$.

Definition (K-k-Schur function)

K-k-Schur functions $g_\lambda^{(k)}$ is characterized by $g_\varnothing^{(k)} = 1$ and

$$h_r g_\lambda^{(k)} = \sum_{(\mu/\lambda, A):\ \text{weak set-valued strip of size } r} (-1)^{|A|-(|\mu|-|\lambda|)} g_\mu^{(k)}.$$

Example. $k=3$. $\lambda =$  $\in \mathcal{P}_3$, $\mathfrak{c}(\lambda) =$  $\in \mathcal{C}_4$, $w_\lambda = s_{203210} \in \tilde{S}_4^\circ$.



weak s-v. strips$/w_\lambda$ of size 2

(w_μ, A)	$(-1)^{\lvert A\rvert-(\lvert\mu\rvert-\lvert\lambda\rvert)}$
$(w_\lambda, \{0,2\})$	$+1$
$(d_1 w_\lambda, \{0,1\})$	-1
$(d_3 w_\lambda, \{2,3\})$	-1
$(d_3 w_\lambda, \{3,0\})$	-1
$(d_{1,2} w_\lambda, \{1,2\})$	$+1$
$(d_{1,3} w_\lambda, \{1,3\})$	$+1$

$$\begin{aligned} h_2 g^{(3)}_\lambda &= g^{(3)}_{d_{\{1,3\}}\lambda} + g^{(3)}_{d_{\{1,2\}}\lambda} \\ &\quad - g^{(3)}_{d_{\{1\}}\lambda} - 2g^{(3)}_{d_{\{3\}}\lambda} \\ &\quad + g^{(3)}_\lambda. \end{aligned}$$

Remark.

$\{g_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ in $\mathbb{Z}$-bases of $\Lambda^{(k)} = \mathbb{Z}[h_1, \ldots, h_k]$.

$g_\lambda^{(k)}$: inhomogeneous, highest degree term $= s_\lambda^{(k)}$

Auxiliary Results (A): On the lattice property on W

For $(P, \leq)$: poset, $a, b \in P$,
their meet $a \wedge b := \max\{c \in P \mid a \geq c \leq b\}$ (if max exists),
their join $a \vee b := \min\{c \in P \mid a \leq c \geq b\}$ (if min exists).

Let W: $\forall$Coxeter group, $x, y \in W$.

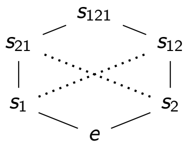
Fact

$x \wedge_L y = \max_{\leq_L}\{z \mid x \geq_L z \leq_L y\}$ exists.
$x \vee_L y = \min_{\leq_L}\{z \mid x \leq_L z \geq_L y\}$ exists if $\{z \mid x \leq_L z \geq_L y\} \neq \varnothing$.

Remark.

$x \wedge y = \max_{\leq}\{z \mid x \geq z \leq y\}$ and
$x \vee y = \min_{\leq}\{z \mid x \leq z \geq y\}$ do not always exist.

e.g. S_3



Auxiliary Results (A): On the lattice property on W

W: $\forall$Coxeter group
$x, y \in W$.

Lemma (T.)

$x\,{}_S{\wedge}_L\, y := \max_{\leq}\{z \in W \mid x \geq z \leq_L y\}$ exists.
$x\,{}_S{\vee}_L\, y := \min_{\leq}\{z \in W \mid x \leq z \geq_L y\}$ exists.

Sketch: present

$$\begin{aligned} x &= s_{i_1} \dots s_{i_n} \quad s_{j_{a_1}} \dots s_{j_{a_t}} \qquad (1 \leq a_1 < \cdots < a_l \leq m) \\ y &= \qquad\qquad s_{j_1} \dots\dots\dots s_{j_m} \end{aligned}$$

so that t is maximal. Then

$$\begin{aligned} x\,{}_L{\wedge}_S\, y &= \qquad\qquad s_{j_{a_1}} \dots s_{j_{a_t}} \\ x\,{}_S{\vee}_L\, y &= s_{i_1} \dots s_{i_n} s_{j_1} \dots\dots\dots s_{j_m} \end{aligned}$$

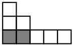
Auxiliary Results (B): On the weak strips

Lemma (T.)

$$d_A\lambda/\lambda,\ d_B\lambda/\lambda\text{: } \textit{w.s.} \implies \begin{cases} d_{A\cup B}\lambda/\lambda,\ d_{A\cap B}\lambda/\lambda\text{: } \textit{w.s.} \\ d_{A\cap B}\lambda = (d_A\lambda)\wedge(d_B\lambda) \text{ under } \le. \end{cases}$$

where "w.s." = weak strip, $\le$: strong order

$$\begin{tikzcd}[column sep=small]
 & s_3s_2s_1\lambda \arrow[dl, dash, "3"'] \arrow[dr, dotted, dash] & & \\
s_2s_1\lambda \arrow[dr, dash, "2"'] & & s_3s_1\lambda \arrow[dl, dash, "3"'] \arrow[dr, dash, "1"] & \\
 & s_1\lambda \arrow[dr, dash, "1"'] & & s_3\lambda \arrow[dl, dash, "3"] \\
 & & \lambda &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \{1,2,3\} \arrow[dl, dash, "3"'] \arrow[dr, dotted, dash] & & \\
\{1,2\} \arrow[dr, dash, "2"'] & & \{1,3\} \arrow[dl, dash, "3"'] \arrow[dr, dash, "1"] & \\
 & \{1\} \arrow[dr, dash, "1"'] & & \{3\} \arrow[dl, dash, "3"] \\
 & & \varnothing &
\end{tikzcd}$$

$k=3$. Weak strips over $\lambda =$  $\in \mathcal{P}_3$, $\mathfrak{c}(\lambda) =$  $\in \mathcal{C}_4$, $w_\lambda = s_{203210} \in \tilde{S}_4^\circ$.

Main Results (A) a Pieri-type formula

For $\lambda \in \mathcal{P}_k$ and $1 \leq r \leq k$, let

$$\widetilde{g}_\lambda^{(k)} = \sum_{\mu \leq \lambda} g_\mu^{(k)}, \qquad \widetilde{h}_r = h_0 + h_1 + \cdots + h_r.$$

Let $\{d_{A_1}\lambda, d_{A_2}\lambda, \dots\}$ be the list of weak strips over λ of size r.

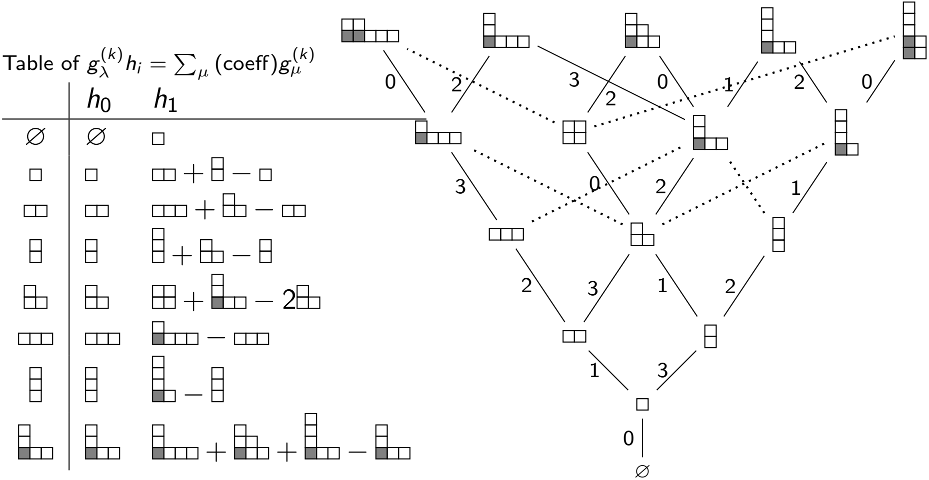
Theorem (T.)

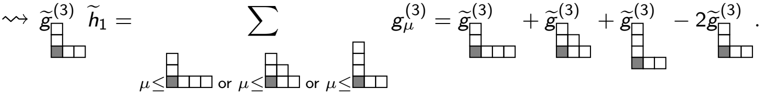
$$\begin{aligned}\widetilde{g}_\lambda^{(k)} \widetilde{h}_r &= \sum_{\mu \leq d_{A_i}\lambda \ (\exists i)} g_\mu^{(k)} \\ &= \sum_a \widetilde{g}^{(k)}_{d_{A_a}\lambda} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{A_a \cap A_b}\lambda} + \sum_{a<b<c} \widetilde{g}^{(k)}_{d_{A_a \cap A_b \cap A_c}\lambda} - \dots.\end{aligned}$$

(The second equiality is from $d_{B \cap C}\lambda = (d_B\lambda) \wedge (d_C\lambda)$ and the Inclusion-Exclusion Principle)

Example. $k = 3$.

Table of $g^{(k)}_\lambda h_i = \sum_\mu (\text{coeff}) g^{(k)}_\mu$





Main Results (B): k-rectangle factorization (1)

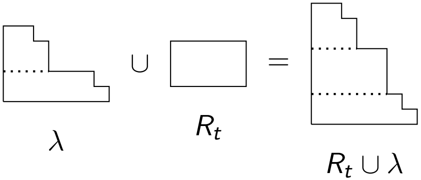
k-rectangle: $R_t = \underbrace{(t, \dots, t)}_{k+1-t} \in \mathcal{P}_k$ for $1 \le t \le k$

Theorem (Lapointe, Morse)

$s^{(k)}_{R_t \cup \lambda} = s^{(k)}_{R_t} s^{(k)}_{\lambda}$ *for* $\forall \lambda \in \mathcal{P}_k$.

Here,

$\mu \cup \nu =$ the partition obtained by reordering $(\mu_1, \dots, \mu_{l(\mu)}, \nu_1, \dots, \nu_{l(\nu)})$



Main Results (B): k-rectangle factorization (2)

Similar formula for $g_\lambda^{(k)}$?

$$\to \begin{cases} g_{R_t}^{(k)} \mid g_{R_t \cup \lambda}^{(k)} \text{ (in } \mathbb{Z}[h_1, \dots, h_k]\text{) is proved} \\ \text{but } g_{R_t \cup \lambda}^{(k)} \neq g_{R_t}^{(k)} g_\lambda^{(k)} \text{ in general} \end{cases}$$

Example.

$$g_{R_t \cup (r)}^{(k)} = \begin{cases} g_{R_t}^{(k)} \cdot g_{(r)}^{(k)} & (\text{if } t < r), \\ g_{R_t}^{(k)} \cdot \left(g_{(r)}^{(k)} + g_{(r-1)}^{(k)} + \cdots + g_{\varnothing}^{(k)} \right) & (\text{if } t \geq r) \end{cases}$$

$$g^{(3)}_{(3,2,1)} = g^{(3)}_{(3)} \left(g^{(3)}_{(2,1)} + g^{(3)}_{(1,1)} \right)$$

$$g^{(3)}_{(2,2,2,1,1)} = g^{(3)}_{(2,2)} \left(g^{(3)}_{(2,1,1)} + g^{(3)}_{(3)} + g^{(3)}_{(2,1)} + g^{(3)}_{(1,1,1)} + g^{(3)}_{(2)} + g^{(3)}_{(1,1)} + g^{(3)}_{(1)} + g^{(3)}_{\varnothing} \right)$$

Main Results (B): *k*-rectangle factorization (3)

Theorem (T.)

For any $\lambda \in \mathcal{P}_k$ and $1 \le t \le k$,

$$\widetilde{g}^{(k)}_{R_t \cup \lambda} = \widetilde{g}^{(k)}_{R_t} \widetilde{g}^{(k)}_{\lambda}.$$

Remark. $s^{(k)}_\lambda \mapsto \widetilde{g}^{(k)}_\lambda$ is not a ring homomorphism.

Outline of the proof: Pieri

Write simply $W = \tilde{S}_{k+1}$ and $W^\circ = \tilde{S}^\circ_{k+1}$. Recall the Pieri rule

$$g^{(k)}_v h_i = \sum_{\substack{A \subset I, |A|=i \\ \phi_{d_A}(v) \in W^\circ}} (-1)^{i-(l(\phi_{d_A}(v))-l(v))} g^{(k)}_{\phi_{d_A}(v)}.$$

Summing over $v \in \{v \in W^\circ \mid v \leq w\}$ and $i = 0, 1, \ldots, r$,

$$\widetilde{g}^{(k)}_w \widetilde{h}_r = \sum_{\substack{v \leq w \\ v \in W^\circ}} \sum_{\substack{A \subset I, |A| \leq r \\ \phi_{d_A}(v) \in W^\circ}} (-1)^{|A|-(l(\phi_{d_A}(v))-l(v))} g^{(k)}_{\phi_{d_A}(w)},$$

and its coefficient of $g^{(k)}_u$ $(u \in W^\circ)$ is

$$\sum_{\substack{v \leq w \\ v \in W^\circ \\ A \subset I, |A| \leq r \\ u = \phi_{d_A}(v)}} (-1)^{|A|-(l(u)-l(v))} = \sum_{|A| \leq r} \sum_{\substack{v \leq w \\ v \in W^\circ \\ u = \phi_{d_A}(v)}} (-1)^{|A|-(l(u)-l(v))}.$$

Outline of the proof: Pieri (cont.)

…and its coefficient of $g_u^{(k)} \in W^\circ$ is

$$\sum_{\substack{v \le w \\ v \in W^\circ \\ A \subset I, |A| \le r \\ u = \phi_{d_A}(v)}} (-1)^{|A| - (l(u) - l(v))} = \sum_{|A| \le r} \sum_{\substack{v \le w \\ v \in W^\circ \\ u = \phi_{d_A}(v)}} (-1)^{|A| - (l(u) - l(v))}$$

Outline of the proof: Pieri (cont.)

…and its coefficient of $g_u^{(k)} \in W^\circ$ is

$$\sum_{\substack{v \leq w \\ v \in W^\circ \\ A \subset I, |A| \leq r \\ u = \phi_{d_A}(v)}} (-1)^{|A|-(l(u)-l(v))} = \sum_{|A| \leq r} \sum_{\substack{v \leq w \\ v \in W^\circ \\ u = \phi_{d_A}(v)}} (-1)^{|A|-(l(u)-l(v))}$$

$$= \sum_{|A| \leq r} \sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))},$$

where $\qquad Y_{A,u} := \{v \in W^\circ \mid \phi_{d_A}(v) = u \text{ and } v \leq w\}.$

Step 1. $Y_{A,u}$ is isomorphic to boolean lattices. Hence

$$\sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))} = \begin{cases} 1 & \text{if } |Y_{A,u}| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Outline of the proof: Pieri

Step 2. $|Y_{A,u}| = 1 \iff d_A^{-1} u = \max\{z \mid w \geq z \leq_L u\} (= w_S \wedge_L u)$.
Hence

$$\sum_{|A| \leq r} \sum_{v \in Y_{A,u}} (-1)^{|A| - (l(u) - l(v))} = 1 \iff \exists A \text{ s.t. } \begin{cases} |A| \leq r, \\ d_A^{-1} u = w_S \wedge_L u. \end{cases}$$

Step 3.

$$\exists A \text{ s.t. } \begin{cases} |A| \leq r, \\ d_A^{-1} u = w_S \wedge_L u \end{cases}$$

$$\iff u \leq d_C w \text{ for } \exists C \text{ s.t. } \begin{cases} |C| \leq r \\ d_C w / w \text{: weak strip} \end{cases}$$

□

Outline of the proof: k-rectangle factorization

(Basically similar to the proof of $s^{(k)}_{R_t \cup \lambda} = s^{(k)}_{R_t} s^{(k)}_{\lambda}$)

Define a linear map $\Theta : \Lambda^{(k)} \longrightarrow \Lambda^{(k)}$ by $\widetilde{g}^{(k)}_{\lambda} \mapsto \widetilde{g}^{(k)}_{R_t \cup \lambda}$ $(\forall \lambda \in \mathcal{P}_k)$.
It suffices to show

$$\Theta(\widetilde{h}_r \widetilde{g}^{(k)}_{\lambda}) = \widetilde{h}_r \Theta(\widetilde{g}^{(k)}_{\lambda}). \tag{1}$$

(since it implies Θ is $\Lambda^{(k)}$-hom and

$$\widetilde{g}^{(k)}_{R_t \cup \lambda} = \Theta(\widetilde{g}^{(k)}_{\lambda}) = \widetilde{g}^{(k)}_{\lambda} \Theta(1) = \widetilde{g}^{(k)}_{\lambda} \Theta(\widetilde{g}^{(k)}_{\varnothing}) = \widetilde{g}^{(k)}_{\lambda} \widetilde{g}^{(k)}_{R_t}$$

)

Outline of the proof: k-rectangle factorization

Let $\{d_{A_1}\lambda, d_{A_2}\lambda, \dots\}$ be the list of weak strips over λ of size r.
From the Pieri rule for $\widetilde{g}^{(k)}_\lambda$,

$$\widetilde{h}_r \widetilde{g}^{(k)}_\lambda = \sum_a \widetilde{g}^{(k)}_{d_{A_a}\lambda} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{A_a \cap A_b}\lambda} + \dots,$$

Applying Θ to this, (LHS) of (1) is

$$\Theta(\widetilde{h}_r \widetilde{g}^{(k)}_\lambda) = \sum_a \widetilde{g}^{(k)}_{R_t \cup d_{A_a}\lambda} - \sum_{a<b} \widetilde{g}^{(k)}_{R_t \cup d_{A_a \cap A_b}\lambda} + \dots.$$

Outline of the proof: k-rectangle factorization

$\{d_{A_1}\lambda, d_{A_2}\lambda, \dots\}$: the list of weak strips over λ of size r.

Fact

(a) $\{d_{A_1+t}(R_t \cup \lambda), d_{A_2+t}(R_t \cup \lambda), \dots\}$ *is the list of weak strips over* $R_t \cup \lambda$ *of size* r.

(b) $d_{A_i+t}(R_t \cup \lambda) = R_t \cup (d_A\lambda)$.

From the Pieri rule for $\widetilde{g}^{(k)}_\lambda$ and Fact(a), (RHS) of (1) is

$$\begin{aligned}\widetilde{h}_r\Theta(\widetilde{g}^{(k)}_\lambda) &= \widetilde{h}_r\widetilde{g}^{(k)}_{R_t\cup\lambda} \\ &= \sum_a \widetilde{g}^{(k)}_{d_{A_a+t}(R_t\cup\lambda)} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{(A_a+t)\cap(A_b+t)}(R_t\cup\lambda)} + \dots .\end{aligned}$$

From Fact (b), this equals to (LHS) of (1).

□