# SYMMETRIES OF SHAMROCKS

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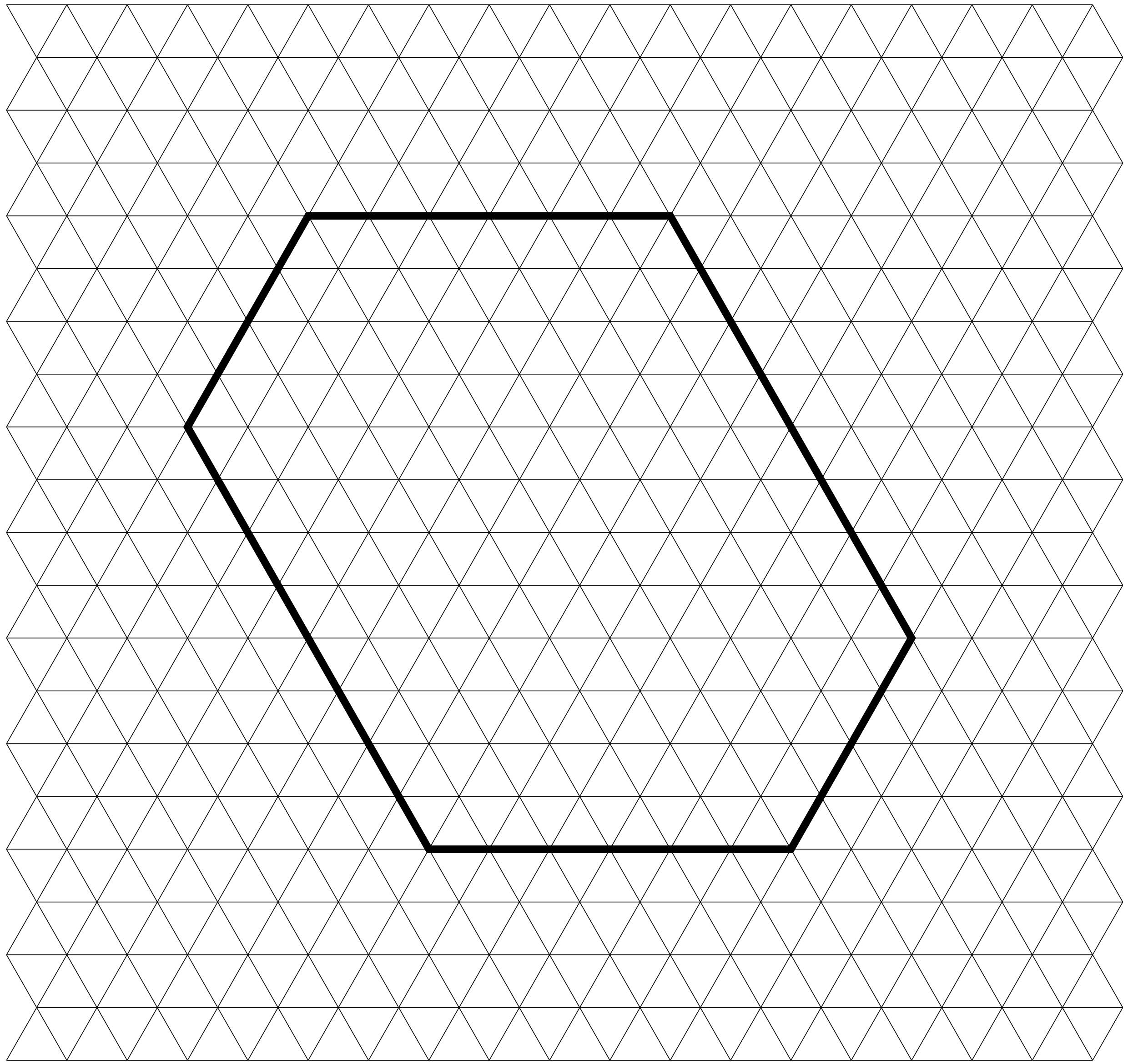
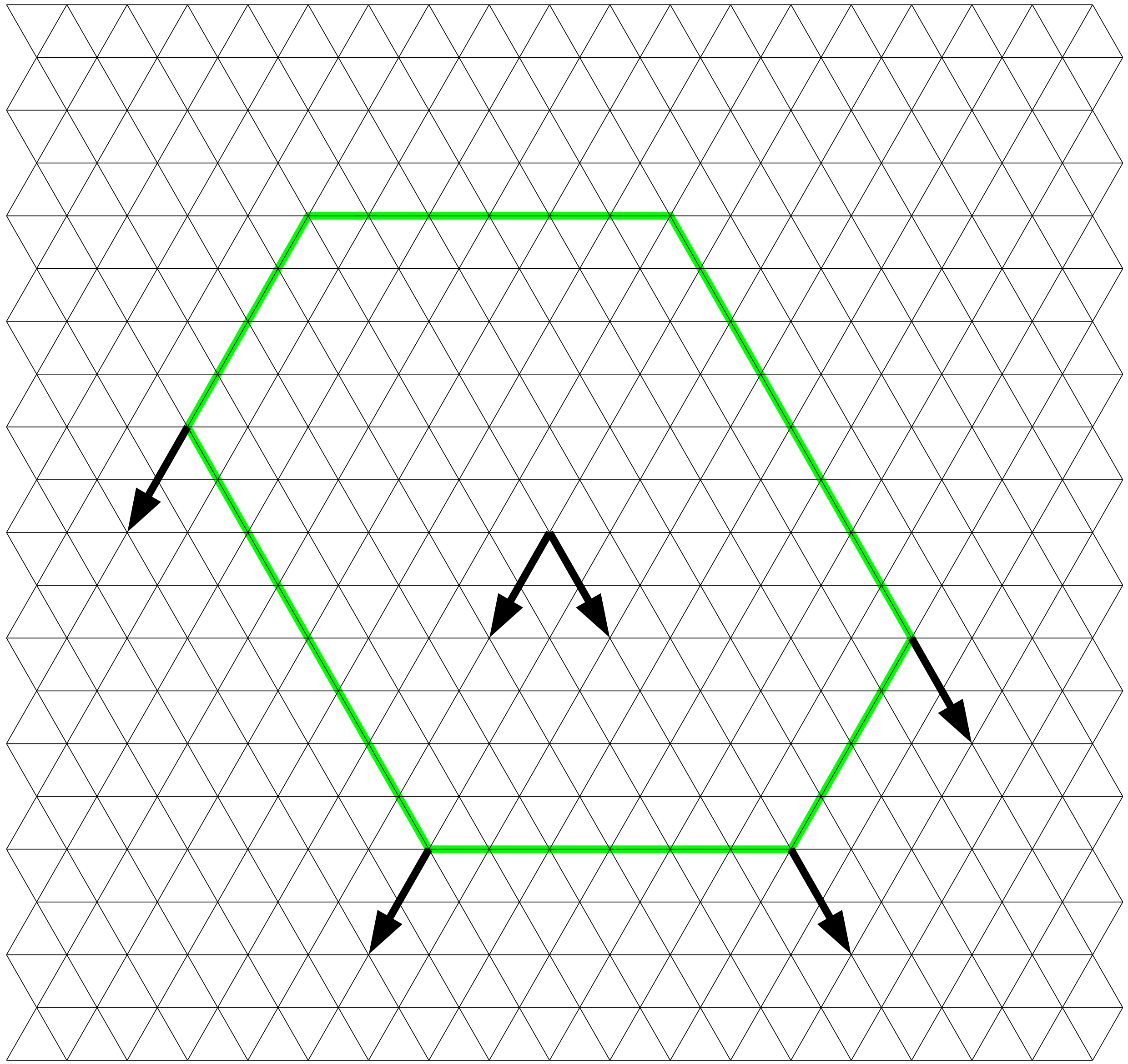
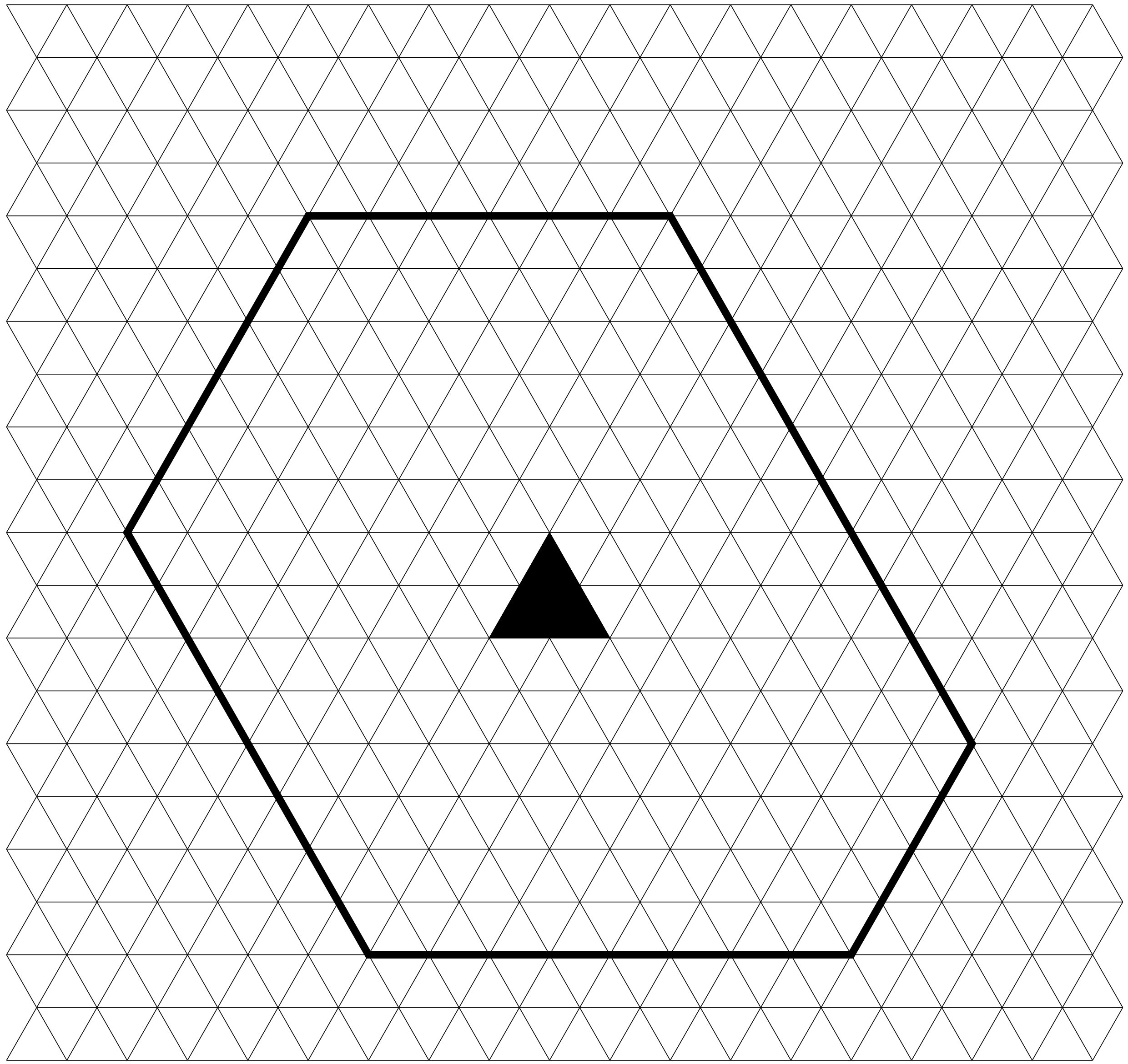
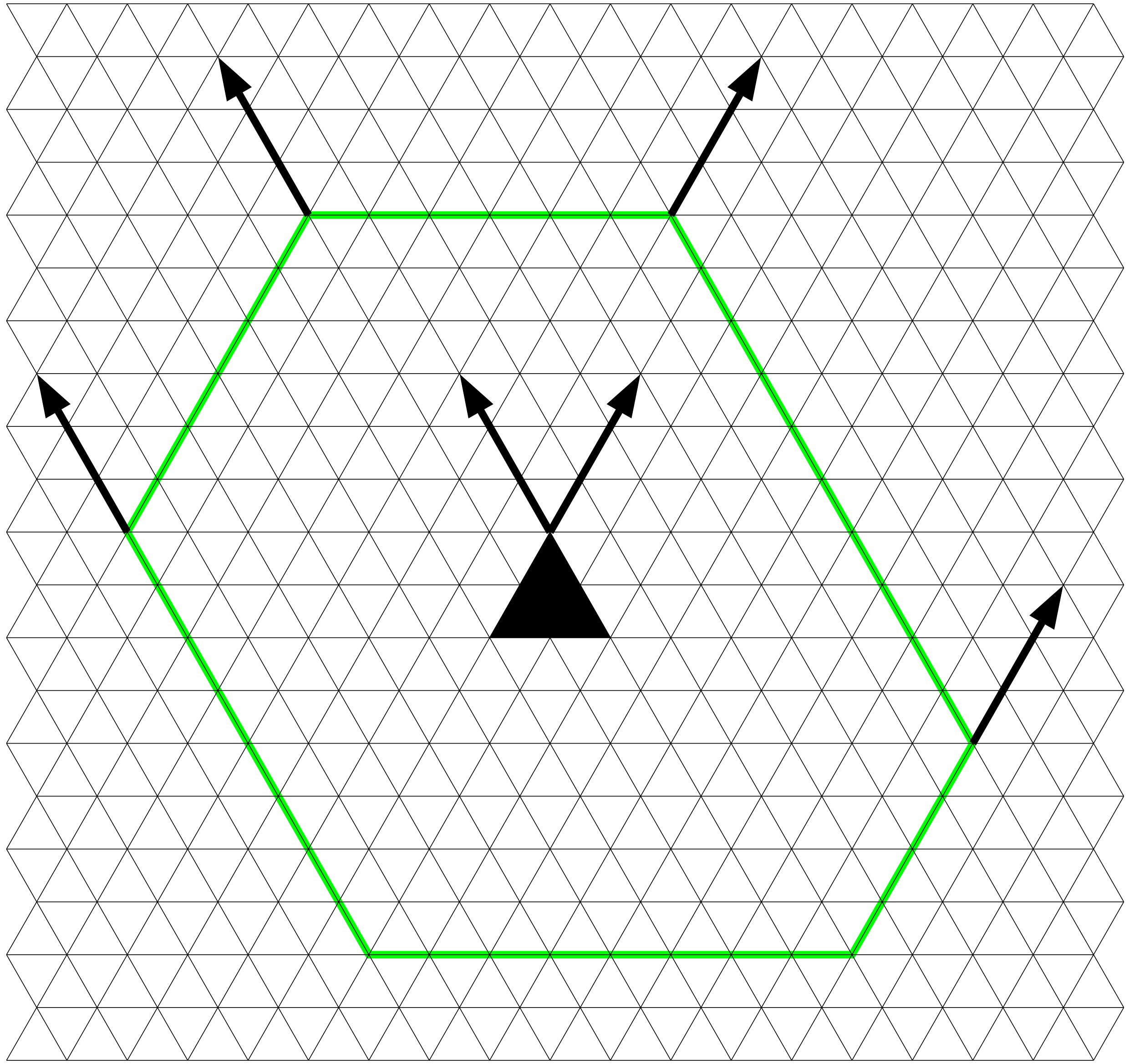


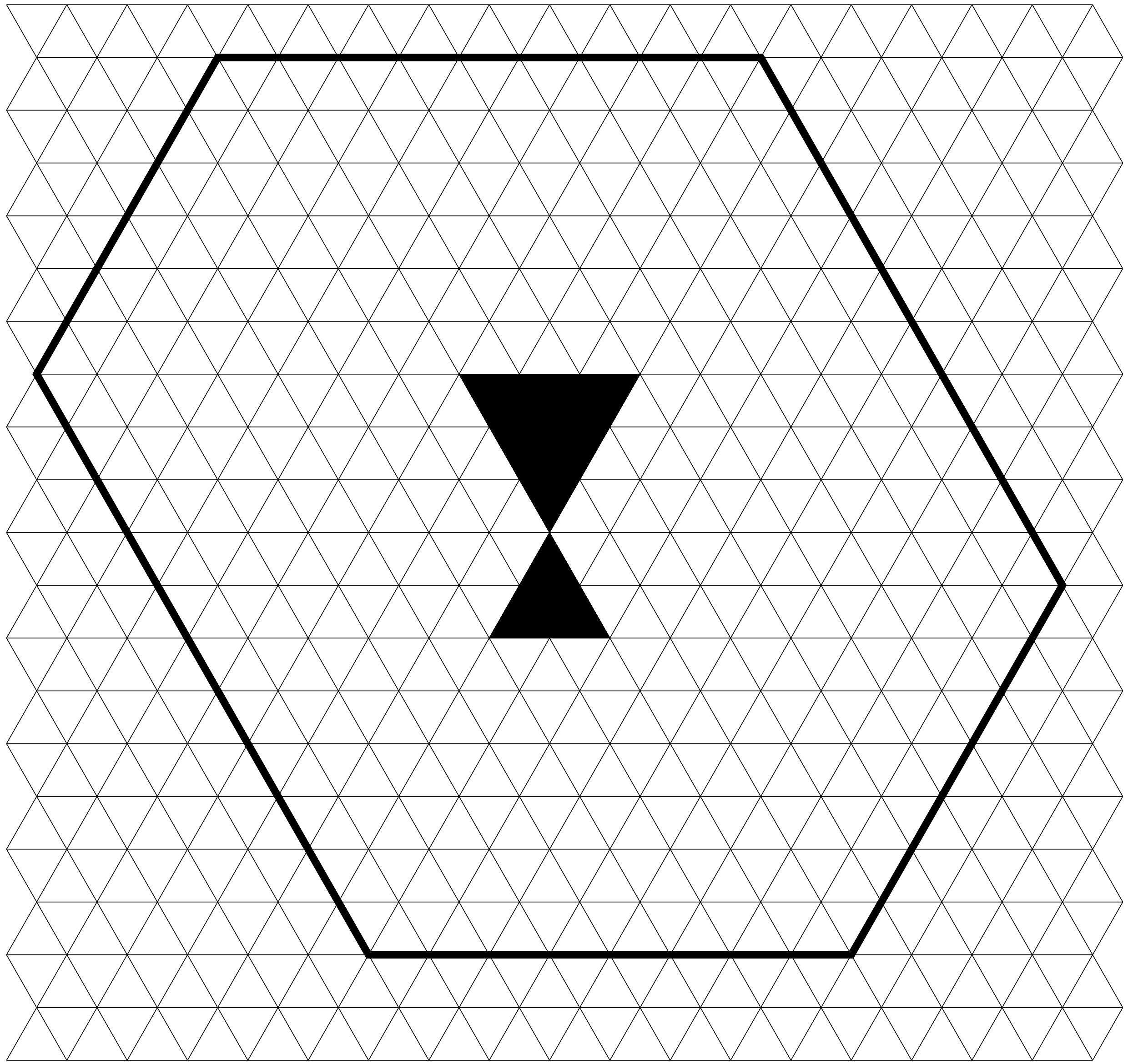
The hexagon $H_{6,8,4}$



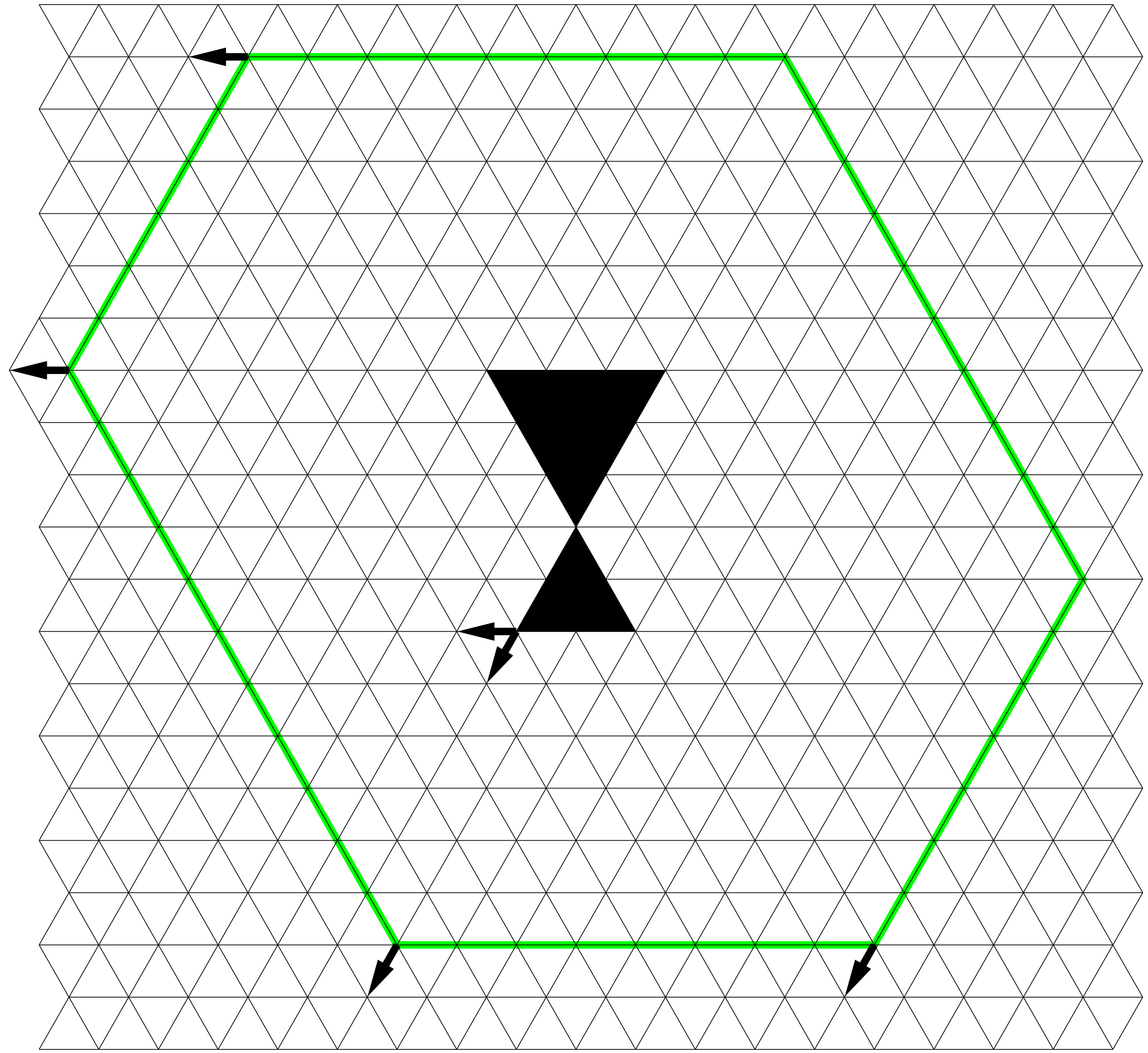


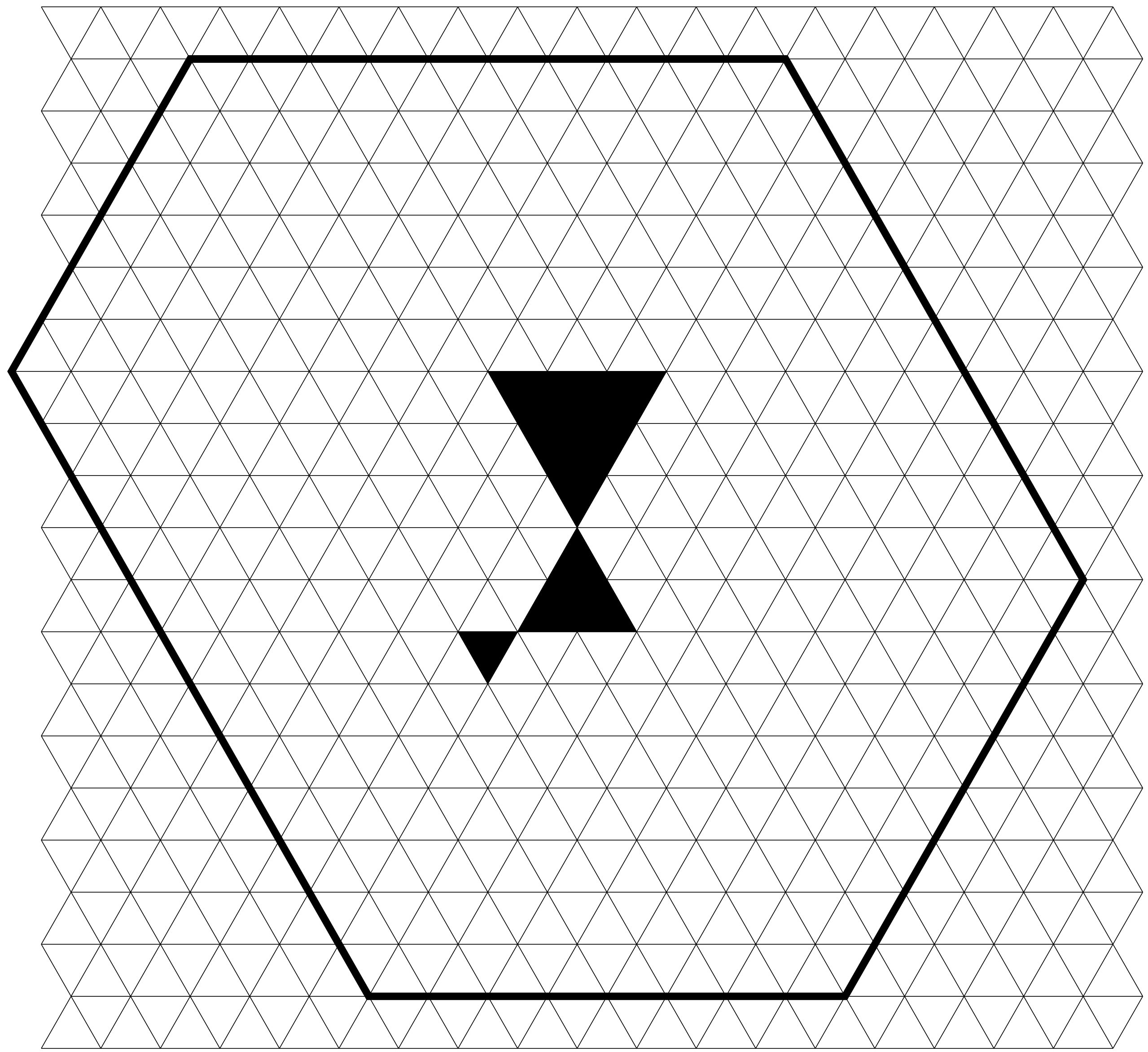
The cored hexagon $C_{6,8,4}(2)$



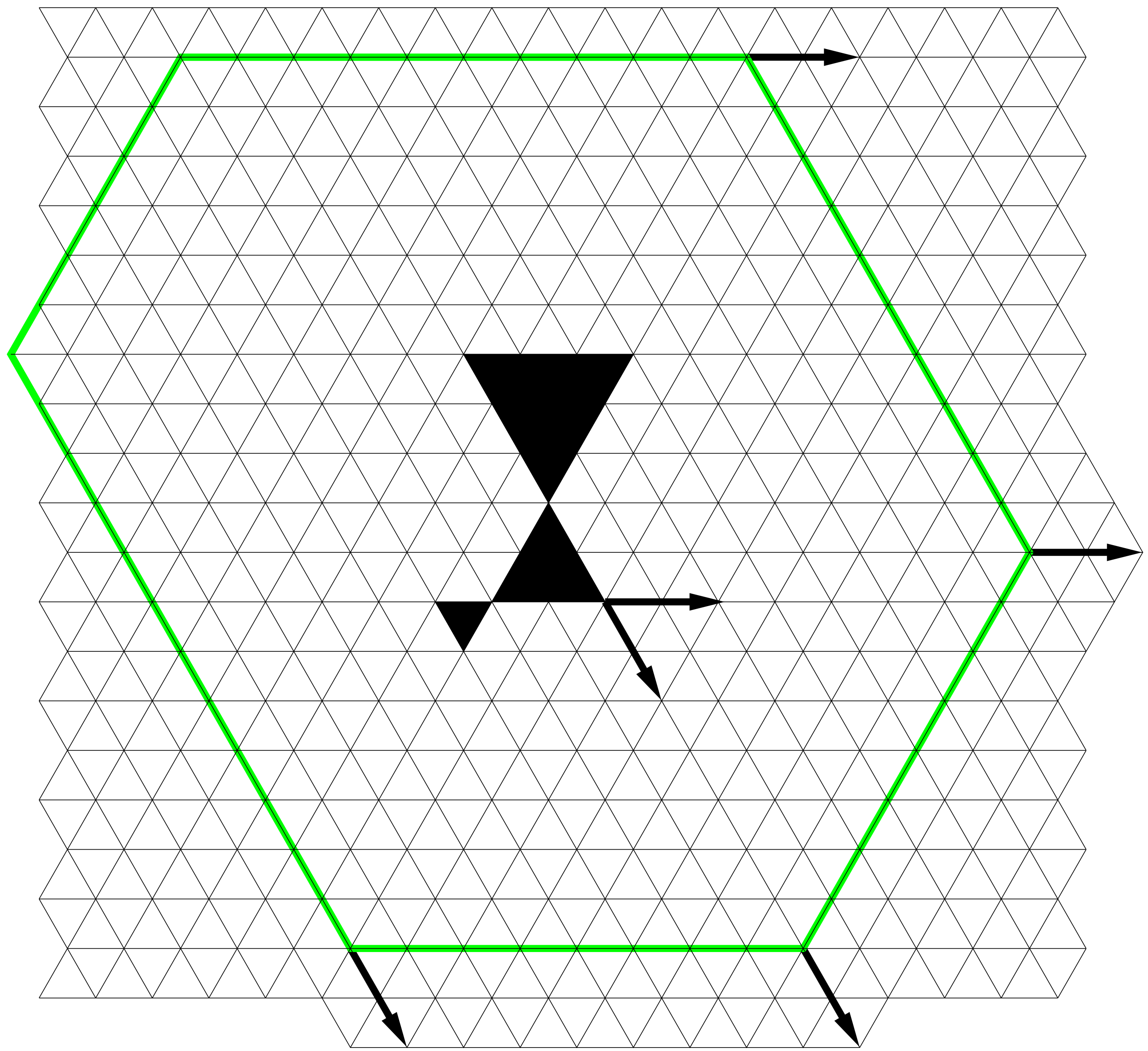


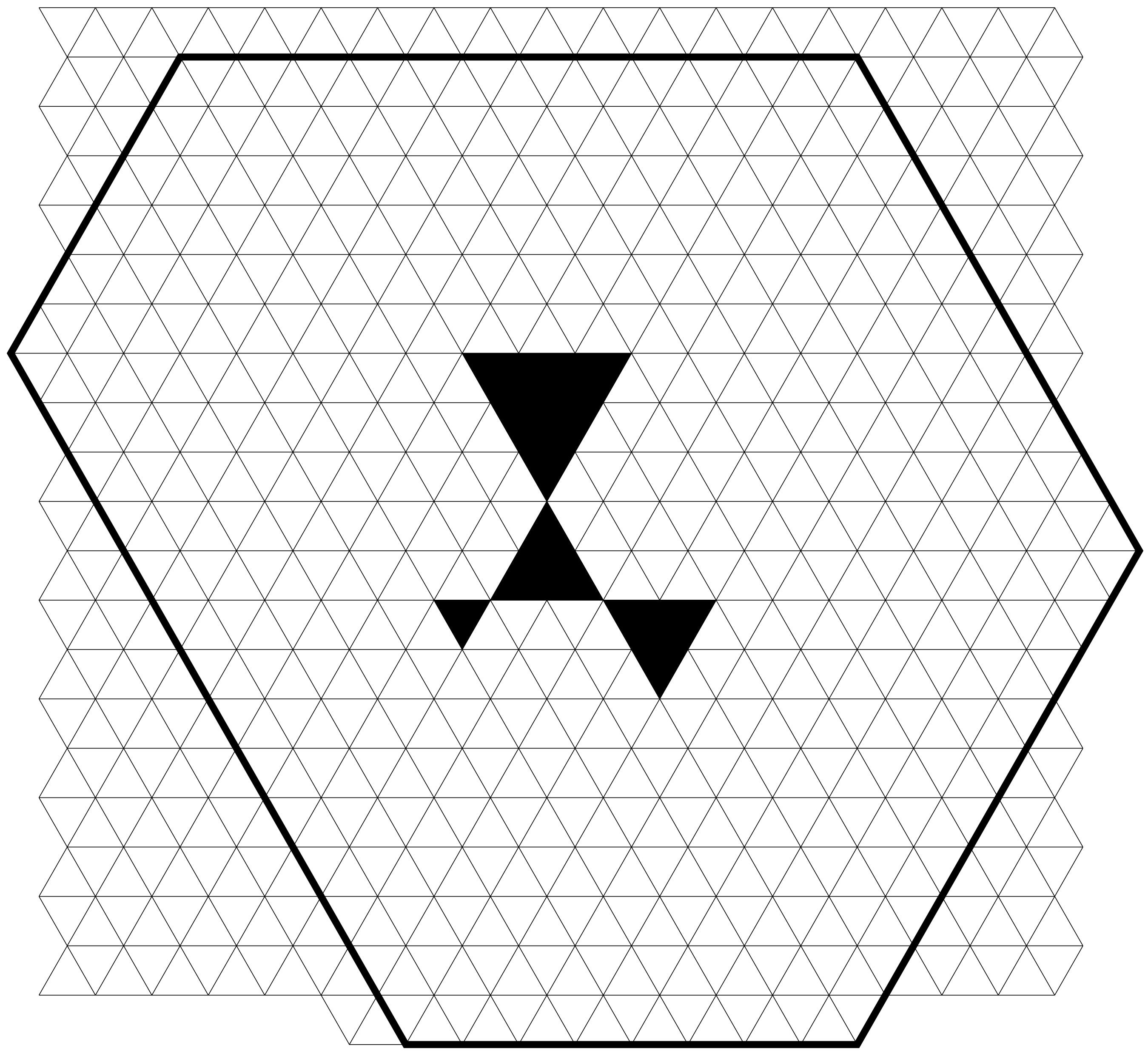
The first lobe





The first two lobes





The $S$-cored hexagon $SC_{6,8,4}(3,1,2,2)$

Define

$$\mathrm{H}(n) := \prod_{k=0}^{n} \Gamma(k),$$

$$\mathrm{H}(n+\frac{1}{2}) := \prod_{k=0}^{n} \Gamma(k+\frac{1}{2})$$

THEOREM 1. *Let $x$, $y$, $z$, $a$, $b$, $c$ and $m$ be nonnegative integers. If $y$ and $z$ have the same parity, we have:*

$$\mathrm{M}(SC_{x,y,z}(a,b,c,m)) = \frac{\mathrm{H}(m)^3\ \mathrm{H}(a)\ \mathrm{H}(b)\ \mathrm{H}(c)}{\mathrm{H}(m+a)\ \mathrm{H}(m+b)\ \mathrm{H}(m+c)}$$

$$\times \frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b)\ \mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+c)\ \mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+m+c)\ \mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+m+b)\ \mathrm{H}(\frac{y+z}{2}+m+a)}$$

$$\times \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+c)\ \mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+b)\ \mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+a+b)\ \mathrm{H}(\lceil\frac{x+z}{2}\rceil+a+c)\ \mathrm{H}(\frac{y+z}{2}+b+c)}$$

$$\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}$$

$$\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)}$$

$$\times \frac{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+m+a+b+c)\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+m+a+b+c)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b+c)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+b+c)\,\mathrm{H}(\frac{y+z}{2}+m+a+b+c)}$$

$$\times \frac{\mathrm{H}(\lceil\frac{x}{2}\rceil)\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\lceil\frac{y}{2}\rceil)}{\mathrm{H}(\lceil\frac{x}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\lceil\frac{y}{2}+\frac{m+a+b+c}{2})}$$

$$\times \frac{\mathrm{H}(\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\lceil\frac{z}{2}\rceil)\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor)}{\mathrm{H}(\lfloor\frac{y}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\lceil\frac{z}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor+\frac{m+a+b+c}{2})}$$

$$\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\ \mathrm{H}(\lceil\frac{x+y}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+\frac{m+a+b+c}{2})}{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\lceil\frac{x+y}{2}\rceil)}$$

$$\times \frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\frac{y+z}{2}+\frac{m+a+b+c}{2})^2}{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor)\,\mathrm{H}(\frac{y+z}{2})}.$$

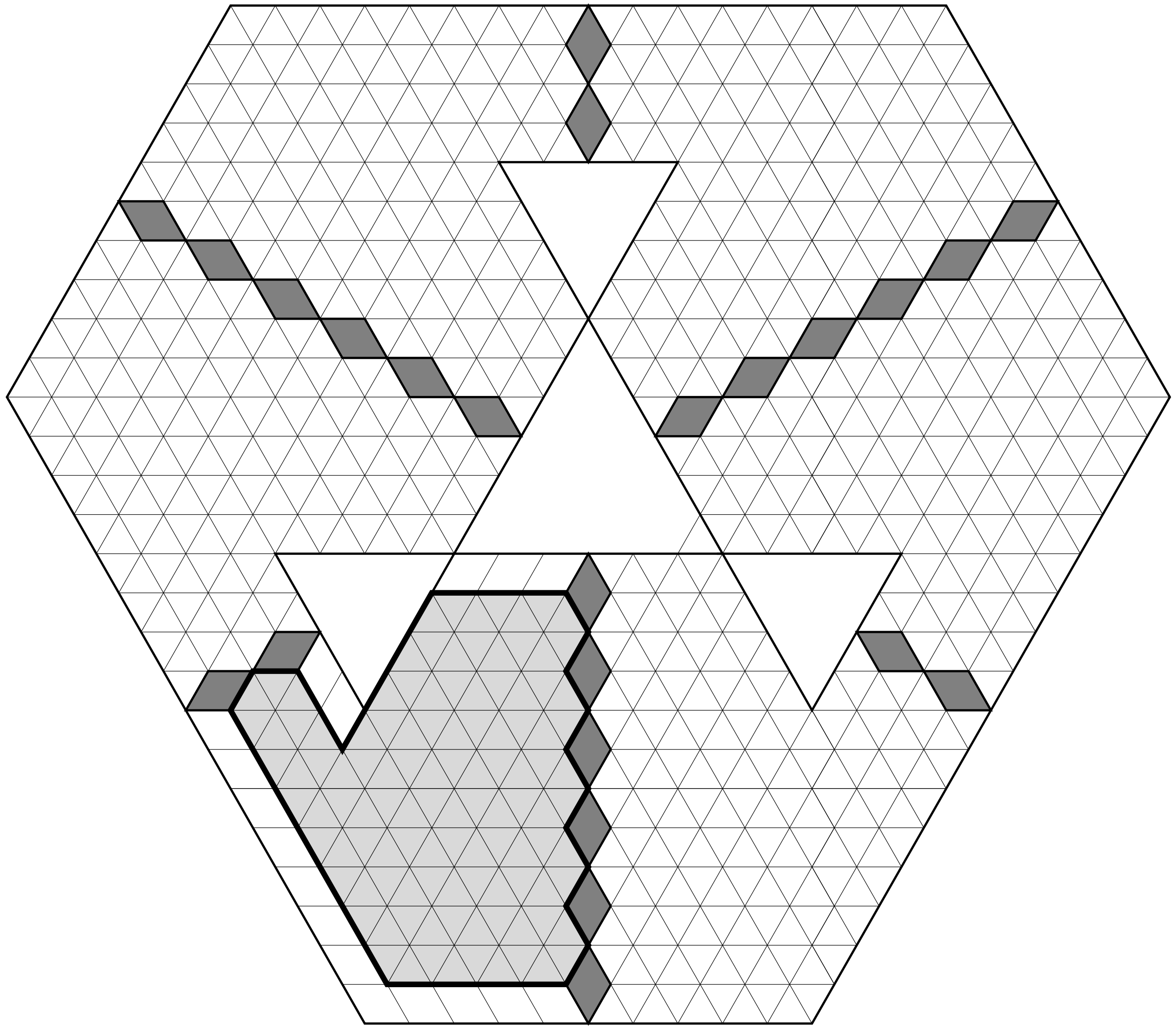
**How about symmetry classes?**

**How about symmetry classes?**

Will talk about two:

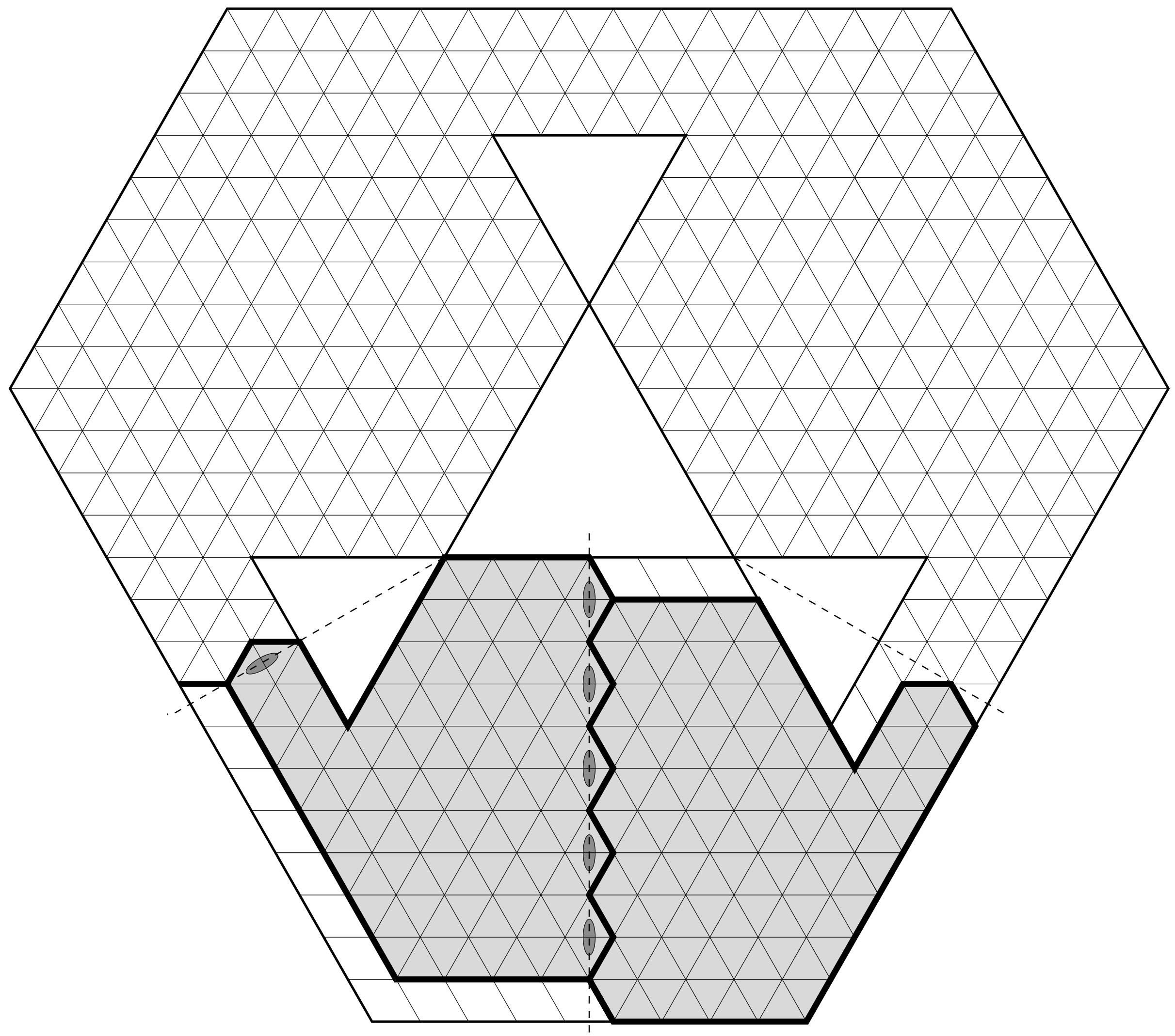
- cyclically symmetric
- cyclically symmetric and transpose complementary

**CSTC**



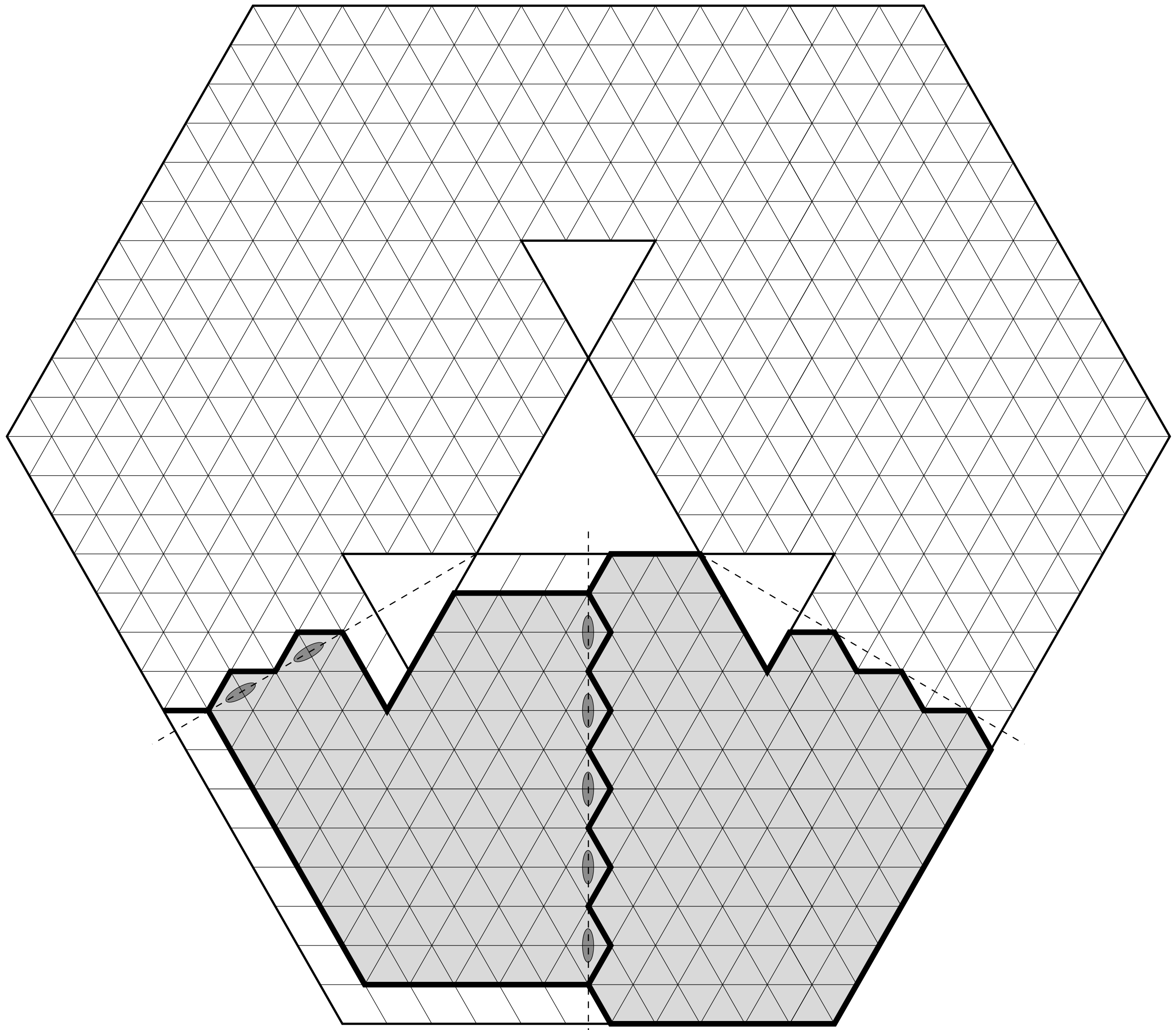
**CS:** $a$ **even,** $m$ **even**

Use factorization theorem for perfect matchings



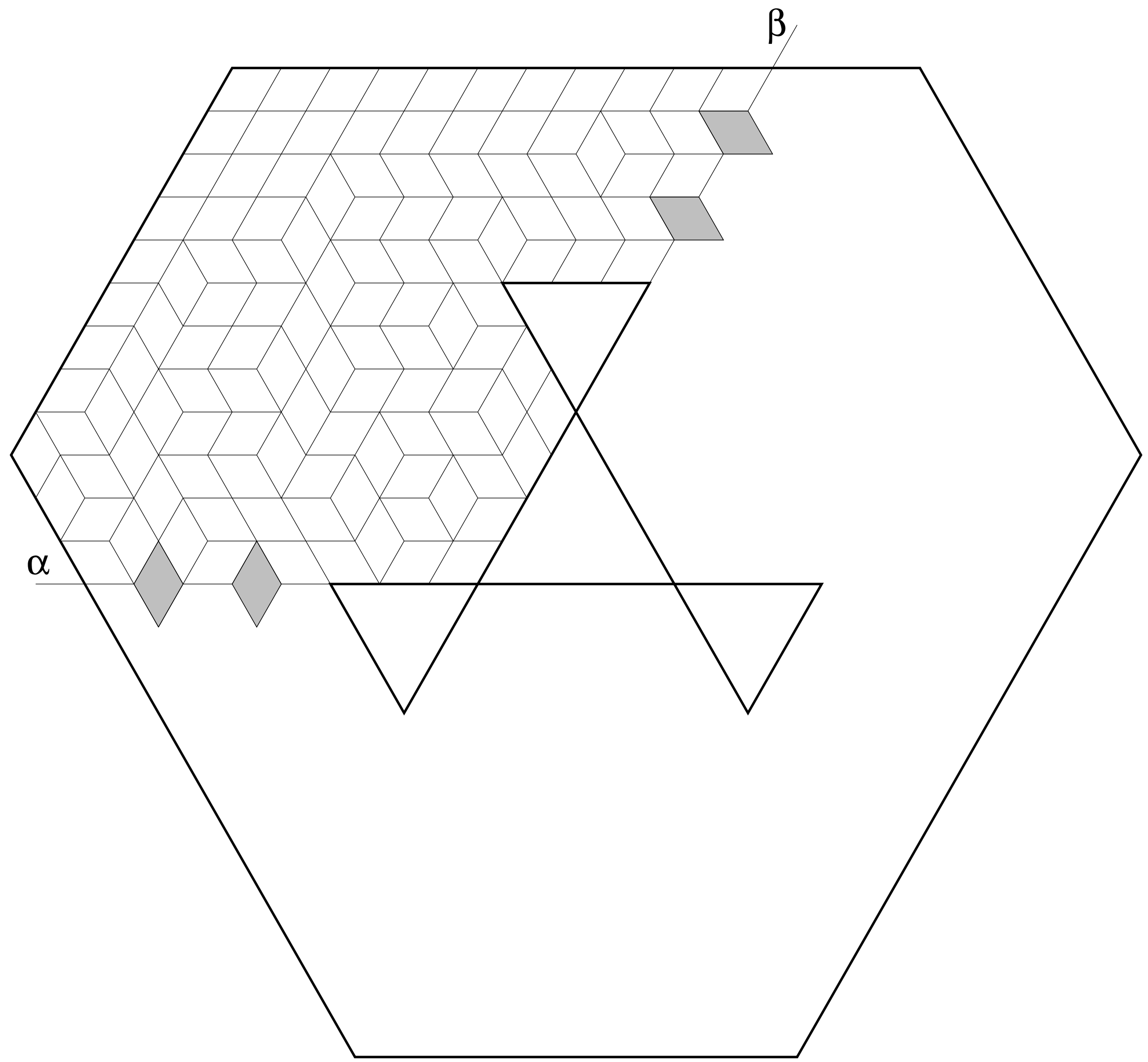
For $a$ even and $m$ even the cyclically symmetric case reduces to regions of these two types (shown here is the region $SC_{x,x,x}(a,a,a,m)$ for $x=3$, $a=4$, $m=6$)

**CS: $a$ odd, $m$ odd**



For $a$ odd and $m$ odd the cyclically symmetric case reduces to regions of these other two types (shown here is the region $SC_{x,x,x}(a,a,a,m)$ for $x=3$, $a=3$, $m=5$)

**Polynomiality in $m$**

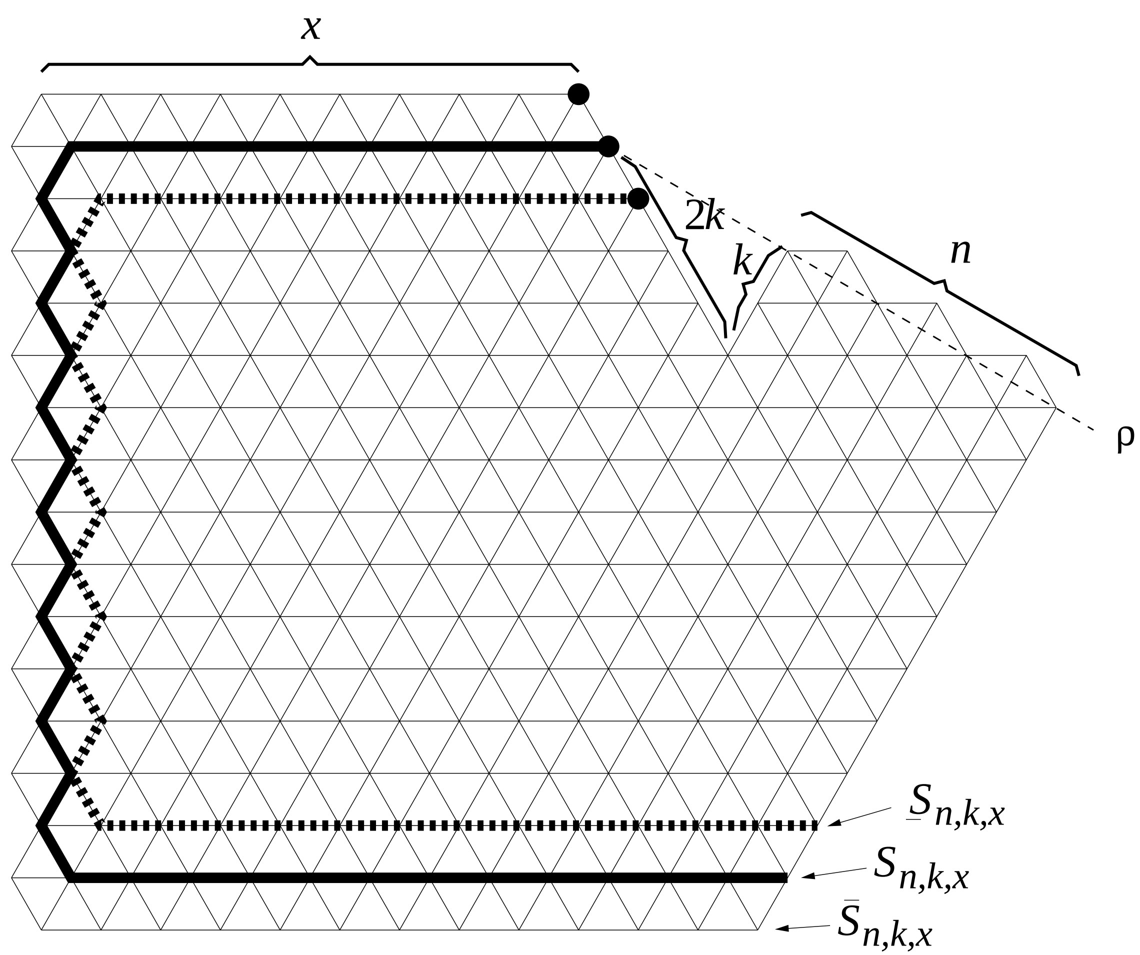



LEMMA. *Set* $CS(x,a,m) = \mathrm{M}(SC_{x,x,x}(a,a,a,m))$. *Then*

$$CS(x,a,m) = \det\left(\delta_{i,a+j} + \binom{m+a+i+j}{a+i}\right)_{0\le i,j\le x-1},$$

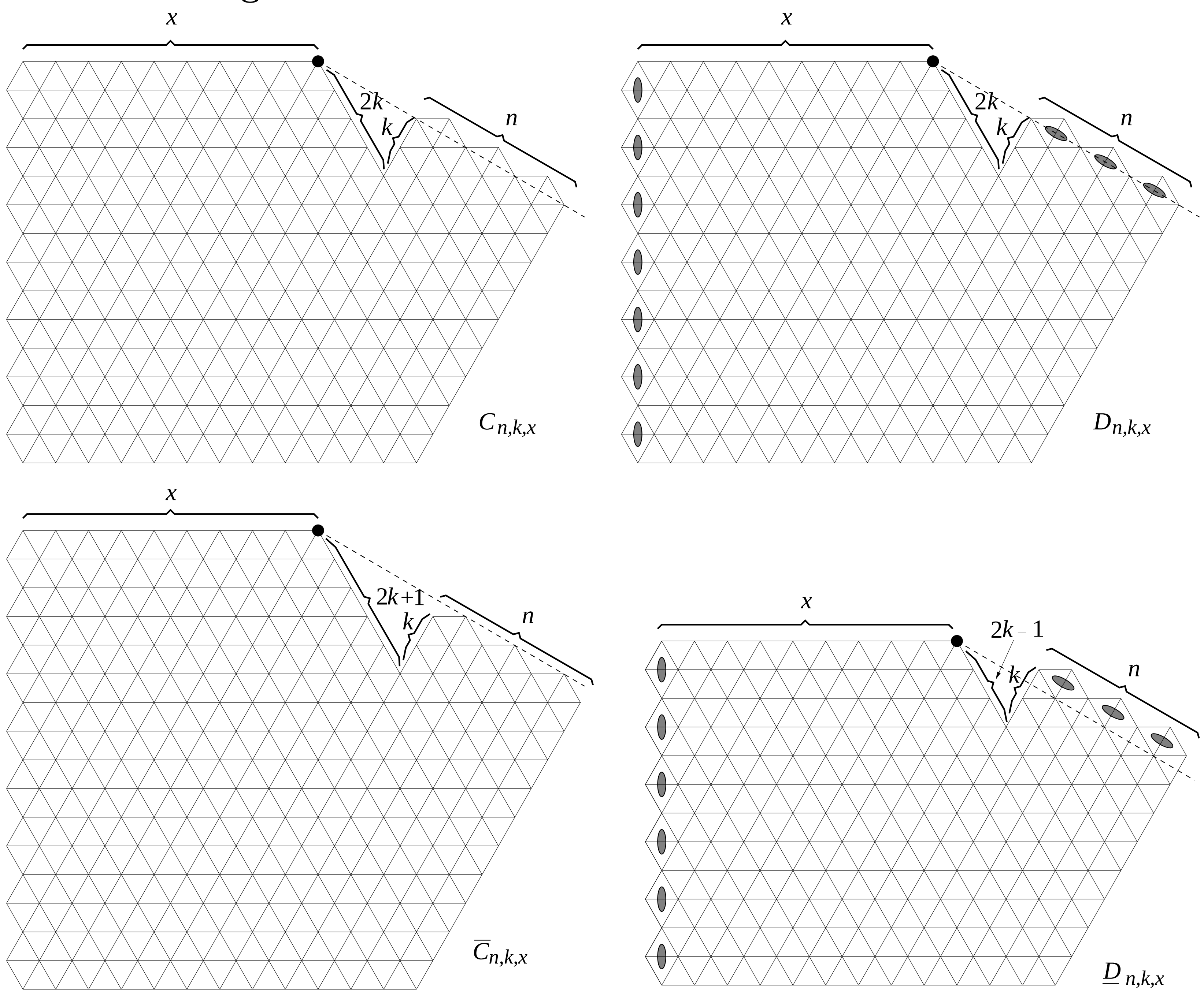
*where* $\delta_{ij}$ *is the Kronecker symbol.*

**Twelve families of regions**



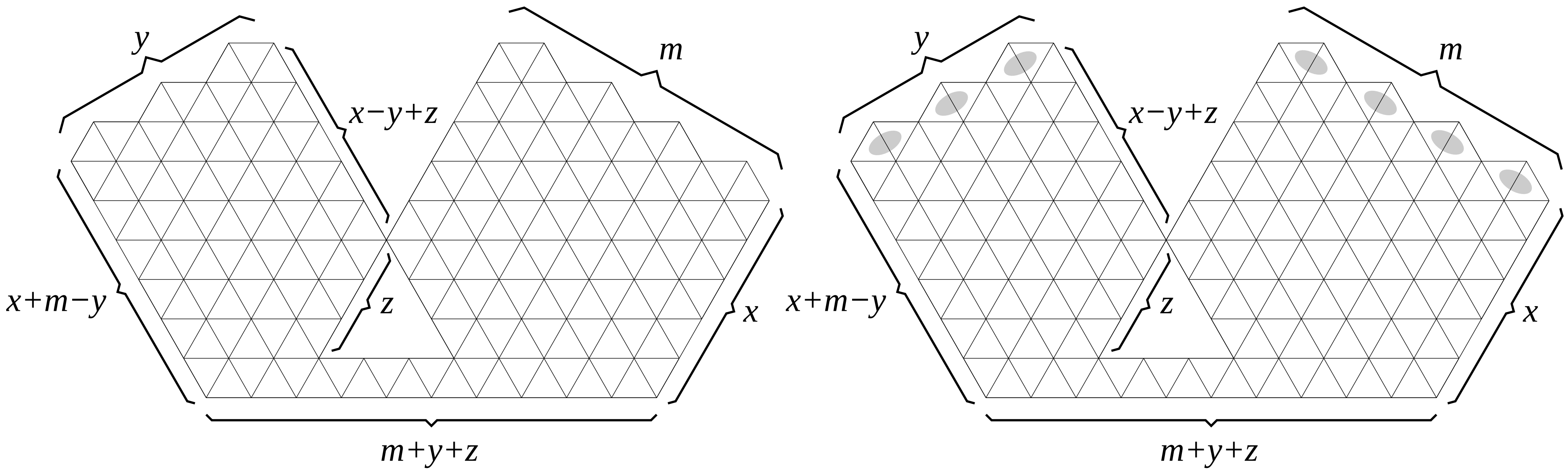

The contours $S_{n,k,x}$, $\overline{S}_{n,k,x}$ and $\underline{S}_{n,k,x}$ for $n = 3$, $k = 2$, $x = 9$.

## The four regions we need

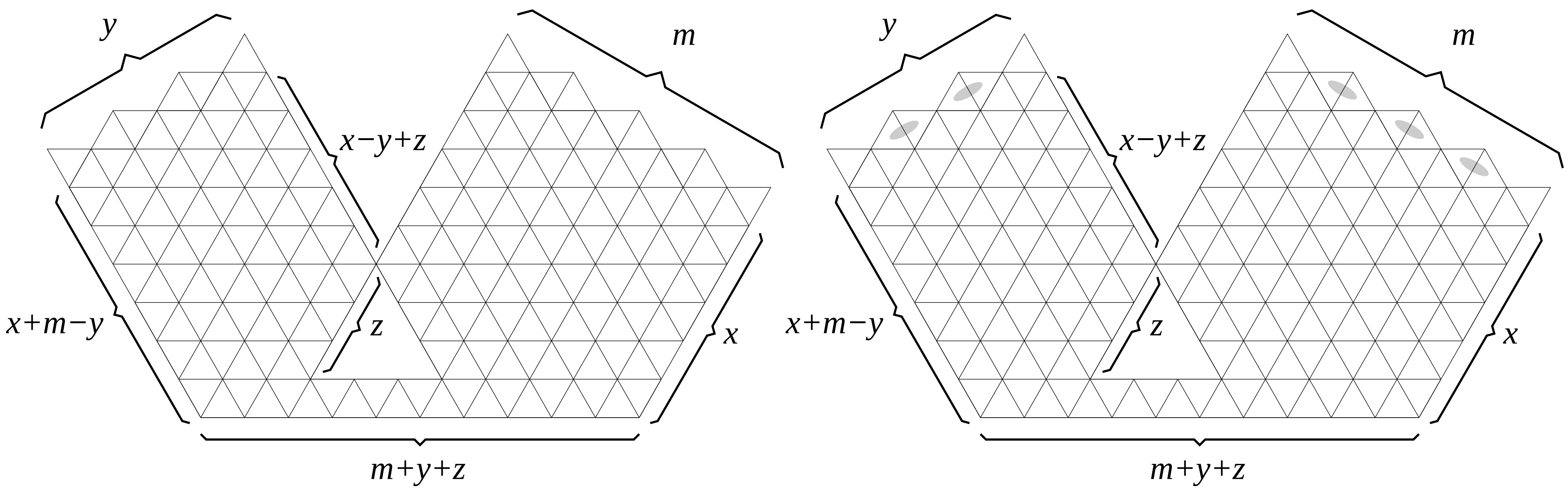



$C_{n,k,x}$, $D_{n,k,x}$, $\overline{C}_{n,k,x}$ and $\underline{D}_{n,k,x}$ ($n = 3$, $k = 2$, $x = 9$)

**Four enumeration results we deduce these from**




The regions $F_{x,y,z,m}$ (left) and $F'_{x,y,z,m}$ (right) for $x = 5$, $y = 3$, $z = 3$, $m = 4$.




The regions $G_{x,y,z,m}$ (left) and $G'_{x,y,z,m}$ (right) for $x = 5$, $y = 3$, $z = 3$, $m = 4$.

**Convention**

$$\prod_{k=m}^{n-1} \mathrm{Expr}(k) = \begin{cases} \prod_{k=m}^{n-1} \mathrm{Expr}(k) & n > m, \\ 1 & n = m, \\ \dfrac{1}{\prod_{k=n}^{m-1} \mathrm{Expr}(k)} & n < m. \end{cases}$$

**Pochhammer symbol**

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 1/((\alpha-1)(\alpha-2)\cdots(\alpha+k)) & \text{if } k < 0. \end{cases}$$

For integers $x$, $y$, $z$ and $m$ define the function $f(x,y,z,m)$ by

$$
\begin{aligned}
f(x,y,z,m) := {} & \frac{1}{2^{\max(y,0)}} \prod_{i=0}^{y-1} \frac{(2x+2z-i-y+1)_{y-i}}{(i+1)_{i+1}} \prod_{i=0}^{m-1} \frac{(2x-2y+2z+2i+2)_{m-i}}{(i+2)_{i+1}} \\
& \times \prod_{i=0}^{m-z-1} \frac{(2x+m+3z+2i-y+3)_{y-m}(y+i-z+1)_z}{(2x-2y+3z+i+2)_{y-z}(i+1)_z} \\
& \times \prod_{i=0}^{\lfloor m/3-z/3-1/3 \rfloor} (2x+y+2z+3i+3)_{3m-3z-2-9i} \\
& \times \prod_{i=0}^{\lfloor m/3-z/3-2/3 \rfloor} (2x-2y+5z+6i+5)_{3m-3z-5-9i} \\
& \times \prod_{i=0}^{\lfloor m/3-z/3-2/3 \rfloor} \frac{1}{(2x+y-z+3m-6i-1)} \prod_{i=0}^{\lfloor m/3-z/3-1 \rfloor} \frac{1}{(2x-2y+5z+6i+6)}.
\end{aligned}
$$

Theorem (C. and Fischer, 2015). *Let $x$, $y$, $z$ and $m$ be non-negative integers with $0 \leq y - z \leq x$ and $z \leq m$. Then the number of lozenge tilings of the regions $F_{x,y,z,m}$ and $F'_{x,y,z,m}$ is given by*

$$\mathrm{M}(F_{x,y,z,m}) = f(x, y, z, m)$$

*and*

$$\mathrm{M}(F'_{x,y,z,m}) = f(x - \frac{1}{2}, y, z, m).$$

THEOREM (C., 2017). *Let $x$, $y$, $z$ and $m$ be non-negative integers with $0 \leq y-z \leq x$ and $z \leq m$. Then the number of lozenge tilings of the regions $G_{x,y,z,m}$ and $G'_{x,y,z,m}$ are given by*

$$\mathrm{M}(G_{x,y,z,m}) = g(x,y,z,m)$$

*and*

$$\mathrm{M}(G'_{x,y,z,m}) = g(x-\frac{1}{2},y,z,m),$$

*where the function $g$ is defined by*

$$g(x,y,z,m) := f(x,y,z,m)\frac{f(x+1,y-z-1,0,m-1)}{f(x,y-z,0,m)} \times \prod_{k=1}^{z} \frac{(4k+4y-4z-2)(2x+3k+y-z)}{(2x+k-2y+2z+1)(2x+3k-2y+2z)}.$$

**Enumeration of tilings of $C_{n,0,x}$ and $D_{n,0,x}$ follows from previous work**

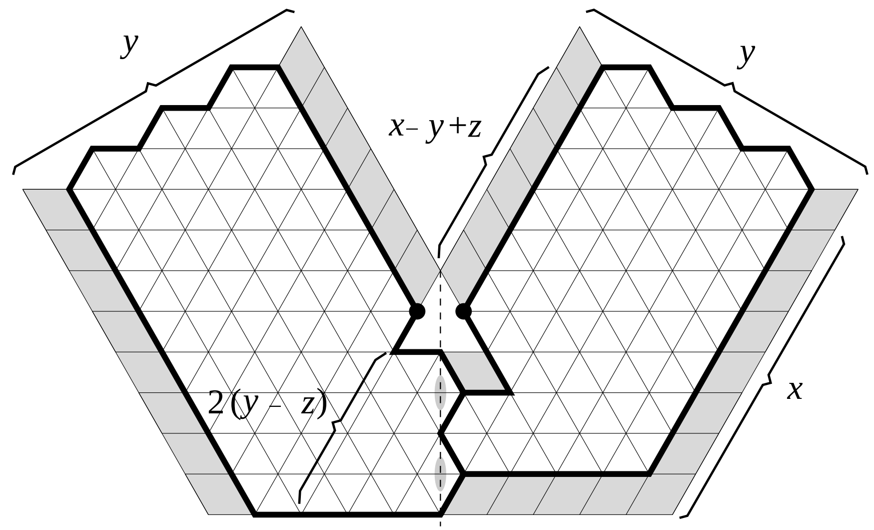
PROPOSITION. *For non-negative integers $n$ and $x$ we have*

$$\mathrm{M}(C_{n,0,x}) = \frac{1}{2^n} \prod_{i=1}^{n} \frac{(2x+2i+2)_i \left(x+2i+\frac{3}{2}\right)_{i-1}}{(i)_i \left(x+i+\frac{3}{2}\right)_{i-1}}$$

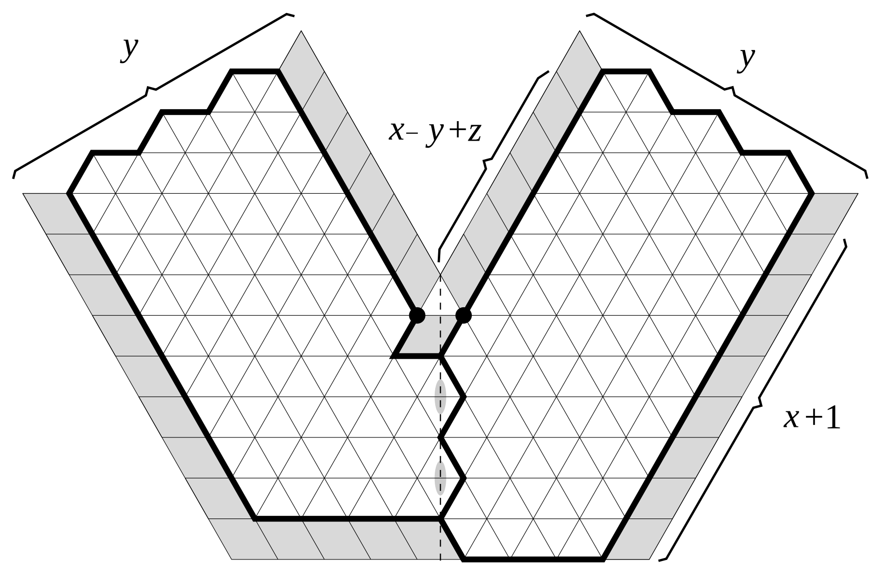
*and*

$$\mathrm{M}(D_{n,0,x}) = \frac{1}{2^n} \prod_{i=0}^{n} \frac{\left(x+2i-\frac{1}{2}\right)_{i+1} (2x+2i-1)_i}{(i+1)_i \left(x+i-\frac{1}{2}\right)_{i+1}}$$

**The $C$-regions**




$$\mathrm{M}(G_{x,y,z,y}) = 2^{y-z}\,\mathrm{M}(\underline{A}_{y-z,\frac{z}{2},x-y+z+1})\,\mathrm{M}(C_{y-z-1,\frac{z}{2},x-y+z+1}).$$




$$\mathrm{M}(G_{x+1,y,z-1,y}) = 2^{y-z+1}\,\mathrm{M}(\underline{A}_{y-z,\frac{z}{2},x-y+z+1})\,\mathrm{M}(C_{y-z+1,\frac{z}{2}-1,x-y+z+1})$$

$\Longrightarrow$

$$\frac{\mathrm{M}(C_{y-z-1,\frac{z}{2},x-y+z+1})}{\mathrm{M}(C_{y-z+1,\frac{z}{2}-1,x-y+z+1})} = 2\ \frac{\mathrm{M}(G_{x,y,z,y})}{\mathrm{M}(G_{x+1,y,z-1,y})}$$

THEOREM (C., 2017). *For non-negative integers $n$, $k$ and $x$ we have*

$$\mathrm{M}(C_{n,k,x}) = \frac{1}{2^n}\binom{2k+x}{2k}\prod_{i=1}^{2k-1}\prod_{j=i}^{2k-1}\frac{2x+i+j+1}{i+j+1}\prod_{i=1}^{n}\frac{(6k+2i+2)_i(3k+2i+3/2)_{i-1}}{(i)_i(3k+i+3/2)_{i-1}}$$

$$\times\prod_{i=1}^{n}\frac{\dfrac{(x+i+k+1)_k(x+2i+3k+3/2)_{i-1}(x+i+k+1/2)_k(x+i+3k+1)_{\lceil i/2\rceil}}{(x+2i+3k-1/2)_{\lceil i/2\rceil-1,\searrow}}}{\dfrac{(i+k+1)_k(2i+3k+3/2)_{i-1}(i+k+1/2)_k(i+3k+1)_{\lceil i/2\rceil}}{(2i+3k-1/2)_{\lceil i/2\rceil-1,\searrow}}},$$

*where*

$$(a)_{k,\searrow} := a(a-1)\cdots(a-k+1).$$

The $D$-, $\overline{C}$- and $\underline{D}$-regions work out similarly.

# SYMMETRIES OF SHAMROCKS, PART I

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Abstract. Hexagons with four-lobed regions called shamrocks removed from their center were introduced in their 2013 paper by Ciucu and Krattenthaler, where product formulas for the number of their lozenge tilings were provided. In analogy with the plane partitions which they generalize, we consider the problem of enumerating the lozenge tilings which are invariant under some symmetries of the underlying region. This leads to six symmetry classes besides the base case of requiring no symmetry. In this paper we provide product formulas for two of these symmetry classes (namely, the ones generalizing cyclically symmetric, and cyclically symmetric and transpose complementary plane partitions).

## 1. Introduction

In [**7**] we presented a generalization of MacMahon's classical counting of boxed plane partitions [**17**, Sect. 495] — which is equivalent to enumerating lozenge tilings of hexagons on the triangular lattice — by providing product formulas for the number of lozenge tilings of hexagons from whose centers four-lobed structures called shamrocks were removed (such regions are called $S$-cored hexagons; see Figure 1.1 for an example).

Motivated by the case of plane partitions (see [**1**][**19**][**16**][**20**][**12**] and the surveys [**2**] and [**13**] for more recent developments) we consider in this paper the symmetry classes of tilings of $S$-cored hexagons. More precisely, for any subgroup $G$ of the group of symmetries of an $S$-cored hexagon, we seek to find how many of its lozenge tilings are invariant under $G$.

This leads to six non-trivial symmetry classes, named after the corresponding symmetries of plane partitions (see [**19**]): (*i*) cyclically symmetric, (*ii*) transpose complementary, (*iii*) cyclically symmetric and transpose complementary, (*iv*) symmetric, (*v*) self-complementary and (*vi*) symmetric and self-complementary.

In this paper we provide product formulas for the first and third of these symmetry classes. The remaining ones require different methods and will be treated in subsequent work.

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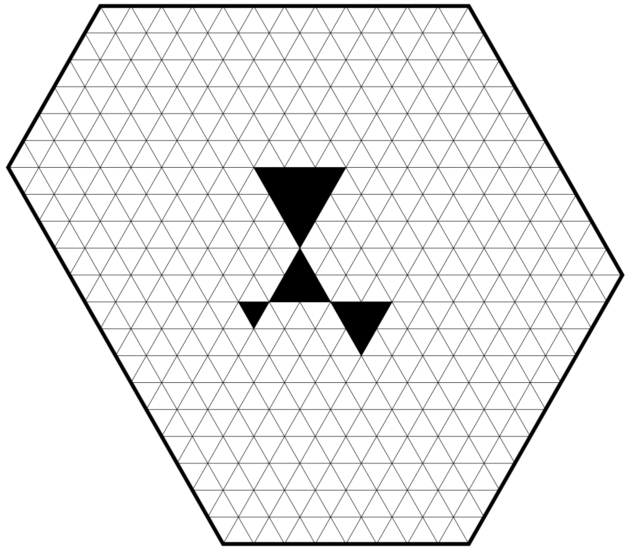

FIGURE 1.1. The $S$-cored hexagon $SC_{6,8,4}(3,1,2,2)$ (see [**7**] for details of its definition).

We point out that Kuo's graphical condensation method [**14**][**15**], which has been very fruitful in previous related work on enumerating tilings of lattice regions (see e.g. [**6**][**7**][**8**][**9**][**10**]), does not seem to provide the proofs here (the method *can* be applied to our regions, and it does lead to interesting recurrences, but they involve two different types of regions, and it does not seem possible to prove our formulas using them). Instead, we deduce our enumerations from two results we proved in [**8**] and two counterparts we present in this paper (we note that we need the full generality of the results in [**8**], and not just the special cases of [**5**] they extend). This constitutes an unforeseen and welcome application of the results of [**8**], which involve regions that did not seem particularly natural to consider, other than the fact that they extended certain regions of [**5**] in a general enough way so that conjectures concerning those regions could be proved by Kuo condensation.

## 2. Twelve families of regions

In this section we define twelve closely related families of regions on the triangular lattice. The enumeration of tilings of both cyclically symmetric and cyclically-symmetric-and-transpose-complementary $S$-cored hexagons will follow from the enumeration of tilings of four of these regions. Four more also come up in our proofs (as a result of applying the factorization theorem [**3**]), but the enumeration of tilings of these regions (which also follow from our arguments) is not necessary for our purposes. Interestingly, the remaining four families do not come up in our arguments; nevertheless, it would be worth finding enumeration formulas for their tilings (even though, cf. the last paragraph of this section, it seems that these will not be simple product formulas), given how closely they are related

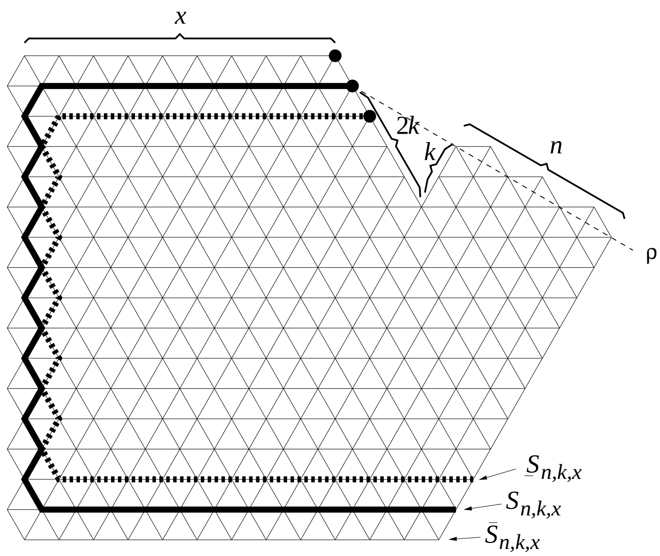


FIGURE 2.1. The contours $S_{n,k,x}$, $\overline{S}_{n,k,x}$ and $\underline{S}_{n,k,x}$ for $n = 3$, $k = 2$, $x = 9$.

to the other eight.

Our twelve families are determined by three types of contours, each of which will be weighted in one of four ways. The three types of contours, denoted $S_{n,k,x}$, $\overline{S}_{n,k,x}$ and $\underline{S}_{n,k,x}$, are shown in Figure 2.1. The portion cut out by the thick solid line is $S_{n,k,x}$; the dotted ray $\rho$ starting from the indicated solid dot in the polar direction $-\pi/6$ cuts in half the $n$ lozenges that would fit in the folds of the northeastern zig-zag boundary. The outermost contour is $\overline{S}_{n,k,x}$; its notation records the fact that the analogous ray for it is half a lattice spacing above $\rho$. Finally, the region cut out by the thick dotted line[1] is $\underline{S}_{n,k,x}$; the notation refers to the fact that the analogous ray in it is half a lattice spacing below $\rho$. The four different weightings concern two sets of lozenge positions — the set of $n$ lozenge positions that fit in the folds of the northeastern boundary, and the set of lozenge positions that fit in the folds of the western boundary (the latter consists of $n+2k$, $n + 2k + 1$, resp. $2k - 1$ lozenge positions, according as the region is $S_{n,k,x}$, $\overline{S}_{n,k,x}$, resp. $\underline{S}_{n,k,x}$). Each of these two sets of lozenge positions will be given weights of 1 or 1/2.

We are now ready to define our twelve families of regions. They are illustrated in Figures 2.2–2.4 (which list them in the order of future use).

We define[2] $A_{n,k,x}$ (resp., $\overline{A}_{n,k,x}$, and $\underline{A}_{n,k,x}$) to be the region inside the contour $S_{n,k,x}$ (resp., $\overline{S}_{n,k,x}$, and $\underline{S}_{n,k,x}$), in which the $n$ tile positions that fit in the folds of the north-

[1]The case $k = 0$ needs special consideration here, as the distance $2k - 1$ we need to travel from the solid dot in polar direction $-\pi/3$ (before we turn and travel distance $k$ in polar direction $\pi/3$; see the region cut out by the thick dotted line in Figure 2.1) is, for $k = 0$, equal to $-1$. What we do in this case is to travel instead a distance of 1 in the opposite direction (polar direction $2\pi/3$).

[2]The choice of the capital letter here, as well as for the $B$-regions, is so as to be in agreement with the related families of regions considered in [**4**].

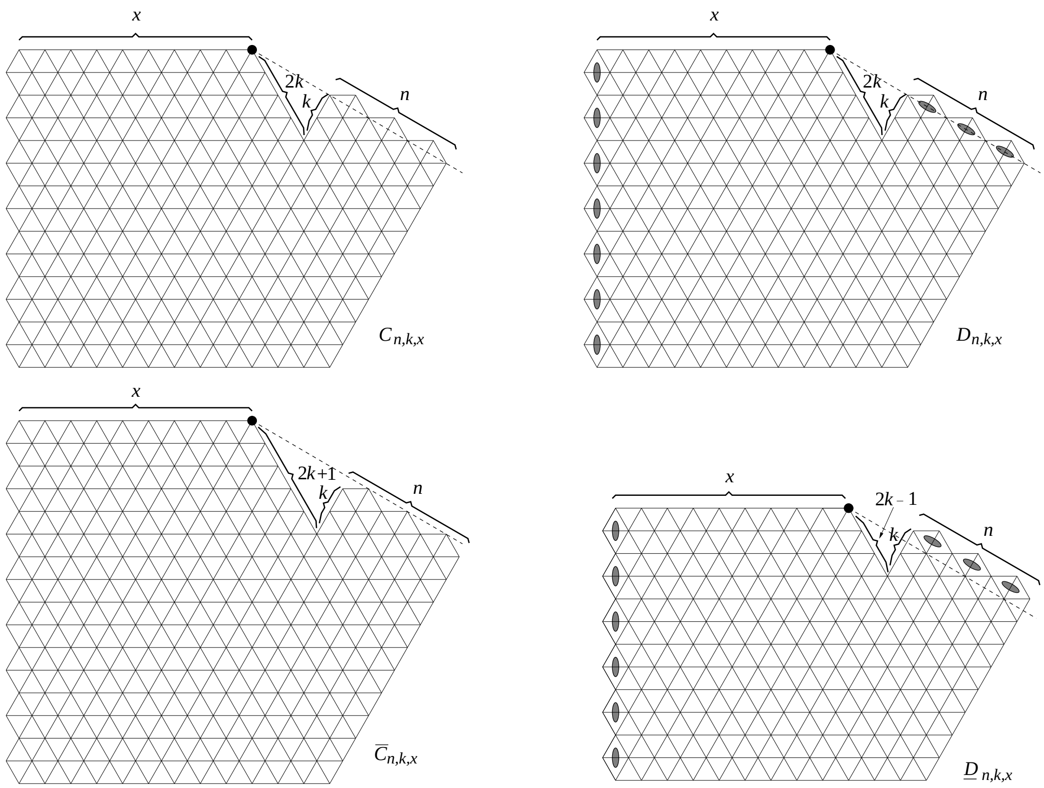


Figure 2.2. The four regions we need: $C_{n,k,x}$ (top left), $D_{n,k,x}$ (top right), $\overline{C}_{n,k,x}$ (bottom left) and $\underline{D}_{n,k,x}$ (bottom right), for $n=3$, $k=2$, $x=9$.

eastern boundary are weighted by $1/2$ (see Figures 2.3 and 2.4).

Define the regions $B_{n,k,x}$, $\overline{B}_{n,k,x}$ and $\underline{B}_{n,k,x}$ to be given by precisely the same contours as their $A$-type counterparts, but now weighting by $1/2$ the $n+2k$ (resp., $n+2k+1$ and $n+2k-1$) westernmost tile positions instead.

The regions $C_{n,k,x}$, $\overline{C}_{n,k,x}$ and $\underline{C}_{n,k,x}$ are the regions inside the contours $S_{n,k,x}$, $\overline{S}_{n,k,x}$ and $\underline{S}_{n,k,x}$, with no specially weighted tile positions. Finally, $D_{n,k,x}$, $\overline{D}_{n,k,x}$ and $\underline{D}_{n,k,x}$ are the regions inside the contours $S_{n,k,x}$, $\overline{S}_{n,k,x}$ and $\underline{S}_{n,k,x}$, with all the tile positions considered in their $A$- or $B$-counterparts weighted by $1/2$.

We note that for our purposes we only need to enumerate the tilings of the four families shown in Figure 2.2. It turns out that four more families — the ones shown in Figure 2.3 — arise in our arguments, and simple product formulas for the number of their tilings also follow from our results. The remaining four families (included for completeness in Figure 2.4) turn out, somewhat unexpectedly, to have a number of tilings that does not seem to be "round" — large prime factors show up in the integer factorization of such numbers even for fairly small regions that belong to these families. However, it would still

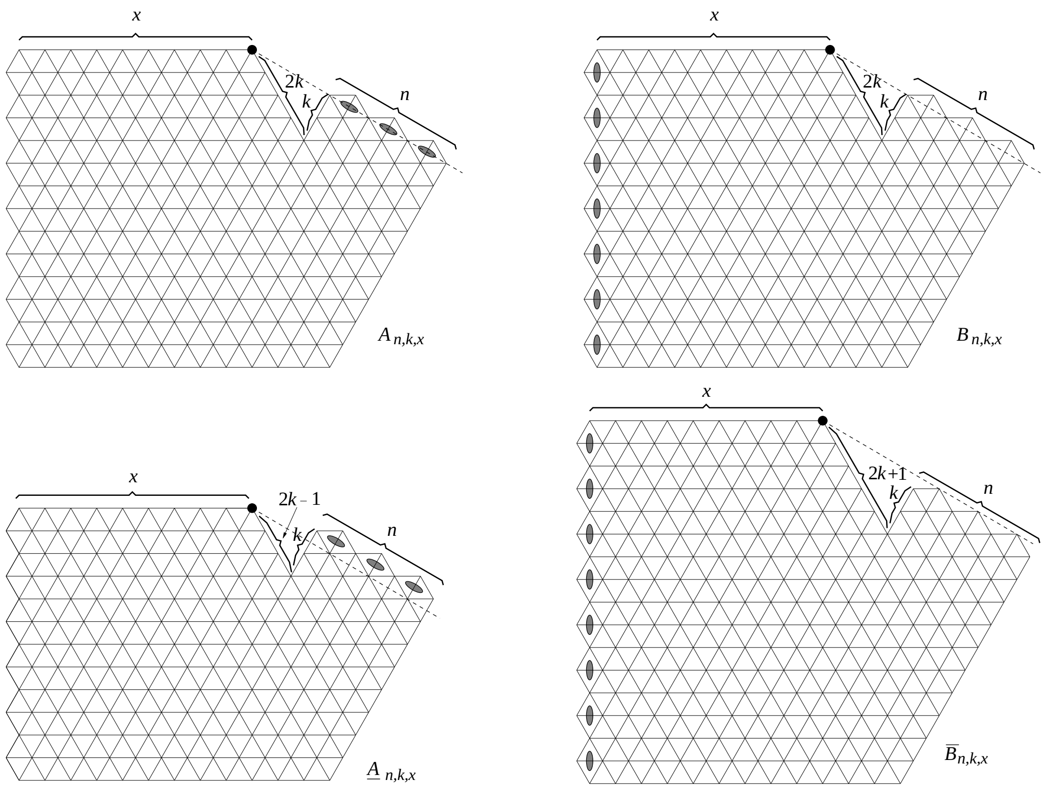


Figure 2.3. Four other regions that arise: $A_{n,k,x}$ (top left), $B_{n,k,x}$ (top right), $\underline{A}_{n,k,x}$ (bottom left) and $\overline{B}_{n,k,x}$ (bottom right), for $n = 3$, $k = 2$, $x = 9$.

be interesting to find enumeration formulas for their lozenge tilings.

## 3. Four enumeration results

In our arguments we employ two results from [**8**], as well as two new counterparts to them, which we present in this section. For completeness, we recall the two results of [**8**] that we will need.

Let $x$, $y$, $z$ and $m$ be non-negative integers with $0 \leq y - z \leq x$ and $z \leq m$, and consider the region $F_{x,y,z,m}$ pictured on the left in Figure 3.1; denote by $F'_{x,y,z,m}$ the region obtained from $F_{x,y,z,m}$ by weighting the tile positions in the folds of the two zig-zag portions of the boundary by $1/2$ (see the picture on the right in Figure 3.1)[3].

[3]In [**8**] we denoted these regions by $D_{x,y,z,m}$ and $D'_{x,y,z,m}$; we change here the notation to $F_{x,y,z,m}$ and $F'_{x,y,z,m}$ in order to distinguish them from the $D$-regions defined in the previous section.

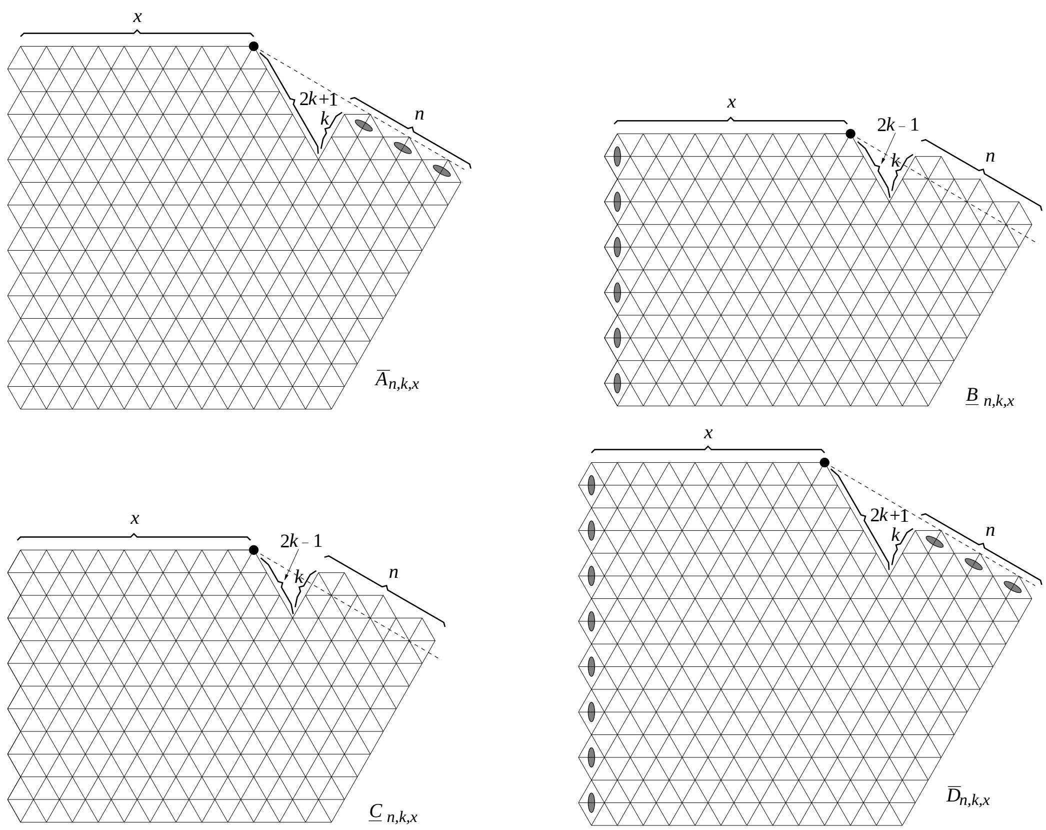


Figure 2.4. The four remaining regions: $\overline{A}_{n,k,x}$ (top left), $\underline{B}_{n,k,x}$ (top right), $\underline{C}_{n,k,x}$ (bottom left) and $\overline{D}_{n,k,x}$ (bottom right), for $n = 3$, $k = 2$, $x = 9$.

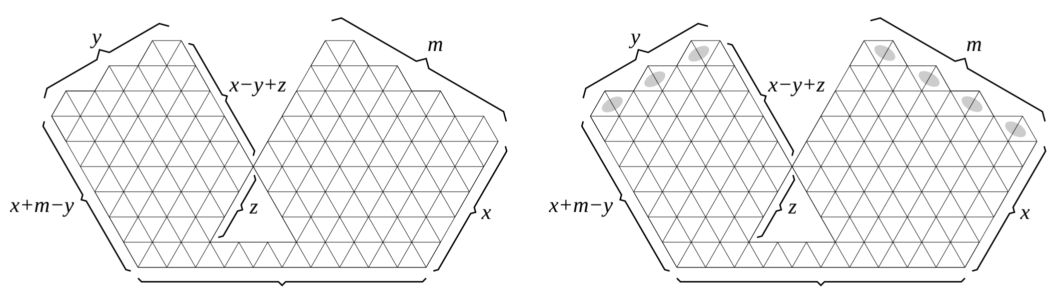


Figure 3.1. The regions $F_{x,y,z,m}$ (left) and $F'_{x,y,z,m}$ (right) for $x = 5$, $y = 3$, $z = 3$, $m = 4$.

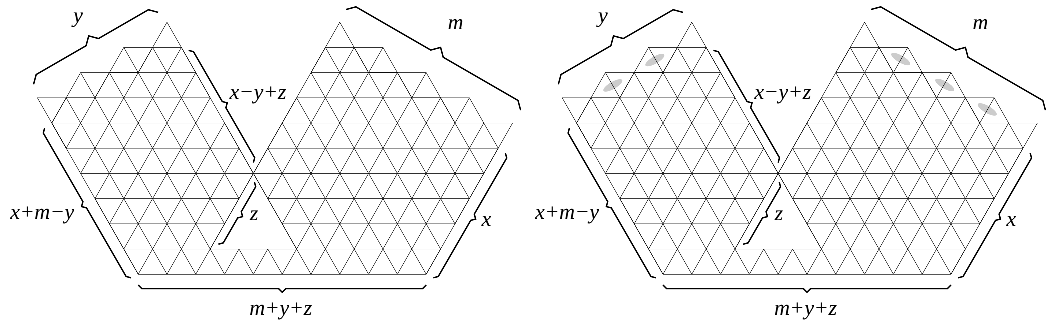


FIGURE 3.2. The regions $G_{x,y,z,m}$ (left) and $G'_{x,y,z,m}$ (right) for $x = 5$, $y = 3$, $z = 3$, $m = 4$.

Throughout this paper we define products according to the convention

$$\prod_{k=m}^{n-1} \mathrm{Expr}(k) = \begin{cases} \prod_{k=m}^{n-1} \mathrm{Expr}(k) & n > m, \\ 1 & n = m, \\ \dfrac{1}{\prod_{k=n}^{m-1} \mathrm{Expr}(k)} & n < m. \end{cases} \tag{3.1}$$

We recall that the Pochhammer symbol $(\alpha)_k$ is defined for any integer $k$ to be $(\alpha)_k := \prod_{i=0}^{k-1}(\alpha + i)$, thus according to (3.1)

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 1/((\alpha-1)(\alpha-2)\cdots(\alpha+k)) & \text{if } k < 0. \end{cases} \tag{3.2}$$

For integers $x$, $y$, $z$ and $m$ define the function $f(x, y, z, m)$ by

$$\begin{aligned} f(x,y,z,m) := {} & \frac{1}{2^{\max(y,0)}} \prod_{i=0}^{y-1} \frac{(2x+2z-i-y+1)_{y-i}}{(i+1)_{i+1}} \prod_{i=0}^{m-1} \frac{(2x-2y+2z+2i+2)_{m-i}}{(i+2)_{i+1}} \\ & \times \prod_{i=0}^{m-z-1} \frac{(2x+m+3z+2i-y+3)_{y-m}(y+i-z+1)_z}{(2x-2y+3z+i+2)_{y-z}(i+1)_z} \\ & \times \prod_{i=0}^{\lfloor m/3-z/3-1/3 \rfloor} (2x+y+2z+3i+3)_{3m-3z-2-9i} \\ & \times \prod_{i=0}^{\lfloor m/3-z/3-2/3 \rfloor} (2x-2y+5z+6i+5)_{3m-3z-5-9i} \\ & \times \prod_{i=0}^{\lfloor m/3-z/3-2/3 \rfloor} \frac{1}{(2x+y-z+3m-6i-1)} \prod_{i=0}^{\lfloor m/3-z/3-1 \rfloor} \frac{1}{(2x-2y+5z+6i+6)}. \end{aligned} \tag{3.3}$$

Denote by $\mathrm{M}(R)$ the weighted count[4] of the lozenge tilings of the region $R$ on the triangular lattice.

In [**8**] we proved the following results.

Theorem 3.1(Ciucu and Fischer [**8**]). *Let $x$, $y$, $z$ and $m$ be non-negative integers with $0 \leq y-z \leq x$ and $z \leq m$. Then the number of lozenge tilings of the regions $F_{x,y,z,m}$ and $F'_{x,y,z,m}$ is given by*

$$\mathrm{M}(F_{x,y,z,m}) = f(x,y,z,m) \tag{3.4}$$

*and*

$$\mathrm{M}(F'_{x,y,z,m}) = f(x-\frac{1}{2},y,z,m), \tag{3.5}$$

*where $f$ is given by* (3.3).

In order to prove the results in the current paper, we will also need the following two results, concerning the enumeration of tilings of the regions $G_{x,y,z,m}$ and $G'_{x,y,z,m}$ shown in Figure 3.2, which are obtained from the regions $F_{x,y,z,m}$ and $F'_{x,y,z,m}$ by adding strips of unit triangles above the zig-zag portions of their boundaries as indicated in Figure 3.2. The proofs of these counterpart results follow in precisely the same way (via Kuo's graphical condensation method [**14**]) as the proofs of the formulas in the above theorem that were presented in [**8**] (remove the forced strips of lozenges along the southwestern and southeastern boundaries of the $G$- and $G'$-regions, and apply Kuo condensation with the four unit triangles chosen precisely as in Figure 3.1 of [**8**]).

Theorem 3.2. *Let $x$, $y$, $z$ and $m$ be non-negative integers with $0 \leq y-z \leq x$ and $z \leq m$. Then the number of lozenge tilings of the regions $G_{x,y,z,m}$ and $G'_{x,y,z,m}$ shown in Figure 3.2 are given by*

$$\mathrm{M}(G_{x,y,z,m}) = g(x,y,z,m) \tag{3.6}$$

*and*

$$\mathrm{M}(G'_{x,y,z,m}) = g(x-\frac{1}{2},y,z,m), \tag{3.7}$$

*where the function $g$ is defined by*

$$\begin{aligned} g(x,y,z,m) := {} & f(x,y,z,m)\frac{f(x+1,y-z-1,0,m-1)}{f(x,y-z,0,m)} \\ & \times \prod_{k=1}^{z} \frac{(4k+4y-4z-2)(2x+3k+y-z)}{(2x+k-2y+2z+1)(2x+3k-2y+2z)}. \end{aligned} \tag{3.8}$$

[4]We weight all lozenge tile positions with positive weights. The weight of a lozenge tiling of a region on the triangular lattice is defined to be the product of the weights of the lozenges that make up that tiling. The weighted count of the tilings of the region $R$ is then the sum of the weights of all the lozenge tilings of $R$. All tile positions of the regions in this paper have weight 1, except the tile positions marked by shaded ellipses, which have weight 1/2.

## 4. The enumeration of tilings of $C_{n,0,x}$ and $D_{n,0,x}$

The enumeration of tilings of the two special cases of the $C$- and $D$-regions presented in this section follow from our earlier results in [**4**], [**5**] and [**3**]. We will use them in the next section.

Theorem 4.1. *For any non-negative integers $n$ and $x$ we have*

$$\mathrm{M}(C_{n,0,x}) = \frac{1}{2^n} \prod_{i=1}^{n} \frac{(2x+2i+2)_i \left(x+2i+\frac{3}{2}\right)_{i-1}}{(i)_i \left(x+i+\frac{3}{2}\right)_{i-1}} \tag{4.1}$$

*and*

$$\mathrm{M}(D_{n,0,x}) = \frac{1}{2^n} \prod_{i=0}^{n} \frac{\left(x+2i-\frac{1}{2}\right)_{i+1} (2x+2i-1)_i}{(i+1)_i \left(x+i-\frac{1}{2}\right)_{i+1}} \tag{4.2}$$

*Proof.* Formula (4.1) follows directly from [**6**, (3.3)] — simply note that our current region $C_{n,0,x}$ is the same as the region $G_{n,x+1}$ of [**6**].

To obtain (4.2), apply the factorization theorem of [**3**] to the special case $a = c$ of the region in [**5**, Theorem 1.7]. One of the regions resulting by the application of the factorization theorem is of the type $D_{n,0,x}$ we want to enumerate, while the other is an instance of the regions $A_{n,x}$ of [**4**, §2]. The desired enumeration follows then by [**5**, Theorem 1.7] and [**4**, (2.4), (2.2)]. It is straightforward to check that the resulting product formula for $\mathrm{M}(D_{n,0,x})$ agrees with (4.2). $\square$

## 5. The number of tilings of regions of type $C$, $D$, $\overline{C}$ and $\underline{D}$

We deduce the numbers mentioned in the title of this section from the special case of Theorems 3.1 and 3.2 when the involved regions are symmetric about a vertical axis. We note that even though this special, symmetric case of Theorems 3.1 and 3.2 is enough for our purposes, *proving* the formulas in these theorems requires the general form of the regions (as Kuo's graphical condensation, when applied to a symmetric region belonging to this family, always gives rise to some regions that are not symmetric).

Let $z$ be even. Consider the region $G_{x,y,z,y}$, and apply to it the factorization theorem of [**3**] (see Figure 5.1). The resulting regions on the left and right are of type $\underline{A}$ and type $C$, respectively. We obtain

$$\mathrm{M}(G_{x,y,z,y}) = 2^{y-z}\, \mathrm{M}(\underline{A}_{y-z,\frac{z}{2},x-y+z+1})\, \mathrm{M}(C_{y-z-1,\frac{z}{2},x-y+z+1}). \tag{5.1}$$

Doing the same for the region $G_{x+1,y,z-1,y}$ (see Figure 5.2), we again obtain regions of type $\underline{A}$ and $C$ on the two sides of the symmetry axis. Crucially, the region of type $\underline{A}$ turns out to be precisely the same as before! Therefore, equation (5.1) and its analog

$$\mathrm{M}(G_{x+1,y,z-1,y}) = 2^{y-z+1}\, \mathrm{M}(\underline{A}_{y-z,\frac{z}{2},x-y+z+1})\, \mathrm{M}(C_{y-z+1,\frac{z}{2}-1,x-y+z+1}), \tag{5.2}$$

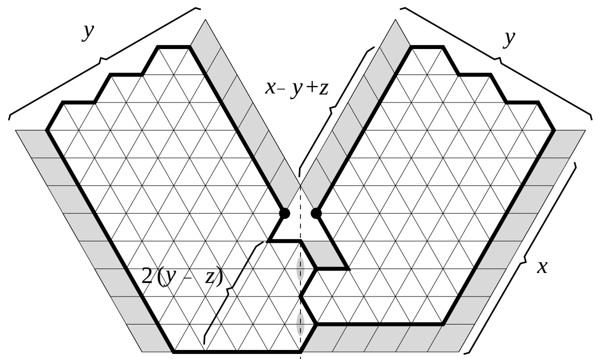


FIGURE 5.1. Regions $\underline{A}_{y-z,\frac{z}{2},x-y+z+1}$ and $C_{y-z-1,\frac{z}{2},x-y+z+1}$ arise when applying the factorization theorem to the region $G_{x,y,z,y}$ (here $x=7$, $y=4$, $z=2$).

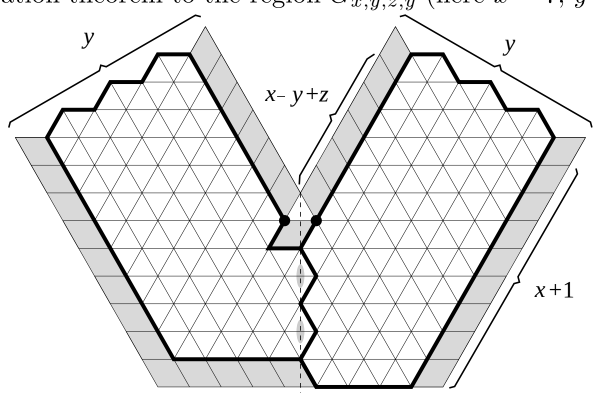


FIGURE 5.2. Regions $\underline{A}_{y-z,\frac{z}{2},x-y+z+1}$ and $C_{y-z+1,\frac{z}{2}-1,x-y+z+1}$ arise when applying the factorization theorem to the region $G_{x+1,y,z-1,y}$ (here $x=7$, $y=4$, $z=2$).

which follows from applying the factorization theorem as shown in Figure 5.2, can be combined to obtain

$$\frac{\mathrm{M}(C_{y-z-1,\frac{z}{2},x-y+z+1})}{\mathrm{M}(C_{y-z+1,\frac{z}{2}-1,x-y+z+1})} = 2\,\frac{\mathrm{M}(G_{x,y,z,y})}{\mathrm{M}(G_{x+1,y,z-1,y})}. \tag{5.3}$$

Using (3.6), changing the parameters $(x,y,z)$ to $(n+x, n+2k+1, 2k)$ and writing for brevity

$$g_0(x,y,z) := g(x,y,z,y), \tag{5.4}$$

we can rewrite (5.3) as

$$\frac{\mathrm{M}(C_{n,k,x})}{\mathrm{M}(C_{n+2,k-1,x})} = 2\,\frac{g_0(n+x,n+2k+1,2k)}{g_0(n+x+1,n+2k+1,2k-1)}. \tag{5.5}$$

Repeated application of (5.5) gives

$$\frac{\mathrm{M}(C_{n,k,x})}{\mathrm{M}(C_{n+2k,0,x})} = 2^k \prod_{i=0}^{k-1} \frac{g_0(n+x+2i, n+2k+1, 2k-2i)}{g_0(n+x+2i+1, n+2k+1, 2k-2i-1)}. \tag{5.6}$$

Using formula (4.1) for $\mathrm{M}(C_{n+2k,0,x})$, we obtain from (5.6) that

$$\mathrm{M}(C_{n,k,x}) = \frac{1}{2^{n+k}} \prod_{i=1}^{n+2k} \frac{(2x+2i+2)_i \left(x+2i+\frac{3}{2}\right)_{i-1}}{(i)_i \left(x+i+\frac{3}{2}\right)_{i-1}}$$
$$\times \prod_{i=0}^{k-1} \frac{g_0(n+x+2i, n+2k+1, 2k-2i)}{g_0(n+x+2i+1, n+2k+1, 2k-2i-1)}, \tag{5.7}$$

where $g_0$ is given by (5.4), (3.8) and (3.3).

After some straightforward manipulations of the expression on the right hand side above we obtain the following result.

Theorem 5.1. *For non-negative integers $n$, $k$ and $x$ we have*[5]

$$\mathrm{M}(C_{n,k,x}) = \frac{1}{2^n}\binom{2k+x}{2k} \prod_{i=1}^{2k-1}\prod_{j=i}^{2k-1} \frac{2x+i+j+1}{i+j+1} \prod_{i=1}^{n} \frac{(6k+2i+2)_i(3k+2i+3/2)_{i-1}}{(i)_i(3k+i+3/2)_{i-1}}$$

$$\times \prod_{i=1}^{n} \frac{\dfrac{(x+i+k+1)_k(x+2i+3k+3/2)_{i-1}(x+i+k+1/2)_k(x+i+3k+1)_{\lceil i/2\rceil}}{(x+2i+3k-1/2)_{\lceil i/2\rceil-1,\searrow}}}{\dfrac{(i+k+1)_k(2i+3k+3/2)_{i-1}(i+k+1/2)_k(i+3k+1)_{\lceil i/2\rceil}}{(2i+3k-1/2)_{\lceil i/2\rceil-1,\searrow}}}. \tag{5.8}$$

Similar reasoning leads to a formula for $\mathrm{M}(D_{n,k,x})$. Indeed, consider the region $F'_{x,y,z,y}$, and apply to it the factorization theorem of [**3**]. The resulting regions turn out to be precisely $D_{y-z,\frac{z}{2},x-y+z}$ and $\overline{B}_{y-z-1,\frac{z}{2},x-y+z}$ (see Figure 5.3). The factorization theorem then gives

$$\mathrm{M}(F'_{x,y,z,y}) = 2^{y-z}\,\mathrm{M}(D_{y-z,\frac{z}{2},x-y+z})\,\mathrm{M}(\overline{B}_{y-z-1,\frac{z}{2},x-y+z}). \tag{5.9}$$

On the other hand, by applying the factorization theorem to $F'_{x-1,y,z+1,y}$ (see Figure 5.4), we obtain in the same way

$$\mathrm{M}(F'_{x-1,y,z+1,y}) = 2^{y-z-1}\,\mathrm{M}(D_{y-z-2,\frac{z}{2}+1,x-y+z})\,\mathrm{M}(\overline{B}_{y-z-1,\frac{z}{2},x-y+z}). \tag{5.10}$$

Combining the above two equations, replacing $(x,y,z)$ by $(n+x+2, n+2k, 2k-2)$, using Theorem 3.1 and setting for brevity

$$f_0(x,y,z) := f(x,y,z,y), \tag{5.11}$$

[5]We denote by $(a)_{k,\searrow}$ the "descending factorial" $a(a-1)\cdots(a-k+1)$, and set $(a)_{0,\searrow}=1$.

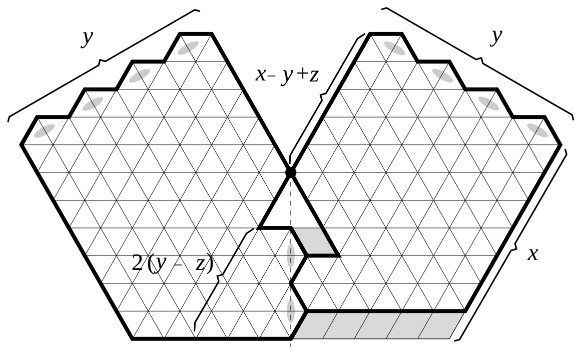


FIGURE 5.3. Regions $D_{y-z,\frac{z}{2},x-y+z}$ and $\overline{B}_{y-z-1,\frac{z}{2},x-y+z}$ arise when applying the factorization theorem to the region $F'_{x,y,z,y}$ (here $x = 7$, $y = 4$, $z = 2$).

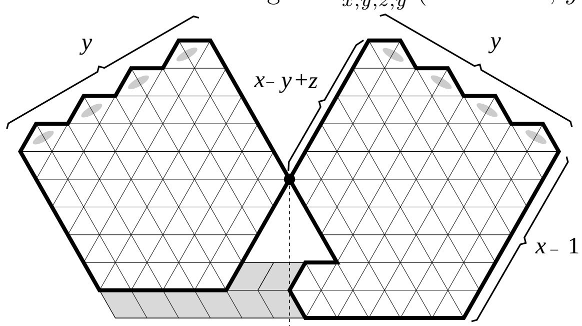


FIGURE 5.4. Regions $D_{y-z-2,\frac{z}{2}+1,x-y+z}$ and $\overline{B}_{y-z-1,\frac{z}{2},x-y+z}$ arise when applying the factorization theorem to the region $F'_{x-1,y,z+1,y}$ (here $x = 7$, $y = 4$, $z = 2$).

we obtain

$$\frac{\mathrm{M}(D_{n,k,x})}{\mathrm{M}(D_{n+2,k-1,x})} = 2\frac{f_0\left(n+x+\frac{1}{2}, n+2k, 2k-1\right)}{f_0\left(n+x+\frac{3}{2}, n+2k, 2k-2\right)}. \tag{5.12}$$

Repeated application of (5.12) gives

$$\frac{\mathrm{M}(D_{n,k,x})}{\mathrm{M}(D_{n+2k,0,x})} = 2^k \prod_{i=0}^{k-1} \frac{f_0\left(n+x+2i+\frac{1}{2}, n+2k, 2k-2i-1\right)}{f_0\left(n+x+2i+\frac{3}{2}, n+2k, 2k-2i-2\right)}. \tag{5.13}$$

Replacing the denominator in the left hand side above with the formula given in Theorem 4.1, we obtain after some manipulation the following result.

Theorem 5.2. *For non-negative integers $n$, $k$ and $x$ we have*[6]

$$\mathrm{M}(D_{n,k,x}) =$$

$$\frac{1}{2^{n+2k}} \frac{\binom{2k+x+2}{2k+1}}{\binom{2k+2}{2k+1}} \frac{(k+2)_{2k} \left(x+\frac{1}{2}\right)_k \left(x+2n+3k+\frac{1}{2}\right)_n \left(x+\frac{3}{2}\right)_k (x+1)_k (x+n+3k)_{k,\searrow}}{(x+k+2)_{2k} \left(\frac{1}{2}\right)_k \left(2n+3k+\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_k (1)_k (n+3k)_{k,\searrow}}$$

$$\times \prod_{i=1}^{2k} \prod_{j=i}^{2k} \frac{2x+i+j+3}{i+j+3} \prod_{i=0}^{n} \frac{\left(3k+2i-\frac{1}{2}\right)_{i+1} (6k+2i-1)_i}{(i+1)_i \left(3k+i-\frac{1}{2}\right)_{i+1}}$$

$$\times \prod_{i=1}^{n-2} \frac{\dfrac{\left(x+i+k+\frac{5}{2}\right)_k \left(x+2i+3k+\frac{7}{2}\right)_i (x+i+k+2)_{k+1} (x+i+3k+3)_{\lfloor i/2 \rfloor}}{\left(x+2i+3k+\frac{3}{2}\right)_{\lfloor i/2 \rfloor,\searrow}}}{\dfrac{\left(i+k+\frac{5}{2}\right)_k \left(2i+3k+\frac{7}{2}\right)_i (i+k+2)_{k+1} (i+3k+3)_{\lfloor i/2 \rfloor}}{\left(2i+3k+\frac{3}{2}\right)_{\lfloor i/2 \rfloor,\searrow}}}. \tag{5.14}$$

The formulas for the remaining regions (those of type $\overline{C}$ and $\underline{D}$) are obtained in a similar fashion. More precisely, application of the factorization theorem to the regions $F_{x,y,z,y}$ and $F_{x+1,y,z-1,y}$ as indicated in Figures 5.5 and 5.6 leads, after combining the two resulting equalities, to the equation

$$\frac{\mathrm{M}(\overline{C}_{n,k,x})}{\mathrm{M}(\overline{C}_{n+2,k-1,x})} = 2 \frac{f_0\,(n+x+1, n+2k+1, 2k)}{f_0\,(n+x+2, n+2k+1, 2k-1)}. \tag{5.15}$$

Repeated application of this gives

$$\frac{\mathrm{M}(\overline{C}_{n,k,x})}{\mathrm{M}(\overline{C}_{n+2k,0,x})} = 2^k \prod_{i=0}^{k-1} \frac{f_0(n+x+2i+1, n+2k+1, 2k-2i)}{f_0(n+x+2i+2, n+2k+1, 2k-2i-1)}. \tag{5.16}$$

However, by definitions the region $\overline{C}_{n,0,x}$ is precisely the same as the region $C_{n+1,0,x-1}$. Therefore, we obtain from (5.16) and Theorem 4.1, after some simplifications in the resulting formula, the following result.

Theorem 5.3. *For non-negative integers $n$, $k$ and $x$ we have*

$$\mathrm{M}(\overline{C}_{n,k,x}) = \frac{1}{2^{n+2k+1}} \frac{\binom{2k+x+1}{2k+1}}{\binom{2k+\frac{1}{2}}{2k+1}} \prod_{i=1}^{2k} \prod_{j=i}^{2k} \frac{2x+i+j+1}{i+j} \prod_{i=0}^{n} \frac{(6k+2i+2)_i (3k+2i+1)_{i+1}}{(i+1)_i (3k+i+1)_{i+1}}$$

$$\times \prod_{i=1}^{n} \frac{\dfrac{(x+i+k+3/2)_k (x+2i+3k+5/2)_i (x+i+k+1)_{k+1} (x+i+3k+2)_{\lfloor i/2 \rfloor}}{(x+2i+3k+1/2)_{\lfloor i/2 \rfloor,\searrow}}}{\dfrac{(i+k+1)_k (2i+3k+2)_i (i+k+1/2)_{k+1} (i+3k+3/2)_{\lfloor i/2 \rfloor}}{(2i+3k)_{\lfloor i/2 \rfloor,\searrow}}}. \tag{5.17}$$

[6] Recall that when the limits of the product index are out of order (this is possible in the last product) we interpret products according to (3.1).

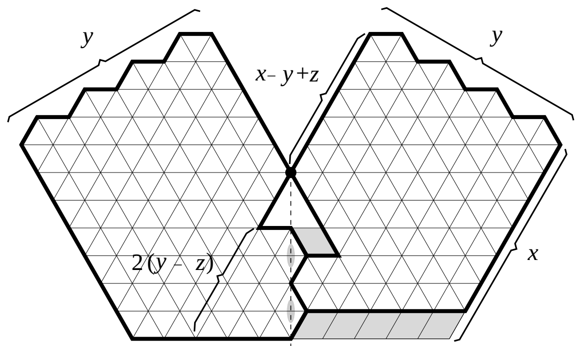


FIGURE 5.5. Regions $A_{y-z,\frac{z}{2},x-y+z}$ and $\overline{C}_{y-z-1,\frac{z}{2},x-y+z}$ arise when applying the factorization theorem to the region $F_{x,y,z,y}$ (here $x=7$, $y=4$, $z=2$).

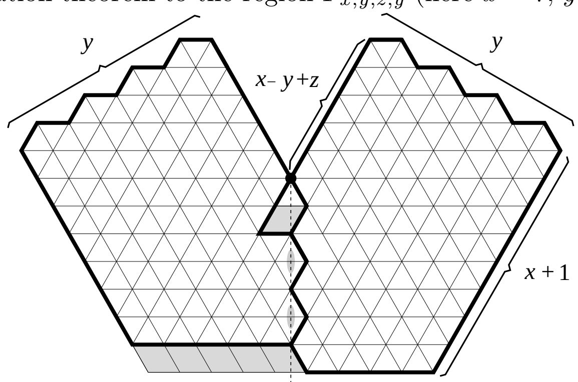


FIGURE 5.6. Regions $A_{y-z,\frac{z}{2},x-y+z}$ and $\overline{C}_{y-z+1,\frac{z}{2}-1,x-y+z}$ arise when applying the factorization theorem to the region $F_{x+1,y,z-1,y}$ (here $x=7$, $y=4$, $z=2$).

Finally, in order to enumerate tilings of the $\underline{D}$-regions, apply the factorization theorem to the regions $G'_{x,y,z,y}$ and $G'_{x-1,y,z+1,y}$ as indicated in Figures 5.7 and 5.8. Combining the resulting equations and applying repeatedly the resulting equation we obtain

$$\frac{\mathrm{M}(\underline{D}_{n,k,x})}{\mathrm{M}(\underline{D}_{n+2k,0,x})} = 2^k \prod_{i=0}^{k-1} \frac{g_0\left(n+x+2i-\frac{1}{2}, n+2k, 2k-2i-1\right)}{g_0\left(n+x+2i+\frac{1}{2}, n+2k, 2k-2i-2\right)}. \tag{5.18}$$

However, by definition (see footnote 1) the region $\underline{D}_{n,0,x}$ becomes, after removing its one forced lozenge (which is weighted 1/2), precisely the region $D_{n-1,0,x+1}$, so in particular

$$\mathrm{M}(\underline{D}_{n,0,x}) = \frac{1}{2}\,\mathrm{M}(D_{n-1,0,x+1}). \tag{5.19}$$

Using (5.18), (5.19) and Theorem 4.1 we obtain after some simplifications the following result.

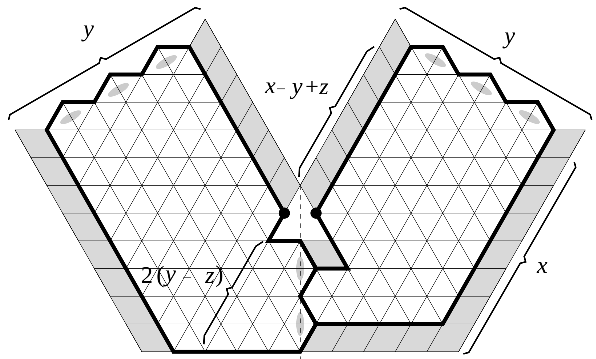


FIGURE 5.7. Regions $\underline{D}_{y-z,\frac{z}{2},x-y+z+1}$ and $B_{y-z-1,\frac{z}{2},x-y+z+1}$ arise when applying the factorization theorem to the region $G'_{x,y,z,y}$ (here $x=7$, $y=4$, $z=2$).

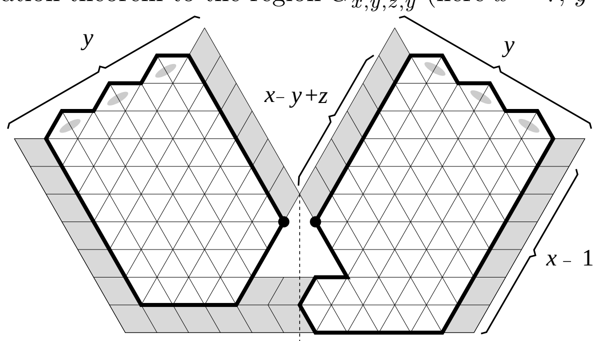


FIGURE 5.8. Regions $\underline{D}_{y-z-2,\frac{z}{2}+1,x-y+z+1}$ and $B_{y-z-1,\frac{z}{2},x-y+z+1}$ arise when applying the factorization theorem to the region $G'_{x-1,y,z+1,y}$ (here $x=7$, $y=4$, $z=2$).

THEOREM 5.4. *For any non-negative integers $n$, $k$ and $x$ we have*

$$\mathrm{M}(\underline{D}_{n,k,x}) = \frac{1}{2^{n+1}} \frac{\binom{2k+x}{2k+1}}{\binom{2k+\frac{1}{2}}{2k+1}} \frac{(x+3k+n-1)_{k,\searrow}\left(\frac{1}{2}\right)_{n+k}\left(k+\frac{1}{2}\right)_{2k}}{(x)_{n+k}(x+k)_{2k}\left(3k+n-\frac{1}{2}\right)_{k,\searrow}}$$

$$\times \prod_{i=0}^{n} \frac{(6k+2i-2)_{i+1}(3k+2i-1)_i}{(3k+i-1)_{i+1}(i+1)_i} \prod_{i=1}^{2k} \frac{(2x+i+1)_{2k}}{(i+1)_{2k}}$$

$$\times \prod_{i=1}^{n-1} \frac{\dfrac{\left(x+i+k+\frac{1}{2}\right)_k\left(x+2i+3k+\frac{3}{2}\right)_i(x+i+k)_{k+1}(x+i+3k+1)_{\lfloor i/2\rfloor}}{\left(x+2i+3k-\frac{1}{2}\right)_{\lfloor i/2\rfloor,\searrow}}}{\dfrac{(i+k+1)_k(2i+3k+2)_i\left(i+k+\frac{1}{2}\right)_{k+1}\left(i+3k+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(2i+3k)_{\lfloor i/2\rfloor,\searrow}}}. \tag{5.20}$$

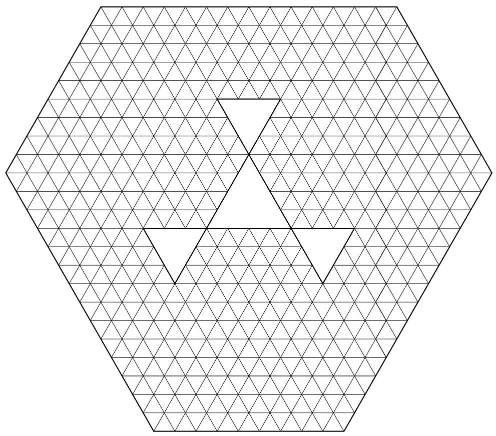

FIGURE 6.1. $SC_{5,5,5}(3,3,3,4)$.

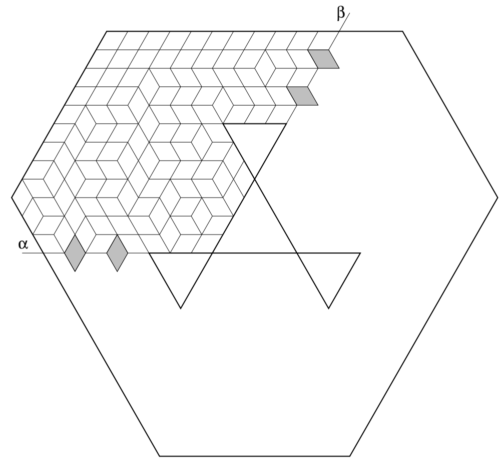


FIGURE 6.2. Paths of lozenges in a fundamental region.

## 6. The cyclically symmetric case

In order for cyclically symmetric (i.e., invariant under rotation by 120°) tilings of the $S$-cored hexagon $SC_{x,y,z}(a,b,c,m)$ to exist we clearly need to have $x=y=z$ and $a=b=c$ (recall that $m$ is the side-length of the triangular core, and $a$, $b$ and $c$ are the sidelengths of the triangular lobes, counterclockwise from top; see [**7**] for the precise definition of $SC_{x,y,z}(a,b,c,m)$). Let us denote by $CS(x,a,m)$ the number of cyclically symmetric tilings of $SC_{x,x,x}(a,a,a,m)$.

We will make use of the following extension of [**4**, Lemma 3.1].

LEMMA 6.1.

$$CS(x,a,m) = \det\left(\delta_{i,a+j} + \binom{m+a+i+j}{a+i}\right)_{0\le i,j\le x-1}, \tag{6.1}$$

*where $\delta_{ij}$ is the Kronecker symbol.*

*Proof.* The proof of [**4**, Lemma 3.1] readily extends to the current set-up (compare Figures 6.1 and 6.2 to [**4**, Figures 3.1 and 3.2]). The only new point is that in the present context the number of cyclically-symmetric lozenge tilings arises as the sum of minors of the Gessel-Viennot matrix that are not principal, but instead the column indices are obtained by increasing each row index by $a$ units. This explains the shift at the index of the Kronecker symbol on the right hand side of (6.1). □

Let $G$ be the planar dual graph of the region $SC_{x,x,x}(a,a,a,m)$ (i.e., the graph whose vertices are the unit triangles contained in the region, and whose edges connect pairs of triangles that share an edge). Since tilings of a region can be viewed as perfect matchings of the planar dual of the region, the lozenge tilings of $SC_{x,x,x}(a,a,a,m)$ can be identified with perfect matchings of $G$. Furthermore, cyclically symmetric tilings of $SC_{x,x,x}(a,a,a,m)$

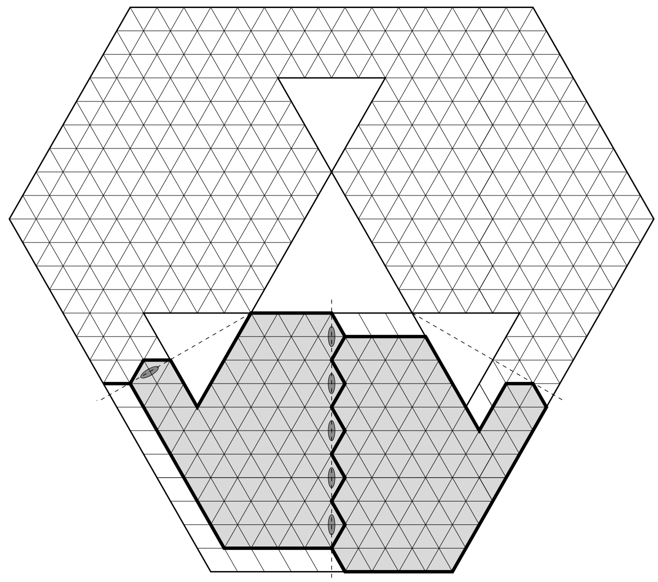

FIGURE 6.3. For $a$ even and $m$ even the cyclically symmetric case reduces to regions of type $C$ and $D$; shown here is the region $SC_{x,x,x}(a,a,a,m)$ for $x=3$, $a=4$, $m=6$.

correspond to perfect matchings of $G$ invariant under rotation by 120°. In turn, these are identified with perfect matchings of the orbit graph of $G$ under the action of this rotation.

Proceeding in a way analogous to the arguments in [**4**, §3], we apply the factorization theorem of [**3**] to this orbit graph and rephrase the result in terms of the regions to which the resulting subgraphs are dual. The details of the resulting regions depend on the parities of $a$ and $m$. As we will see, by Lemma 6.1 it suffices to solve the problem for one parity of $m$, and the case of the other parity of $m$ will follow.

When $a+m$ is even, it turns out that one of the families of regions that results from the factorization theorem is precisely the one whose tiling enumeration we need in order to solve the cyclically symmetric and transpose complementary case (which we treat in Section 7). For this reason, we work out first the cases $a$ even, $m$ even and $a$ odd, $m$ odd; we then deduce the general case from these using Lemma 6.1.

Writing the arguments in a way that shows their parity, we obtain this way (see Figure 6.3)

$$CS(2x+1,2a,2m) = 2^{2x+2a+1}\,\mathrm{M}(C_{x,a,m})\,\mathrm{M}(D_{x,a,m}), \tag{6.2}$$

where the families of regions $C_{n,k,x}$ and $D_{n,k,x}$ are described in Section 2 (see top half of Figure 2.2).

A figure similar to Figure 3.3 but with the $x$-parameter even shows that

$$CS(2x+2,2a,2m) = 2^{2x+2a+2}\,\mathrm{M}(C_{x,a,m})\,\mathrm{M}(D_{x+1,a,m}). \tag{6.3}$$

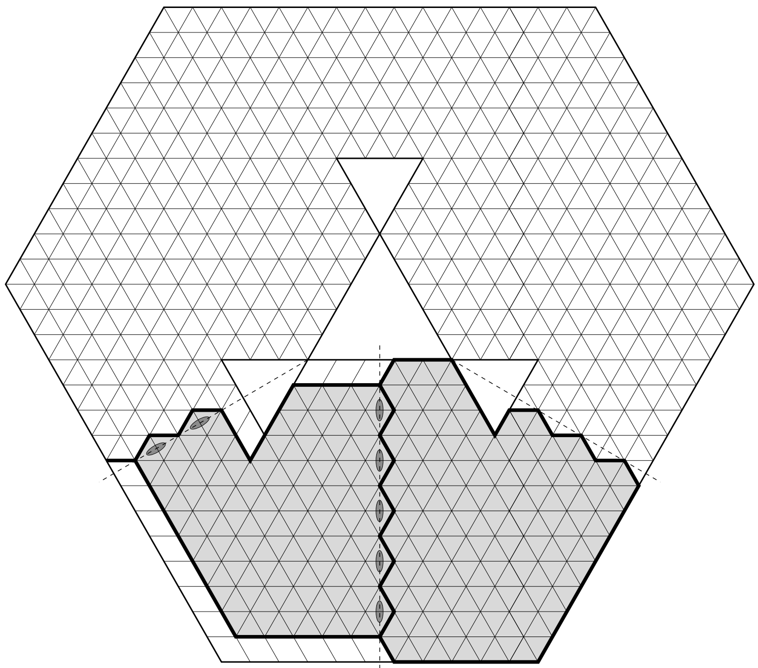

Figure 6.4. For $a$ odd and $m$ odd the cyclically symmetric case reduces to regions of type $\overline{C}$ and $\underline{D}$; shown here is the region $SC_{x,x,x}(a,a,a,m)$ for $x=3$, $a=3$, $m=5$.

In the same fashion, Figure 6.4 and the bottom half of Figure 2.2 show that

$$CS(2x+1,2a+1,2m+1) = 2^{2x+2a+2}\,\mathrm{M}(\overline{C}_{x,a,m})\,\mathrm{M}(\underline{D}_{x,a+1,m+1}) \tag{6.4}$$

and a figure analogous to Figure 6.4 but with the $x$-parameter even yields

$$CS(2x+2,2a+1,2m+1) = 2^{2x+2a+3}\,\mathrm{M}(\overline{C}_{x+1,a,m+1})\,\mathrm{M}(\underline{D}_{x,a+1,m+1}). \tag{6.5}$$

We have thus proved the following result.

Theorem 6.2. *For non-negative integers $x$, $a$ and $m$ we have*

$$CS(2x+1,2a,2m) = 2^{2x+2a+1}\,\mathrm{M}(C_{x,a,m})\,\mathrm{M}(D_{x,a,m}) \tag{6.6}$$

$$CS(2x+2,2a,2m) = 2^{2x+2a+2}\,\mathrm{M}(C_{x,a,m})\,\mathrm{M}(D_{x+1,a,m}) \tag{6.7}$$

$$CS(2x+1,2a+1,2m+1) = 2^{2x+2a+2}\,\mathrm{M}(\overline{C}_{x,a,m})\,\mathrm{M}(\underline{D}_{x,a+1,m+1}) \tag{6.8}$$

$$CS(2x+2,2a+1,2m+1) = 2^{2x+2a+3}\,\mathrm{M}(\overline{C}_{x+1,a,m+1})\,\mathrm{M}(\underline{D}_{x,a+1,m+1}), \tag{6.9}$$

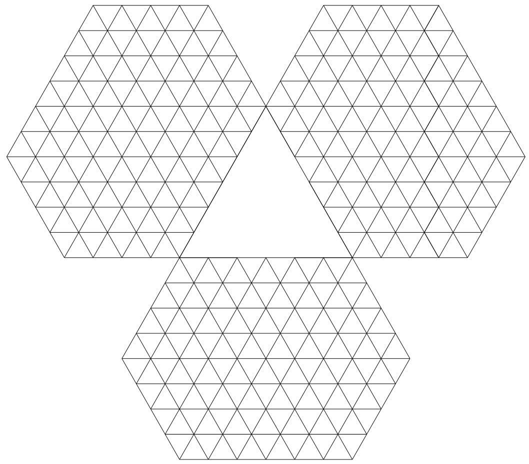

FIGURE 6.5. The region $SC_{0,0,0}(a,a,a,m)$ is the union of three independent hexagons.

*where the quantities on the right hand sides are given by the formulas of Theorems 5.1–5.4.*

By Lemma 6.1, for fixed $x$ and $a$, the expressions on the right hand side in (6.6)–(6.9) are polynomials in $m$. Define $P_{2x+1,2a}(m)$ and $P_{2x+2,2a}(m)$ to be the polynomials obtained from the expressions on the right hand sides of (6.6) and (6.7), respectively, by replacing $m$ with $m/2$. Similarly, define $P_{2x+1,2a+1}(m)$ and $P_{2x+2,2a+1}(m)$ to be the polynomials obtained from the expressions on the right hand sides of (6.8) and (6.9), respectively, by replacing $m$ with $(m-1)/2$.

COROLLARY 6.3. *With the above definition of the polynomials $P_{x,a}(m)$, for non-negative integers $a$, $m$ and $x$ with $x \geq 1$ we have*

$$CS(x,a,m) = P_{x,a}(m). \tag{6.10}$$

*Proof.* By Theorem 6.2, when $a$ is even, (6.10) holds if $m$ is even and $m \geq 0$. Since the two sides of (6.10) are polynomials in $m$ (the left hand side by Lemma 6.1), they must be equal. The case when $a$ is odd follows in the same way, since according to Theorem 6.2, (6.10) holds if $m$ is odd and $m \geq 1$. $\square$

The case of cyclically symmetric tilings of the $S$-cored hexagon $SC_{x,x,x}(a,a,a,m)$ is covered by Theorem 6.2 and Corollary 6.3, provided $x \geq 1$. The remaining case of $x = 0$ follows directly, due to the fact that the region $SC_{0,0,0}(a,a,a,m)$ is the union of three independent hexagons, each of sides $a$, $a$, $m$, $a$, $a$, $m$ (see Figure 6.5), and thus its number of 120°-rotation invariant tilings is simply the number of tilings of one of those hexagons. By MacMahon's classical theorem [**17**, Sect. 495] that enumerates tilings of such hexagons, we thus obtain

$$CS(0,a,m) = \frac{\mathrm{H}(a)^2\,\mathrm{H}(m)\,\mathrm{H}(2a+m)}{\mathrm{H}(a+m)^2\,\mathrm{H}(2a)}, \tag{6.11}$$

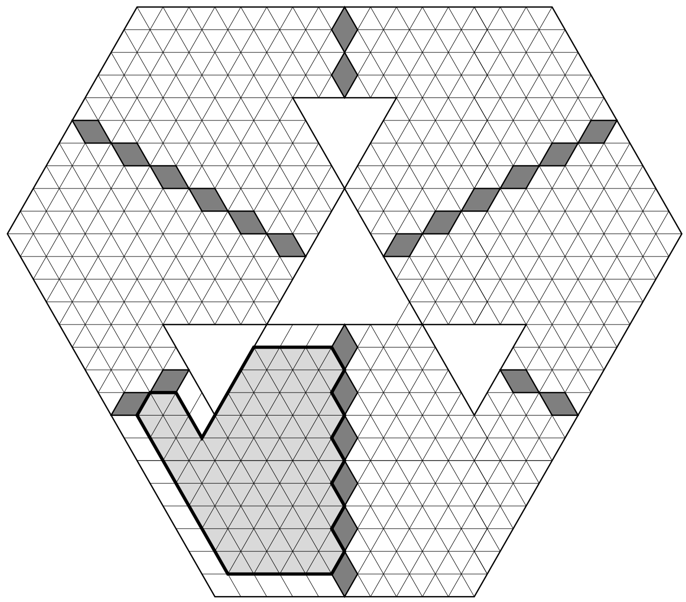

Figure 7.1. Reduction of the CSTC case to a $C$-region.

where the hyperfactorials $\mathrm{H}(n)$ are defined by

$$\mathrm{H}(n) := 0!\,1!\cdots(n-1)!. \tag{6.12}$$

This completes the case of cyclically symmetric tilings.

## 7. The cyclically symmetric and transpose complementary case

This case turns out to reduce to the enumeration of tilings of the regions of type $C$ defined in Section 2.

In order for the $S$-cored hexagon $SC_{x,y,z}(a,b,c,m)$ to have tilings that are both cyclically symmetric and transpose complementary, we need to have $x=y=z$, $a=b=c$, and all of $x$, $a$ and $m$ must be even. Let us denote by $CSTC(2x,2a,2m)$ the number of tilings of $SC_{2x,2x,2x}(2a,2a,2a,2m)$ that are both cyclically symmetric and transpose complementary.

Theorem 7.1. *For non-negative integers $x$, $a$ and $m$ with $x \geq 1$ we have*

$$CSTC(2x,2a,2m) = \mathrm{M}(C_{x-1,a,m}), \tag{7.1}$$

*where the quantity on the right hand side is given by the product formula of Theorem 5.1.*

*Proof.* Consider the region $SC_{2x,2x,2x}(2a,2a,2a,2m)$, illustrated in Figure 7.1 for $x=2$, $a=2$ and $m=3$. In any tiling of it that is transpose complementary, all vertical lozenge positions along its vertical symmetry axis must be occupied by lozenges (see Figure 7.1). Moreover, the lozenges obtained from these by rotating them 120° and 240° must be part of any tiling that is in addition also cyclically symmetric. Upon removal of all these forced lozenges from $SC_{2x,2x,2x}(2a,2a,2a,2m)$, the leftover region becomes the union of six independent and congruent regions. Clearly, the tilings of any one of these are in one to one correspondence with the tilings of $SC_{2x,2x,2x}(2a,2a,2a,2m)$ that are cyclically symmetric and transpose complementary. However, after removing the forced lozenges from this fundamental region, one obtains precisely the region $C_{x-1,a,m}$. This completes the proof. □

As seen at the end of the previous section (see Figure 6.5), when $x=0$ the $S$-cored hexagon reduces to three independent hexagons, each of side-lengths $2a$, $2a$, $2m$, $2a$, $2a$, $2m$ (in cyclic order). Thus $CSTC(0,2a,2m)$ is just the number of tilings of such a hexagon that are invariant under reflection across its symmetry axis perpendicular to the sides of length $2m$. However, the latter can be identified (for example, via the bijection described in [**11**]) with the transpose complementary plane partitions fitting inside a $2a\times 2a\times 2m$ box. Therefore, by Proctor's formula [**18**] we obtain

$$CSTC(0,2a,2m)=\binom{2a+m-1}{2a-1}\prod_{i=1}^{2a-2}\prod_{j=i}^{2a-2}\frac{2m+i+j+1}{i+j+1}. \tag{7.2}$$

This completes the cyclically symmetric and transpose complementary case.

## 8. Concluding remarks

In this paper we considered the problem of enumerating the symmetry classes of lozenge tilings of the $S$-cored hexagons introduced in [**7**], and solved two of the six new cases that arise. Our solution is based on previous results we obtained in [**8**], as well as new counterparts to them presented in the current paper.

The remaining cases require different techniques and will be addressed in subsequent papers.

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