On an involution on the set of Littlewood–Richardson tableaux
A module model for Azenhas’ bijection

Itaru Terada

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Outline

1 Overview

2 Preliminaries

3 Main Theorem
Azenhas’ procedure: $\mathcal{LR}(\lambda/\mu, \nu) \sim LR(\lambda/\nu, \mu)$, $T \mapsto T^\vee$

(1999 or 2000)

She expressed hope to interpret her procedure using $R$-modules of the following form ($R$: PID, $p \in R$ prime):

$$R/(p^{\lambda_1}) \oplus R/(p^{\lambda_2}) \oplus \cdots \oplus R/(p^{\lambda_l}).$$

We give a possible answer for $R = \mathbb{C}[t]$, $p = t$ (an indet.).

**Thm. (T)**

Set $M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l})$.

Then, for most submodules $N$ yielding a given LR-tableau $T$, the submodule $N^\perp \subset M^*$ yields $T^\vee$.

A precise result is phrased using the irreducible components of certain submodule varieties.
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Overview
Preliminaries
Main Theorem

**Littlewood-Richardson (LR) tableaux**

\[
\begin{array}{cccc}
 & & & 1 1 \\
 & & 1 & \\
 & 1 & 1 & 2 \\
2 & 2 & 2 & 3
\end{array}
\]

\[\in \mathcal{LR}((8, 6, 2, 1)/(6, 5, 2), (5, 4, 1))\]

Conditions: \(\leq, \land\), lattice permutation condition (rephrased):

- before row 1
- row 1
- rows 1-2
- rows 1-3
- rows 1-4

<table>
<thead>
<tr>
<th># 1's</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>5</th>
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</tr>
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<tbody>
<tr>
<td># 2's</td>
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<td># 3's</td>
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Itaru Terada
A module model
Littlewood-Richardson (LR) tableaux

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</table>

\[ \wedge \]

\[ \leq \]
Littlewood-Richardson (LR) tableaux

\[ \begin{array}{cccc}
1 & 1 \\
1 & 1 & 2 \\
2 & 2 & 2 & 3 \\
\end{array} \]

\[ \in LR((8, 6, 2, 1)/ (6, 5, 2), (5, 4, 1)) \]

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Littlewood-Richardson (LR) tableaux

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Itaru Terada  A module model
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\[ \begin{array}{cccc}
1 & 1 & \quad & \\
1 & 1 & 2 & \\
2 & 2 & 2 & 3
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Littlewood-Richardson (LR) tableaux

\[ \begin{array}{cccc}
2 & 2 & 2 & 3 \\
1 & 1 & 1 & 2 \\
1 & 1 & & 1 \\
& & & 1 \\
\end{array} \]

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before row 1  row 1  rows 1-2  rows 1-3  rows 1-4

\[
\begin{array}{ccccccc}
\# 1 \text{'s} & 0 & 2 & 3 & 5 & 5 \\
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\# 3 \text{'s} & 0 & 0 & 0 & 0 & 1 \\
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\[ \begin{array}{ccccccc} & & & & & 1 \leq & 1 \\ & & & & 1 \leq & 1 \leq & 2 \\ & & & 2 \leq & 2 \leq & 2 \leq & 3 \end{array} \]

\[ \in LR((8, 6, 2, 1)/(6, 5, 2), (5, 4, 1)) \]

Conditions: \( \leq \), \( \wedge \), lattice permutation condition (rephrased):

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$$\in \mathcal{LR}\left((8, 6, 2, 1) / (6, 5, 2, 5, 4, 1)\right)$$

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Conditions: \( \leq, \wedge, \) lattice permutation condition (rephrased):

\[
\begin{array}{ccccccc}
& & & & 1 & \leq 1 & \\
& & 1 & \leq 1 & \leq 2 & \\
& 2 & \leq 1 & \leq 2 & \leq 3 & \\
\end{array}
\]

before row 1 row 1 rows 1-2 rows 1-3 rows 1-4

\[
\begin{array}{ccccccc}
\# 1's & 0 & 2 & 3 & 5 & 5 \\
\# 2's & 0 & 0 & 0 & 1 & 3 \\
\# 3's & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]
Littlewood-Richardson sequences

Such a sequence of partitions is called a Littlewood-Richardson (LR) sequence.
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Azenhas’ procedure

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \]

\[ T = \begin{array}{cccc}
1 & 1 & 1 \\
1 & & \\
1 & 2 & 2 \\
1 & 2 & 2 & 3
\end{array} \]

\[ \mapsto \]

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(5,4,1),(6,5,2)}) \]

\[ T = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3
\end{array} \]

\[ T(0) = \emptyset \]

\[ T^{(1)} = \begin{array}{cc}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2
\end{array} \]

\[ T^{(2)} = \begin{array}{cc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2
\end{array} \]

\[ T^{(3)} = \begin{array}{cc}
1 & 1 & 1 & 1 \\
1 & 2 & 2
\end{array} \]

full 4-deletion, full 3-deletion, full 2-deletion, full 1-deletion.
Azenhas’ procedure

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \]

\[ T = \begin{array}{cccc}
1 & 1 & & \\
1 & & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \Downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \begin{array}{cccc}
1 & 1 & 1 & \\
1 & 2 & & \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \Downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \Downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} \]

\[ \Downarrow \text{full 1-deletion} \]

\[ T(0) = \emptyset \]

\[ T(1) = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} \]

\[ \Downarrow \text{full 1-deletion} \]

\[ T(0) = \emptyset \]
Azenhas’ procedure

\[ \mathcal{LR}(\langle 8,6,2,1 \rangle / \langle 6,5,2 \rangle, \langle 5,4,1 \rangle) \]

\[ T = \]

\[ \mathcal{LR}(\langle 8,6,2,1 \rangle / \langle 5,4,1 \rangle, \langle 6,5,2 \rangle) \]

\[ T = T^\vee \]

\[ T^{(3)} \]

\[ T^{(2)} \]

\[ T^{(1)} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\[ \mathcal{LR}(\langle 8,6,2,1 \rangle, \langle 6,5,2 \rangle, \langle 5,4,1 \rangle) \]

\[ T = \]

\[
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
1 & 2 & 2 & 3 \\
\end{array}
\]

\[ \Downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \]

\[
\begin{array}{cccc}
1 & 1 & & \\
1 & & & \\
1 & 2 & 2 & 3 \\
1 & 2 & 2 & 3 \\
\end{array}
\]

\[ \Downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \]

\[
\begin{array}{cccc}
1 & 1 & & \\
1 & & & \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 \\
\end{array}
\]

\[ \Downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \]

\[
\begin{array}{cccc}
1 & & & \\
1 & & & \\
1 & 1 & 1 & 1 \\
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\[ \Downarrow \text{full 1-deletion} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\( \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \)

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T = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 2 & 3
\end{bmatrix}
\]

\( \Downarrow \) full 4-deletion

\( T^{(3)} = \begin{bmatrix}
1 & 1 & 1 \\
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\end{bmatrix} \)

\( \Downarrow \) full 3-deletion

\( T^{(2)} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
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1 & 2 & 2 & 2
\end{bmatrix} \)

\( \Downarrow \) full 2-deletion

\( T^{(1)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1
\end{bmatrix} \)

\( \Downarrow \) full 1-deletion

\( T^{(0)} = \emptyset \)

\( \mathcal{LR}(\frac{(8,6,2,1)}{(5,4,1),(6,5,2)}) \)

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3 & 3
\end{bmatrix}
\]

\( \Downarrow \) full 4-deletion

\( T^{(3)} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 \\
1 & 2 & 2 & 2
\end{bmatrix} \)

\( \Downarrow \) full 3-deletion

\( T^{(2)} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2
\end{bmatrix} \)

\( \Downarrow \) full 2-deletion

\( T^{(1)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1
\end{bmatrix} \)

Itaru Terada

A module model
Azenhas’ procedure

\[ \mathcal{LR}( (8,6,2,1) / (6,5,2), (5,4,1) ) \]

\[
\begin{array}{cccc}
 & & & 1 & 1 \\
 & & & 1 & \\
 & 1 & 2 & 2 & \\
1 & 2 & 2 & 3
\end{array}
\]

\[
\uparrow \text{full 4-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 \\
 & & & 1 & 2 & \\
 & 1 & 2 & 2 & 2 & \\
1 & 2 & 2 & 2 & 3
\end{array}
\]

\[
\uparrow \text{full 3-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 \\
 & 1 & 2 & 2 & 2 & \\
1 & 2 & 2 & 3 & 3
\end{array}
\]

\[
\uparrow \text{full 2-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 & \\
 & & & 1 & 2 & 2 & 2
\end{array}
\]

\[
\uparrow \text{full 1-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 & \\
 & & & 1 & 2 & 2 & 2
\end{array}
\]

\[
\uparrow \text{full 0-deletion} \Rightarrow T(0) = \emptyset
\]

\[
\mathcal{LR}( (8,6,2,1) / (5,4,1), (6,5,2) )
\]

\[
\uparrow \text{full 4-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 \\
 & & & 1 & 2 & \\
 & 1 & 2 & 2 & 2 & \\
1 & 2 & 3 & 3
\end{array}
\]

\[
\uparrow \text{full 3-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 \\
 & 1 & 2 & 2 & 2 & 2 \\
1 & 2 & 3 & 3
\end{array}
\]

\[
\uparrow \text{full 2-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 & 2 \\
 & & & 1 & 2 & 2 & 2 & 2
\end{array}
\]

\[
\uparrow \text{full 1-deletion}
\]

\[
\begin{array}{cccc}
 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 2 & 2 & 2 \\
 & & & 1 & 2 & 2 & 2 & 2
\end{array}
\]

\[
\uparrow \text{full 0-deletion} \Rightarrow T(0) = \emptyset
\]
Azenhas’ procedure

\[ L(R((8,6,2,1)/(6,5,2),(5,4,1))) \]

\[ T = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ \Rightarrow \text{full 1-deletion} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\( \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \)

Unit

\[ T = \frac{1 \ 1 \\
1 \ 2 \ 2 \\
1 \ 2 \ 2 \ 3}{1 \ 2 \ 2 \ 3} \]

Full 4-deletion

\[ T^{(3)} = \frac{1 \ 1 \ 1 \\
1 \ 2 \\
1 \ 2 \ 2 \ 2 \ 3}{1 \ 2 \ 2 \ 2 \ 3} \]

Full 3-deletion

\[ T^{(2)} = \frac{1 \ 1 \ 1 \ 1 \\
1 \ 2 \ 2 \ 2 \\
1 \ 2 \ 2 \ 2}{1 \ 2 \ 2 \ 3} \]

Full 2-deletion

\[ T^{(1)} = \frac{1 \ 1 \ 1 \ 1 \ 1}{1 \ 1 \ 1 \ 1 \ 1} \]

Full 1-deletion

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\[ \mathcal{L}\mathcal{R}((8,6,2,1)/ (6,5,2),(5,4,1)) \]

\[
T = \begin{array}{cccc}
1 & 1 & \downarrow \text{full 4-deletion} \\
& & \\
1 & 1 & 1 & 1 \\
& & & \downarrow \text{full 3-deletion} \\
1 & 1 & 1 & 1 & 1 \\
& & & & \downarrow \text{full 2-deletion} \\
1 & 1 & 1 & 1 & 1 & 1 & \downarrow \text{full 1-deletion} \\
& & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \downarrow \text{full 0-deletion} \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

\[ \mathcal{L}\mathcal{R}((8,6,2,1)/(5,4,1),(6,5,2)) \]

\[
T = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
& & & \downarrow \text{full 4-deletion} \\
1 & 1 & 1 & 1 & 1 \\
& & & & \downarrow \text{full 3-deletion} \\
1 & 1 & 1 & 1 & 1 & 1 \\
& & & & & \downarrow \text{full 2-deletion} \\
1 & 1 & 1 & 1 & 1 & 1 & \downarrow \text{full 1-deletion} \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

\[ \Rightarrow \quad T = T^\vee \]

Itaru Terada

A module model
Azenhas’ procedure

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \]

\[ T = \begin{array}{cccc}
| & | & | & 1 1 \\
| | | | 1 \\
| 1 2 2 \\
| 1 2 2 3 \\
\end{array} \]

\[ \downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \begin{array}{cccc}
| & | & | & 1 1 1 \\
| | | | 1 2 \\
| 1 2 2 2 3 \\
\end{array} \]

\[ \downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \begin{array}{cccc}
| & | & | & 1 1 1 1 \\
| | | | 1 2 2 2 \\
\end{array} \]

\[ \downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \begin{array}{cccc}
| & | & | & 1 1 1 1 1 \\
\end{array} \]

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(5,4,1),(6,5,2)}) \]

\[ T = \begin{array}{cccc}
| & | & | & 1 1 1 \\
| | | | 1 2 2 \\
| 1 2 3 3 \\
\end{array} \]

\[ \downarrow \text{full 4-deletion} \]

\[ \downarrow \text{full 3-deletion} \]

\[ \downarrow \text{full 2-deletion} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\[ LR((8,6,2,1)/(6,5,2),(5,4,1)) \]

\[ T = \]

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array}
\]

\[ \Downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array}
\]

\[ \Downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \]

\[
\begin{array}{cccc}
1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 \\
\end{array}
\]

\[ \Downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ \Downarrow \text{full 1-deletion} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\[ \mathcal{LR}(\langle 8,6,2,1 \rangle / \langle 6,5,2 \rangle, \langle 5,4,1 \rangle) \]

\[ T = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 1 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array} \]

\[ \mathcal{LR}(\langle 8,6,2,1 \rangle / \langle 5,4,1 \rangle, \langle 6,5,2 \rangle) \]

\[ T = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ \downarrow \text{full 1-deletion} \]

\[ T^{(0)} = \emptyset \]
Azenhas’ procedure

\[
\mathcal{LR}((8,6,2,1)/(6,5,2),(5,4,1))
\]

\[
\mathcal{LR}((8,6,2,1)/(5,4,1),(6,5,2))
\]

\[
\mathcal{T} =
\begin{array}{cccc}
1 & 1 & & \\
& 1 & & \\
1 & 2 & 2 & \\
1 & 2 & 2 & 3
\end{array}
\]

\[
\mathcal{T}^\uparrow = \mathcal{T}^\lor
\]

\[
\mathcal{T}^{(3)} =
\begin{array}{cccc}
1 & 1 & 1 & \\
1 & 2 & & \\
1 & 2 & 2 & 2 & \downarrow \text{full 4-deletion}
\end{array}
\]

\[
\mathcal{T}^{(2)} =
\begin{array}{cccccc}
1 & 1 & 1 & 1 & & \\
1 & 2 & 2 & 2 & & \downarrow \text{full 3-deletion}
\end{array}
\]

\[
\mathcal{T}^{(1)} =
\begin{array}{cccc}
1 & 1 & 1 & 1 & 1 & \downarrow \text{full 2-deletion}
\end{array}
\]

\[
\mathcal{T}^{(0)} = \emptyset
\]
Azenhas’ procedure

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(6,5,2),(5,4,1)}) \]

\[ \mathcal{T} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & \\
1 & 2 & 2 & \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 4-deletion} \]

\[ \mathcal{T}^{(3)} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 3-deletion} \]

\[ \mathcal{T}^{(2)} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \downarrow \text{full 2-deletion} \]

\[ \mathcal{T}^{(1)} = \]

\[ \begin{array}{cccc}
\square & \square & \square & 1 \\
\square & \square & \square & 1 \\
\square & \square & \square & 1 \\
\square & \square & \square & 1 \\
\end{array} \]

\[ \Rightarrow \text{full 1-deletion} \]

\[ \mathcal{T}^{(0)} = \emptyset \]

\[ \mathcal{LR}(\frac{(8,6,2,1)}{(5,4,1),(6,5,2)}) \]

\[ \mathcal{T} = \]

\[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ \Rightarrow \mathcal{T}^{\uparrow} \]
Azenhas’ procedure

\[ \mathcal{LR}(8,6,2,1)/(6,5,2),(5,4,1)) \]

\[ T = \begin{array}{cccc}
& & & 1 1 \\
& & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \text{full 4-deletion} \]

\[ T^{(3)} = \begin{array}{cccc}
& & & 1 1 1 \\
& & 1 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \text{full 3-deletion} \]

\[ T^{(2)} = \begin{array}{cccc}
& & & 1 1 1 1 \\
& & 1 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 \\
\end{array} \]

\[ \text{full 2-deletion} \]

\[ T^{(1)} = \begin{array}{cccc}
& & & 1 1 1 1 1 \\
& & 1 1 1 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ T(0) = \emptyset \]

\[ \mathcal{LR}(8,6,2,1)/(5,4,1),(6,5,2)) \]

\[ \bigcup \]

\[ T = \begin{array}{cccc}
& & & 1 1 1 \\
& & 1 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ \text{full 4-deletion} \]

\[ T^{(3)} = \begin{array}{cccc}
& & & 1 1 1 \\
& & 1 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \text{full 3-deletion} \]

\[ T^{(2)} = \begin{array}{cccc}
& & & 1 1 1 1 \\
& & 1 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 3 \\
\end{array} \]

\[ \text{full 2-deletion} \]

\[ T^{(1)} = \begin{array}{cccc}
& & & 1 1 1 1 1 \\
& & 1 1 1 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
\end{array} \]

\[ T(0) = \emptyset \]
Azenhas’ procedure

\[ LR(\left(\frac{8,6,2,1}{6,5,2}\right), (5,4,1)) \]

\[ T = \]

\[
\begin{array}{cccc}
& & 1 & 1 \\
1 & & 1 \\
1 & 2 & 2 & 3 \\
1 & 2 & 2 & 3 \\
\end{array}
\]

\[ \downarrow \text{full 4-deletion} \]

\[ T^{(3)} = \]

\[
\begin{array}{cccc}
& & 1 & 1 & 1 \\
& & 1 & 2 \\
1 & 2 & 2 & 2 & 3 \\
1 & 2 & 2 & 2 & 3 \\
\end{array}
\]

\[ \downarrow \text{full 3-deletion} \]

\[ T^{(2)} = \]

\[
\begin{array}{cccc}
& & 1 & 1 & 1 & 1 \\
& & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 & \end{array}
\]

\[ \downarrow \text{full 2-deletion} \]

\[ T^{(1)} = \]

\[
\begin{array}{cccc}
& & & & 1 & 1 & 1 & 1 \\
& & & & 1 & 1 & 1 & 1 \\
& & & & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ \rightarrow \text{full 1-deletion} \]

\[ T^{(0)} = \emptyset \]
Hall varieties

- Let \( \mathbb{C}[t] \) be the polynomial ring in \( t \) over \( \mathbb{C} \).
- Consider \( \mathbb{C}[t] \)-modules only of the form
  \[ M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l}), \]
  \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) being a partition.
- Call it a (nilpotent) \( \mathbb{C}[t] \)-module of type \( \lambda \), write type \( M = \lambda \).
- \( \dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l \).
- A submodule or a quotient of \( M \) is also of that kind.
- Fix partitions \( \lambda, \mu, \nu \) with \( |\lambda| = |\mu| + |\nu| \), and \( M \) of type \( \lambda \).
- Tentatively call
  \[ G^M_{\mu\nu} := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \} \]
  a Hall variety.
- It is a locally closed subvariety of a Grassmannian.
- If \( \mathbb{C} \) is replaced by \( \mathbb{F}_q \), then \( \#G^M_{\mu\nu} = g^\lambda_{\mu\nu}(q) \), the Hall polynomial evaluated at \( q \). (P. Hall, T. Klein, I. G. Macdonald)
Let $\mathbb{C}[t]$ be the polynomial ring in $t$ over $\mathbb{C}$.

Consider $\mathbb{C}[t]$-modules only of the form

$$M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l}),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition.

Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.

$$\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l.$$

A submodule or a quotient of $M$ is also of that kind.

Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.

Tentatively call

$$G^M_{\mu\nu} := \{ N \subset M \text{ submodule} | \text{type } M/N = \mu, \text{ type } N = \nu \}$$

a Hall variety.

It is a locally closed subvariety of a Grassmannian.

If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then $\#G^M_{\mu\nu} = g^\lambda_{\mu\nu}(q)$, the Hall polynomial evaluated at $q$. (P. Hall, T. Klein, I. G. Macdonald)
Hall varieties

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  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ being a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.
- $\dim_\mathbb{C} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of $M$ is also of that kind.
- Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
- Tentatively call
  \[ G^M_{\mu\nu} := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \} \]
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Hall varieties

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  $M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l})$,  
  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ being a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.
- $\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of $M$ is also of that kind.
- Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
- Tentatively call $G^M_{\mu\nu} := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \}$ a Hall variety.
- It is a locally closed subvariety of a Grassmannian.
- If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then $\#G^M_{\mu\nu} = g^\lambda_{\mu\nu}(q)$, the Hall polynomial evaluated at $q$. (P. Hall, T. Klein, I. G. Macdonald)
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- $\dim_\mathbb{C} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of $M$ is also of that kind.
- Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
- Tentatively call
  \[ G_{\mu \nu}^M := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \} \]
  a Hall variety.
- It is a locally closed subvariety of a Grassmannian.
- If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then $\# G_{\mu \nu}^M = g_{\mu \nu}^\lambda(q)$, the Hall polynomial evaluated at $q$. (P. Hall, T. Klein, I. G. Macdonald)
Hall varieties

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- Consider $\mathbb{C}[t]$-modules only of the form
  \[ M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l}), \]
  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ being a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.
- $\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of $M$ is also of that kind.
- Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
- Tentatively call
  \[ G_M^{\mu\nu} := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \} \]
  a Hall variety.
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- Let $\mathbb{C}[t]$ be the polynomial ring in $t$ over $\mathbb{C}$.
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  $$M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l}),$$
  where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.
- $\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of $M$ is also of that kind.
- Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
- Tentatively call
  $$G_{\mu \nu}^M := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \}$$
  a Hall variety.
- It is a locally closed subvariety of a Grassmannian.
- If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then $\#G_{\mu \nu}^M = g_{\mu \nu}^\lambda(q)$, the Hall polynomial evaluated at $q$. (P. Hall, T. Klein, I. G. Macdonald)
Let $C[t]$ be the polynomial ring in $t$ over $C$.
Consider $C[t]$-modules only of the form
\[ M = C[t]/(t^{\lambda_1}) \oplus C[t]/(t^{\lambda_2}) \oplus \cdots \oplus C[t]/(t^{\lambda_l}), \]
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ being a partition.
Call it a (nilpotent) $C[t]$-module of type $\lambda$, write type $M = \lambda$.
$\dim_{C} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
A submodule or a quotient of $M$ is also of that kind.
Fix partitions $\lambda, \mu, \nu$ with $|\lambda| = |\mu| + |\nu|$, and $M$ of type $\lambda$.
Tentatively call
\[ G^M_{\mu\nu} := \{ N \subset M \text{ submodule} \mid \text{type } M/N = \mu, \text{ type } N = \nu \} \]
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Hall varieties

- Let $\mathbb{C}[t]$ be the polynomial ring in $t$ over $\mathbb{C}$.
- Consider $\mathbb{C}[t]$-modules only of the form 
  \[ M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l}), \]
  where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type $\lambda$, write type $M = \lambda$.
- $\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
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Green-Klein tableaux, Green-Klein varieties

Let \( \prime \) denote the conjugate partition, e.g. 

- To each \( N \in G^M_{\mu \nu} \), J. A. Green associated a LR tableau 
  \[ T_M(N) \in \mathcal{LR}(\lambda'/\mu', \nu'). \]
- Setting \( \mu^{(s)} = \text{type } M/t^s N \) for all \( s \), he showed that 
  \( ((\mu^{(0)})', (\mu^{(1)})', \ldots, (\mu^{(u)})')' \) is a LR sequence \( (u = \nu_1) \).
- \( T_M(N) \) is the corresponding LR tableau.
- Note \( \mu^{(0)} = \mu, \mu^{(u)} = \lambda \).
- For each \( T \in \mathcal{LR}(\lambda'/\mu', \nu') \), set 
  \[ G^M_T := \{ N \in G^M_{\mu \nu} \mid T_M(N) = T \}. \]
- \[ G^M_{\mu \nu} = \bigsqcup_{T \in \mathcal{LR}(\lambda'/\mu', \nu')} G^M_T. \]
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u}^M \mid T_M(N) = T \} \).
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u}^M = \bigsqcup_{T \in LR(\lambda'/\mu',\nu')} G_T^M \).
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Let \( \prime \) denote the conjugate partition, e.g. \( = \) .

To each \( N \in G^{\mu \nu}_M \), J. A. Green associated a LR tableau \( T_M(N) \in \mathcal{LR}(\lambda'/\mu',\nu') \).

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\( T_M(N) \) is the corresponding LR tableau.

Note \( \mu^{(0)} = \mu, \mu^{(u)} = \lambda \).

For each \( T \in \mathcal{LR}(\lambda'/\mu',\nu') \), set \( G^M_T := \{ N \in G^{\mu \nu}_M | T_M(N) = T \} \).

\( G^{\mu \nu}_M = \bigsqcup_{T \in \mathcal{LR}(\lambda'/\mu',\nu')} G^M_T \).

Temporarily call each \( G^M_T \) a Green-Klein variety.
Let \( ' \) denote the conjugate partition, e.g. \( \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} = \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \).

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Green-Klein tableaux, Green-Klein varieties

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\[
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array} \quad = \quad \\
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array}
\]

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- \( T_M(N) \) is the corresponding LR tableau.
- Note \( \mu^{(0)} = \mu, \mu^{(u)} = \lambda \).
- For each \( T \in \mathcal{LR}(\lambda'/\mu', \nu') \), set \( \mathcal{G}_T^M := \{ N \in \mathcal{G}_{\mu\nu}^M \mid T_M(N) = T \} \).
- \( \mathcal{G}_{\mu\nu}^M = \bigsqcup_{T \in \mathcal{LR}(\lambda'/\mu', \nu')} \mathcal{G}_T^M \).
- Temporarily call each \( \mathcal{G}_T^M \) a Green-Klein variety.
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Let ‘ denote the conjugate partition, e.g. $\lambda' = \mu'$. 

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- $T_M(N)$ is the corresponding LR tableau. 
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- For each $T \in LR(\lambda'/\mu', \nu')$, set $G^M_T := \{N \in G^M_{\mu\nu} \mid T_M(N) = T\}$. 
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Green-Klein tableaux, Green-Klein varieties

Let $'$ denote the conjugate partition, e.g. $\begin{array}{cccc} & & & \end{array} = \begin{array}{cccc} & & & \end{array}$.

- To each $N \in G_{\mu\nu}^M$, J. A. Green associated a LR tableau $T_M(N) \in \mathcal{LR}(\lambda'/\mu', \nu')$.
- Setting $\mu^{(s)} = \text{type } M/t^sN$ for all $s$, he showed that $((\mu^{(0)}'), (\mu^{(1)}'), \ldots, (\mu^{(u)})')$ is a LR sequence $(u = \nu_1)$.
- $T_M(N)$ is the corresponding LR tableau.
- Note $\mu^{(0)} = \mu$, $\mu^{(u)} = \lambda$.
- For each $T \in \mathcal{LR}(\lambda'/\mu', \nu')$, set $G_T^M := \{ N \in G_{\mu\nu}^M \mid T_M(N) = T \}$.
- $G_{\mu\nu}^M = \bigsqcup_{T \in \mathcal{LR}(\lambda'/\mu', \nu')} G_T^M$.
- Temporarily call each $G_T^M$ a Green-Klein variety.
Green-Klein tableaux, Green-Klein varieties

Let ′ denote the conjugate partition, e.g. 

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\quad \quad \quad \quad =
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

- To each \( N \in G_{\mu \nu}^M \), J. A. Green associated a LR tableau \( T_M(N) \in LR(\lambda'/\mu', \nu') \).
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- \( G_{\mu \nu}^M = \bigsqcup_{T \in LR(\lambda'/\mu', \nu')} G_T^M \).
- Temporarily call each \( G_T^M \) a Green-Klein variety.
More facts about the Green-Klein varieties

- Each $G^M_T$ is irreducible, nonsingular, locally closed in $G^M_{\mu\nu}$.
- $\dim G^M_T = n(\lambda) - n(\mu) - n(\nu)$, where $n(\lambda) = \sum_{i=1}^{l} (i - 1)\lambda_i$.
- $\dim G^M_T$ is constant for fixed $\lambda$, $\mu$ and $\nu$.
- $\overline{G^M_T}$, $T \in \mathcal{LR}(\lambda'/\mu', \nu')$, are the irreducible components of $G^M_{\mu\nu}$.

The above is sufficient to state the main theorem, but here are some more facts useful for the proof.

- $N \mapsto (N, tN, t^2N, \ldots)$ embeds $G^M_T$ into a slightly larger variety $\hat{G}^M_T = \{ (N_0, N_1, \ldots, N_u) \text{ submodules} \mid tN_{s-1} \subset N_s, \text{ type } M/N_s = (\mu^{(s)})' (\forall s) \}$ as an open subvariety.
- $\hat{G}^M_T$ has an open covering by subsets isomorphic to affine spaces.
More facts about the Green-Klein varieties

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- $\dim G^M_T$ is constant for fixed $\lambda, \mu$ and $\nu$.
- $\overline{G^M_T}, T \in L\mathcal{R}(\lambda'/\mu', \nu')$, are the irreducible components of $G^M_{\mu\nu}$.

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- $\dim G^M_T$ is constant for fixed $\lambda, \mu$ and $\nu$.
- $\overline{G^M_T}, \ T \in LR(\lambda'/\mu', \nu')$, are the irreducible components of $G^M_{\mu \nu}$.

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More facts about the Green-Klein varieties

- Each $\mathcal{G}_T^M$ is irreducible, nonsingular, locally closed in $\mathcal{G}_{\mu\nu}^M$.
- $\dim \mathcal{G}_T^M = n(\lambda) - n(\mu) - n(\nu)$, where $n(\lambda) = \sum_{i=1}^{l} (i - 1)\lambda_i$.
- $\dim \mathcal{G}_T^M$ is constant for fixed $\lambda, \mu$ and $\nu$.
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- $\hat{\mathcal{G}}_T^M$ has an open covering by subsets isomorphic to affine spaces.
More facts about the Green-Klein varieties

- Each $G^M_T$ is irreducible, nonsingular, locally closed in $G^M_{\mu\nu}$.
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More facts about the Green-Klein varieties

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- $\dim \mathcal{G}_T^M$ is constant for fixed $\lambda, \mu$ and $\nu$.
- $\mathcal{G}_T^M$, $T \in \mathcal{LR}(\lambda'/\mu', \nu')$, are the irreducible components of $\mathcal{G}_{\mu\nu}^M$.

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- $\widehat{\mathcal{G}_T^M}$ has an open covering by subsets isomorphic to affine spaces.
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ ($t \rightleftharpoons M^*$ as the transpose of $t \rightleftharpoons M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G^M_{\mu\nu} \sim G^{M^*}_{\nu\mu}$ switching $\mu$ and $\nu$.
- $\perp$ induces a bijection between the irreducible components of $G^M_{\mu\nu}$ and $G^{M^*}_{\nu\mu}$.

**Thm. (T)**

$\perp(G^M_T) = G^{M^*}_{T^\perp}$. In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^\perp \in G^{M^*}_{T^\perp}$.

**Lem.**

Let $T^b$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1-1} \in G^{\ker t^{\lambda_1-1}}_{T^b}$. 
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C})$ ($t \mapsto M^*$ as the transpose of $t \mapsto M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G_{\mu\nu}^M \sim G_{\nu\mu}^{M^*}$ switching $\mu$ and $\nu$.
- $\perp$ induces a bijection between the irreducible components of $G_{\mu\nu}^M$ and $G_{\nu\mu}^{M^*}$.

**Thm. (T)**

$\perp(G_T^M) = G_T^{M^*}$. In particular, for most $N \in G_T^M$, i.e. for all $N$ in some dense open subset of $G_T^M$, we have $N^\perp \in G_T^{M^*}$.

**Lem.**

Let $T^b$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G_T^M$, we have $N \cap \ker t^{\lambda_1-1} \in G_{T^b}^{\ker t^{\lambda_1-1}}$. 

Itaru Terada  
A module model
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ ($t \curvearrowright M^*$ as the transpose of $t \curvearrowright M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G^M_{\mu\nu} \sim G^M_{\nu\mu}^*$ switching $\mu$ and $\nu$.
- $\perp$ induces a bijection between the irreducible components of $G^M_{\mu\nu}$ and $G^M_{\nu\mu}^*$.

Thm. (T)

$\perp(G^M_T) = G^M_{TV}^*$. In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^\perp \in G^M_{TV}^*$.

Lem.

Let $T^\flat$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1-1} \in G^\ker t^{\lambda_1-1}_{TV}$. 

Itaru Terada
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C})$ ($t \mapsto M^*$ as the transpose of $t \mapsto M$).
- $N \mapsto N^⊥ = \{ \alpha \in M^* | \alpha|_N = 0 \}$ gives an isomorphism of varieties $⊥ : G^M_{\mu\nu} \sim G^{M^*}_{\nu\mu}$ switching $\mu$ and $\nu$.
- $⊥$ induces a bijection between the irreducible components of $G^M_{\mu\nu}$ and $G^{M^*}_{\nu\mu}$.

**Thm. (T)**

$⊥(\overline{G^M_T}) = \overline{G^{M^*}_{T^\nu}}$. In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^⊥ \in G^{M^*}_{T^\nu}$.

**Lem.**

Let $T^♭$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1 - 1} \in G^{\ker t^{\lambda_1 - 1}}_{T^♭}$. 
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C})$ ($t \mapsto M^*$ as the transpose of $t \mapsto M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp: G_{\mu\nu}^M \xrightarrow{\sim} G_{\nu\mu}^{M^*}$ switching $\mu$ and $\nu$.
- $\perp$ induces a bijection between the irreducible components of $G_{\mu\nu}^M$ and $G_{\nu\mu}^{M^*}$.

**Thm. (T)**

$\perp(G_T^M) = G_T^{M^*}$. In particular, for most $N \in G_T^M$, i.e. for all $N$ in some dense open subset of $G_T^M$, we have $N^\perp \in G_T^{M^*}$.

**Lem.**

Let $T^\flat$ denote the result of applying the full $\lambda_1$-deletion to $T$.

Then for most $N \in G_T^M$, we have $N \cap \ker t^{\lambda_1-1} \in G_T^{\ker t^{\lambda_1-1}}$. 
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ ($t \blacktriangleleft M^*$ as the transpose of $t \blacktriangleleft M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G^M_{\mu\nu} \cong G^M_{\nu\mu}^* \text{ switching } \mu \text{ and } \nu$.
- $\perp$ induces a bijection between the irreducible components of $G^M_{\mu\nu}$ and $G^M_{\nu\mu}^*$.

**Thm. (T)**

$$\perp (\overline{G^M_T}) = \overline{G^M_T^*}.$$ In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^\perp \in G^M_{T^\nu}$.

**Lem.**

Let $T^b$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1 - 1} \in G^\ker_{T^b} t^{\lambda_1 - 1}$.
If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C})$ ($t \circ M^*$ as the transpose of $t \circ M$).

$N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G^M_{\mu\nu} \sim G^{M*}_{\nu\mu}$ switching $\mu$ and $\nu$.

$\perp$ induces a bijection between the irreducible components of $G^M_{\mu\nu}$ and $G^{M*}_{\nu\mu}$.

**Thm. (T)**

$\perp(G^M_T) = G^{M*}_{T^\perp}$. In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^\perp \in G^{M*}_{T^\perp}$.

**Lem.**

Let $T^\flat$ denote the result of applying the full $\lambda_1$-deletion to $T$.

Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1 - 1} \in G^\ker_{T^\flat} t^{\lambda_1 - 1}$.
Main Theorem

- If $M$ is a nilpotent $\mathbb{C}[t]$-module of type $\lambda$, so is $M^* = \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ ($t \curvearrowright M^*$ as the transpose of $t \curvearrowright M$).
- $N \mapsto N^\perp = \{ \alpha \in M^* \mid \alpha|_N = 0 \}$ gives an isomorphism of varieties $\perp : G^M_{\mu \nu} \sim G^M_{\nu \mu}^*$ switching $\mu$ and $\nu$.
- $\perp$ induces a bijection between the irreducible components of $G^M_{\mu \nu}$ and $G^M_{\nu \mu}^*$.

Thm. (T)

$\perp(\overline{G}^M_T) = \overline{G}^M_{T^\vee}$. In particular, for most $N \in G^M_T$, i.e. for all $N$ in some dense open subset of $G^M_T$, we have $N^\perp \in G^M_{T^\vee}$.

Lem.

Let $T^b$ denote the result of applying the full $\lambda_1$-deletion to $T$. Then for most $N \in G^M_T$, we have $N \cap \ker t^{\lambda_1-1} \in G^\ker_{T^b} t^{\lambda_1-1}$. 
The isomorphism from an affine space to a piece of an open covering of $\hat{G}_{M}^{T}$ can be recursively given as follows.

- Consider $\pi: \hat{G}_{M}^{T} \rightarrow G_{\nu'}(\ker t), (N_s)_s^{u=0} \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^aM)_a^{r=0}$ ($r = \lambda_1$).
- The subvariety of $G_{\nu'}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \rightarrow U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $\hat{G}_{M/N_{u-1}}^{T}$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
The isomorphism from an affine space to a piece of an open covering of $\widehat{G}_T^M$ can be **recursively** given as follows.

- Consider $\pi: \widehat{G}_T^M \rightarrow G_{v'_u}(\ker t)$, $(N_s)_s^{u=0} \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^a M)_a^{r=0}$ ($r = \lambda_1$).
- The subvariety of $G_{v'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \rightarrow U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $\widehat{G}_{T/N_{u-1}}^M$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
The isomorphism from an affine space to a piece of an open covering of $\hat{G}^M_T$ can be recursively given as follows.

- Consider $\pi : \hat{G}^M_T \to G_{\nu'_u}(\ker t)$, $(N_s)_{s=0}^u \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu(u-1)')$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^a M)_{a=0}^r (r = \lambda_1)$.
- The subvariety of $G_{\nu'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \to U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $\hat{G}^M_T/N_{u-1}$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
The isomorphism from an affine space to a piece of an open covering of $\hat{G}^M_T$ can be recursively given as follows.

- Consider $\pi : \hat{G}^M_T \to G_{\nu'_u}(\ker t), \ (N_s)_s^{u=0} \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^aM)^r_{a=0} \ (r = \lambda_1)$.
- The subvariety of $G_{\nu'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \to U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $G^M_T/N_{u-1}$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
The isomorphism from an affine space to a piece of an open covering of $\hat{G}_{\hat{T}}^M$ can be recursively given as follows.

- Consider $\pi : \hat{G}_{\hat{T}}^M \rightarrow G_{\nu'_{\hat{u}}}(\ker t), (N_s)^u_{s=0} \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^a M)^r_{a=0} (r = \lambda_1)$.
- The subvariety of $G_{\nu'_{\hat{u}}}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \rightarrow U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $G_{\hat{T}}^{M/N_{u-1}}$, which allows an open covering by affine spaces by induction ($\overline{T} = T \backslash \{u\}$).
The isomorphism from an affine space to a piece of an open covering of $\hat{G}_M^T$ can be recursively given as follows.

- Consider $\pi : \hat{G}_M^T \to G_{\nu'_u}(\ker t), (N_s)_s^{u=0} \mapsto N_{u-1}$.

- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^a M)^r_{a=0} (r = \lambda_1)$.

- The subvariety of $G_{\nu'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_{\alpha}$.

- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \to U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \to \pi^{-1}(U_\alpha)$.

- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $\hat{G}_{M/N_{u-1}}^T$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
Coordinates

The isomorphism from an affine space to a piece of an open covering of $\hat{G}^M_T$ can be recursively given as follows.

- Consider $\pi : \hat{G}^M_T \rightarrow G_{\nu'_u}(\ker t)$, $(N_s)_s^{u=0} \mapsto N_{u-1}$.
- The condition type $M/N_{u-1} = (\mu^{(u-1)})'$ can be specified by dimensions of the intersections of $N_{u-1}$ with the various components of the partial flag $(\ker t \cap t^a M)^r_{a=0} (r = \lambda_1)$.
- The subvariety of $G_{\nu'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each $U_\alpha$ and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \rightarrow U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \sim \pi^{-1}(U_\alpha)$.
- The fiber $\pi^{-1}(N_{u-1})$ is isomorphic to $G^M_T/N_{u-1}$, which allows an open covering by affine spaces by induction ($\overline{T} = T \setminus \{u\}$).
The pieces of the open covering of $\hat{G}_T^M$ are parametrized by the fillings $\Xi$ of the Young diagram of $\lambda'$ which are column increasing and rowwise permutations of $T$.

If $T = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \end{array}$, then one such $\Xi$ is $\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & \end{array}$.

In this example the dimension is 4, and $U_\Xi \cap G_T^M = U_\Xi$.

The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.
The pieces of the open covering of $\Gamma M_T$ are parametrized by the fillings $\Xi$ of the Young diagram of $\lambda'$ which are column increasing and rowwise permutations of $T$.

If $T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, then one such $\Xi$ is $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.

In this example the dimension is 4, and $U_\Xi \cap \Gamma M_T = U_\Xi$.

The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.
The pieces of the open covering of $\mathcal{G}_M^T$ are parametrized by the fillings $\Xi$ of the Young diagram of $\lambda'$ which are column increasing and rowwise permutations of $T$.

If $T = \begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ 2 & 2 \end{array}$, then one such $\Xi$ is $\begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ 2 & 2 \end{array}$.

In this example the dimension is 4, and $U_\Xi \cap \mathcal{G}_M^T = U_\Xi$.

The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.
Coordinates, example

- The pieces of the open covering of $\mathcal{G}^M_T$ are parametrized by the fillings $\Xi$ of the Young diagram of $\lambda'$ which are column increasing and rowwise permutations of $T$.

- If $T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, then one such $\Xi$ is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$.

- In this example the dimension is 4, and $U_\Xi \cap \mathcal{G}^M_T = U_\Xi$.

- The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.
The pieces of the open covering of $\hat{G}_T^M$ are parametrized by the fillings $\Xi$ of the Young diagram of $\lambda'$ which are column increasing and rowwise permutations of $T$.

If $T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$, then one such $\Xi$ is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$.

In this example the dimension is 4, and $U_\Xi \cap G_T^M = U_\Xi$.

The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.
Coordinate, example (continued)

\[
\begin{pmatrix}
  t^3 \\
  t^3 \\
  t^2 \\
  t^2
\end{pmatrix}
\begin{pmatrix}
  1 \\
  1 \\
  a \\
  1
\end{pmatrix}
\begin{pmatrix}
  t^{-1} \\
  t^{-1} \\
  1 \\
  t^{-1}
\end{pmatrix}
\begin{pmatrix}
  1 \\
  x \\
  y \\
  z
\end{pmatrix}
\begin{pmatrix}
  t^{-1} \\
  t^{-1} \\
  1 \\
  t^{-1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  t \\
  t \\
  xt \\
  yt \\
  t^2 \\
  zt + a \\
  1
\end{pmatrix}
\]

\[\tilde{N}_1 = \langle \begin{pmatrix}
  t \\
  0 \\
  xt \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  t \\
  yt \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  t^2 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  zt + a \\
  1
\end{pmatrix} \rangle \subset \mathbb{C}[t]^4\]

\[N_1 = \tilde{N}_1 / \langle \begin{pmatrix}
  t^3 \\
  0 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  t^3 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  t^2 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  t^2
\end{pmatrix} \rangle \subset \mathbb{C}[t]^4 / \langle \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  t^3 \\
  0 \\
  0 \\
  0
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  t^2
\end{pmatrix}, \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix} \rangle\]
Even though Lemma is an essential ingredient, the proof of the Theorem still requires a technique similar to Steinberg’s result on the Steinberg variety, showing that $T^\perp \prec T^\vee$ holds for a certain ordering $\prec$ on $LR(\lambda'/\nu', \mu')$ and then using the fact that both $T \mapsto T^\perp$ and $T \mapsto T^\vee$ are bijections.

The parametrization of the irreducible components of $G^M_{\mu\nu}$ and their dimension were given by M. van Leeuwen (2000).

The affine coordinates of the open covering of $G^M_T$ was given in the form of “generic vectors” by T. Maeda (2003 and later).

This also proves the involutiveness of Azenhas’ procedure. (A combinatorial proof of the involutiveness has been given by Azenhas, R. C. King and T (2017).)
Remarks

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- The parametrization of the irreducible components of $\mathcal{G}^M_{\mu\nu}$ and their dimension were given by M. van Leeuwen (2000).

- The affine coordinates of the open covering of $\mathcal{G}^M_T$ was given in the form of “generic vectors” by T. Maeda (2003 and later).

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