On an involution on the set of Littlewood–Richardson tableaux

A module model for Azenhas' bijection

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version with corrections in red, Feb 22, 2018

Outline

Very rough overview

- Azenhas' procedure: $\mathcal{LR}(\lambda/\mu,\nu) \xrightarrow{\sim} \mathcal{LR}(\lambda/\nu,\mu)$, $T \mapsto T^\vee$ (1999 or 2000)
- She expressed hope to interpret her procedure using R-modules of the following form (R: PID, $p \in R$ prime): $R/(p^{\lambda_1}) \oplus R/(p^{\lambda_2}) \oplus \cdots \oplus R/(p^{\lambda_l})$.
- We give a possible answer for $R = \mathbb{C}[t]$, $p = t$ (an indet.).

Thm. (T)

Set $M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l})$. *Then, for most submodules N yielding a given LR-tableau T, the submodule $N^\perp \subset M^*$ yields $T^\vee$.*

A precise result is phrased using the irreducible components of certain submodule varieties.

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Littlewood-Richardson (LR) tableaux

						1	1
					1		
		1	1	2			
2	2	2	3				

$$\in \mathcal{LR}(\underset{\text{outer shape}}{\underset{\uparrow}{(8,6,2,1)}}/\underset{\text{inner shape}}{\underset{\uparrow}{(6,5,2)}},\underset{\text{weight}}{\underset{\uparrow}{(5,4,1)}})$$

Conditions: $\leq$, $\wedge$, lattice permutation condition (rephrased):

	before row 1	row 1	rows 1-2	rows 1-3	rows 1-4
# $\boxed{1}$'s	0	2	3	5	5
# $\boxed{2}$'s	0	0	0	1	3
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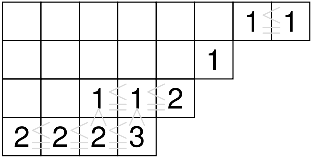
Littlewood-Richardson (LR) tableaux



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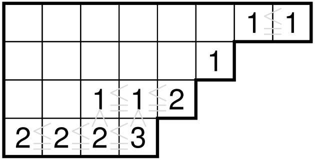

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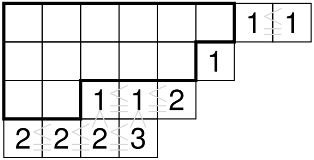

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		$\geqq$	$\geqq$	$\geqq$	$\geqq$
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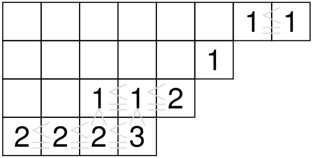

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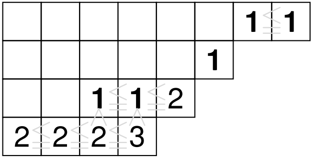

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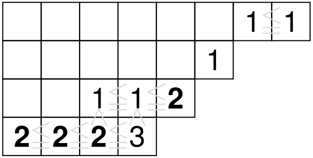

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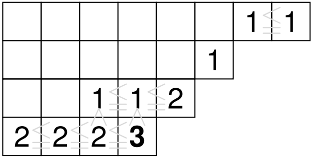

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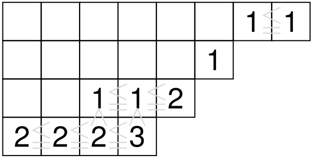

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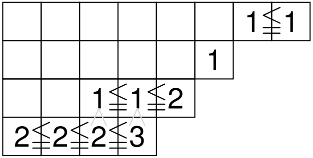

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 & & \geqq & & \geqq & & \geqq & & \geqq & \\
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\end{array}$$

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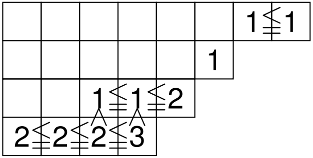

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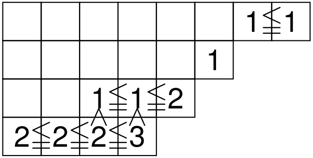

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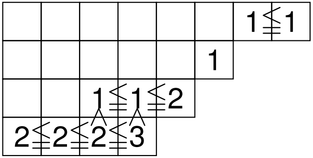

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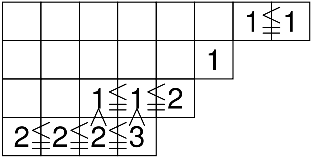

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\# \boxed{2}\text{'s} & 0 & & 0 & & 0 & & 1 & & 3 \\
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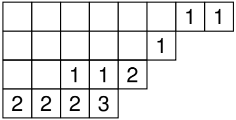
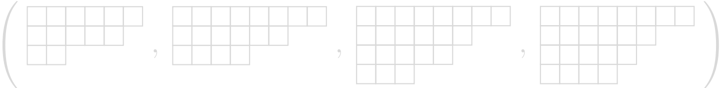
Littlewood-Richardson sequences





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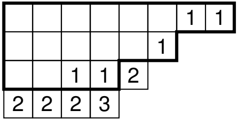

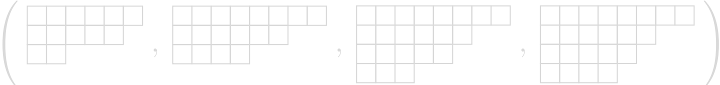

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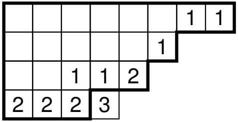

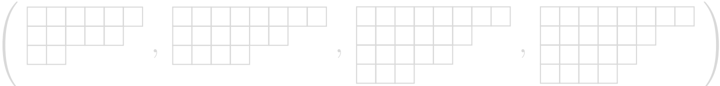

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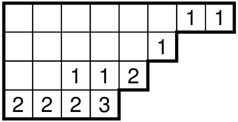

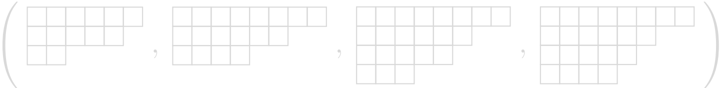

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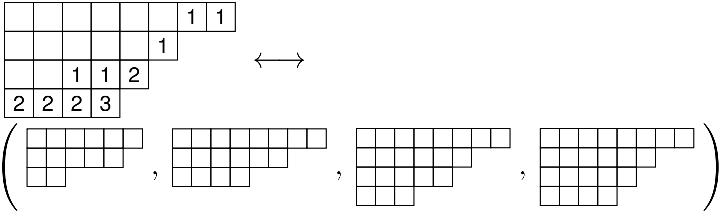

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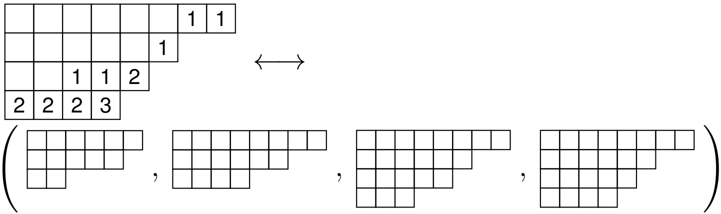

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Azenhas' procedure

$$\mathcal{LR}({}^{(8,6,2,1)}/{}_{(6,5,2)},(5,4,1)) \qquad\qquad \mathcal{LR}({}^{(8,6,2,1)}/{}_{(5,4,1)},(6,5,2))$$

$$\Downarrow \qquad\qquad \Downarrow$$

$T =$

						1	1
					1		
		1	2	2			
1	2	2	3				

$\longmapsto$

					1	1	1
				1	2		
	1	2	2	2			
1	2	3	3				

$= T^{\vee}$

$\downarrow$ full 4-deletion

$T^{(3)} =$

					1	1	1
				1	2		
1	2	2	2	3			

$\downarrow$ full 3-deletion

$T^{(2)} =$

				1	1	1	1
	1	2	2	2	2		

$\downarrow$ full 2-deletion

$T^{(1)} =$

			1	1	1	1	1

$\xmapsto{\text{full 1-deletion}}$ $T^{(0)} = \varnothing$

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$$\mathcal{LR}({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)) \qquad\qquad \mathcal{LR}({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2))$$

$$\Cup \qquad\qquad \Cup$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & 2 & 2 \\ \cline{1-5} 1 & 2 & 2 & 3 \\ \cline{1-4} \end{array} \longmapsto \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4} \end{array} = T^{\vee}$$

$$\downarrow \text{ full 4-deletion}$$

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5} \end{array}$$

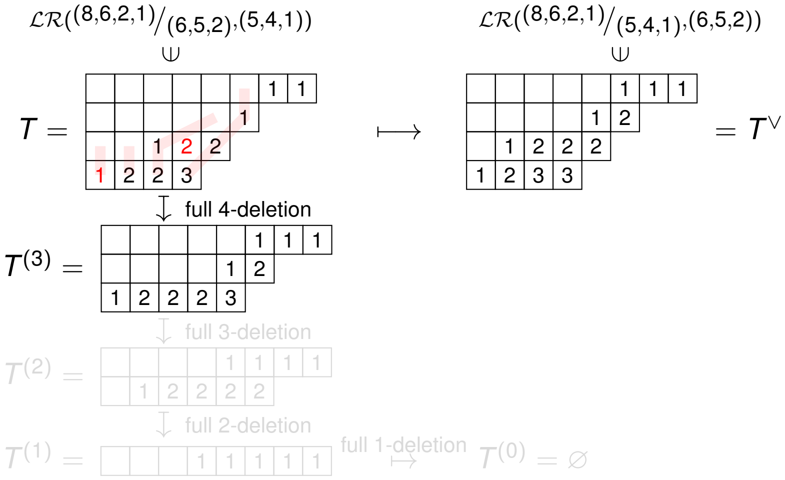
$$\downarrow \text{ full 3-deletion}$$

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6} \end{array}$$

$$\downarrow \text{ full 2-deletion}$$

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \xmapsto{\text{full 1-deletion}} T^{(0)} = \varnothing$$

Azenhas' procedure



$$\mathcal{LR}\left({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)\right) \qquad\qquad \mathcal{LR}\left({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2)\right)$$

$$\Cup \qquad\qquad \Cup$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & \color{red}{2} & 2 \\ \cline{1-5} \color{red}{1} & 2 & 2 & 3 \\ \cline{1-4} \end{array} \longmapsto \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4} \end{array} = T^\vee$$

$\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5} \end{array}$$

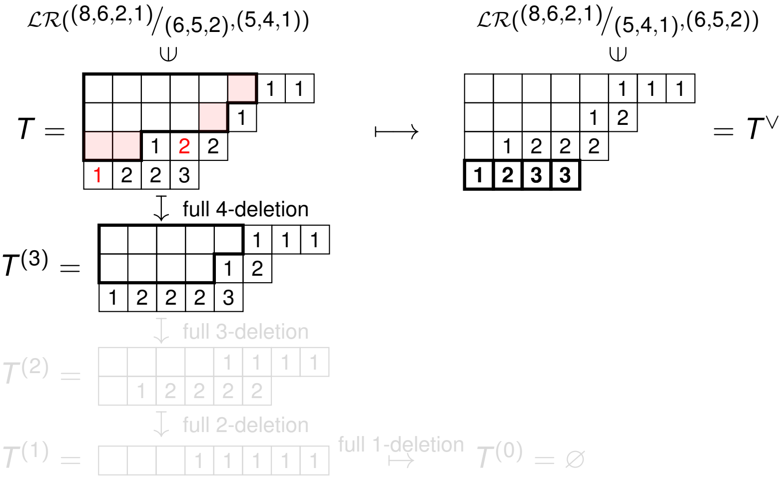
$\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6} \end{array}$$

$\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \overset{\text{full 1-deletion}}{\longmapsto} T^{(0)} = \varnothing$$

Azenhas' procedure



$\mathcal{LR}((8,6,2,1)/(6,5,2),(5,4,1))$ $\Cup$ $T =$

$\longmapsto$

$\mathcal{LR}((8,6,2,1)/(5,4,1),(6,5,2))$ $\Cup$ $= T^\vee$

$\downarrow$ full 4-deletion

$T^{(3)} =$

$\downarrow$ full 3-deletion

$T^{(2)} =$

$\downarrow$ full 2-deletion

$T^{(1)} =$ $\overset{\text{full 1-deletion}}{\longmapsto}$ $T^{(0)} = \varnothing$

Azenhas' procedure

$$\mathcal{LR}({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)) \qquad\qquad \mathcal{LR}({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2))$$

$$\Cup \qquad\qquad\qquad\qquad \Cup$$

$$T = \begin{array}{cccccccc} \ & \ & \ & \ & \ & \ & 1 & 1 \\ \ & \ & \ & \ & \ & 1 & & \\ \ & \ & 1 & \color{red}{2} & 2 & & & \\ \color{red}{1} & 2 & 2 & 3 & & & & \end{array} \quad \longmapsto \quad \begin{array}{cccccccc} \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \ & \ & \ & \ & 1 & 2 & & \\ \ & 1 & 2 & 2 & 2 & & & \\ 1 & 2 & 3 & 3 & & & & \end{array} = T^{\vee}$$

$$\downarrow \text{ full 4-deletion}$$

$$T^{(3)} = \begin{array}{cccccccc} \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \ & \ & \ & \ & 1 & 2 & & \\ 1 & 2 & 2 & 2 & 3 & & & \end{array}$$

$$\downarrow \text{ full 3-deletion}$$

$$T^{(2)} = \begin{array}{cccccccc} \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \ & 1 & 2 & 2 & 2 & 2 & & \end{array}$$

$$\downarrow \text{ full 2-deletion}$$

$$T^{(1)} = \begin{array}{cccccccc} \ & \ & \ & 1 & 1 & 1 & 1 & 1 \end{array} \ \overset{\text{full 1-deletion}}{\longmapsto} \ T^{(0)} = \emptyset$$

Azenhas' procedure

$$\mathcal{LR}({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)) \qquad\qquad \mathcal{LR}({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2))$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & \textcolor{red}{2} & 2 \\ \cline{1-5} \textcolor{red}{1} & 2 & 2 & 3 \\ \cline{1-4}\end{array} \longmapsto \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4}\end{array} = T^{\vee}$$

$\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5}\end{array}$$

$\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6}\end{array}$$

$\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array} \xmapsto{\text{full 1-deletion}} T^{(0)} = \varnothing$$

Azenhas' procedure

$$\begin{array}{ccc}
\mathcal{LR}((8,6,2,1)/(6,5,2),(5,4,1)) & & \mathcal{LR}((8,6,2,1)/(5,4,1),(6,5,2)) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
T = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & \ & 1 & 1 \\
\hline
\ & \ & \ & \ & \ & 1 \\
\cline{1-6}
\ & \ & 1 & \color{red}{2} & 2 \\
\cline{1-5}
\color{red}{1} & 2 & 2 & 3 \\
\cline{1-4}
\end{array}
& \longmapsto &
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & 1 & 1 & 1 \\
\hline
\ & \ & \ & \ & 1 & 2 \\
\cline{1-6}
\boxed{\ } & 1 & 2 & 2 & 2 \\
\cline{1-5}
1 & 2 & 3 & 3 \\
\cline{1-4}
\end{array} = T^{\vee}
\end{array}$$

$$\downarrow \text{ full 4-deletion}$$

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & \ & 1 & 1 & 1 \\
\hline
\ & \ & \ & \ & 1 & 2 \\
\cline{1-6}
1 & 2 & 2 & 2 & \boxed{\mathbf{3}} \\
\cline{1-5}
\end{array}$$

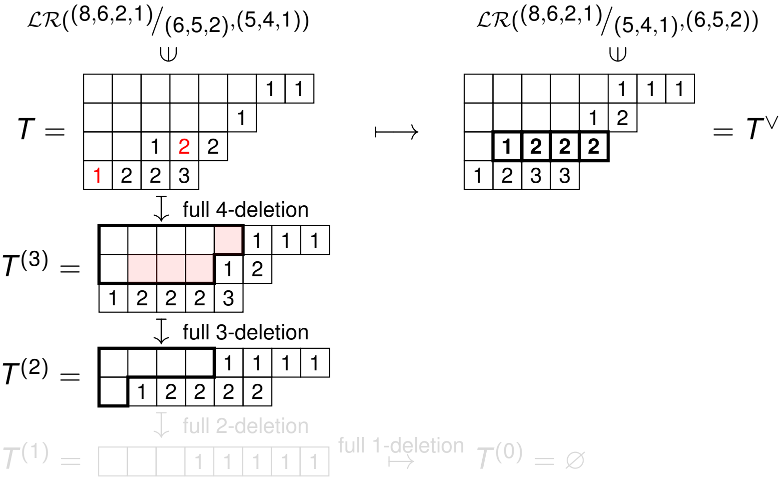
$$\downarrow \text{ full 3-deletion}$$

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & \ & 1 & 1 & 1 & 1 \\
\hline
\ & 1 & 2 & 2 & 2 & 2 \\
\cline{1-6}
\end{array}$$

$$\downarrow \text{ full 2-deletion}$$

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\ & \ & \ & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array} \xmapsto{\text{full 1-deletion}} T^{(0)} = \varnothing$$

Azenhas' procedure



$\mathcal{LR}((8,6,2,1)/(6,5,2),(5,4,1)) \ni T \longmapsto T^{\vee} \in \mathcal{LR}((8,6,2,1)/(5,4,1),(6,5,2))$

Azenhas' procedure

$$\mathcal{LR}\left({}^{(8,6,2,1)}/_{(6,5,2)}, (5,4,1)\right) \qquad\qquad \mathcal{LR}\left({}^{(8,6,2,1)}/_{(5,4,1)}, (6,5,2)\right)$$

$$\Cup \qquad\qquad\qquad\qquad \Cup$$

$$T = \begin{ytableau} \; & & & & & & 1 & 1 \\ & & & & & 1 \\ & & 1 & {\color{red}2} & 2 \\ {\color{red}1} & 2 & 2 & 3 \end{ytableau} \quad \longmapsto \quad \begin{ytableau} \; & & & & & 1 & 1 & 1 \\ & & & & 1 & 2 \\ & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{ytableau} = T^{\vee}$$

$$\downarrow \text{ full 4-deletion}$$

$$T^{(3)} = \begin{ytableau} \; & & & & & 1 & 1 & 1 \\ & & & & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \end{ytableau}$$

$$\downarrow \text{ full 3-deletion}$$

$$T^{(2)} = \begin{ytableau} \; & & & & 1 & 1 & 1 & 1 \\ & 1 & 2 & 2 & 2 & 2 \end{ytableau}$$

$$\downarrow \text{ full 2-deletion}$$

$$T^{(1)} = \begin{ytableau} \; & & & 1 & 1 & 1 & 1 & 1 \end{ytableau} \quad \overset{\text{full 1-deletion}}{\longmapsto} \quad T^{(0)} = \varnothing$$

Azenhas' procedure

$$\mathcal{LR}\left({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)\right) \qquad\qquad \mathcal{LR}\left({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2)\right)$$

$$\rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & & & & 1 & 1 \\ \hline
 & & & & & 1 \\ \cline{1-6}
 & & 1 & \textcolor{red}{2} & 2 \\ \cline{1-5}
\textcolor{red}{1} & 2 & 2 & 3 \\ \cline{1-4}
\end{array}
\quad \longmapsto \quad
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & & & 1 & 1 & 1 \\ \hline
 & & & & 1 & 2 \\ \cline{1-6}
 & 1 & 2 & 2 & 2 \\ \cline{1-5}
1 & 2 & 3 & 3 \\ \cline{1-4}
\end{array} = T^{\vee}$$

↧ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & & & 1 & 1 & 1 \\ \hline
 & & & & 1 & 2 \\ \cline{1-6}
1 & 2 & 2 & 2 & 3 \\ \cline{1-5}
\end{array}$$

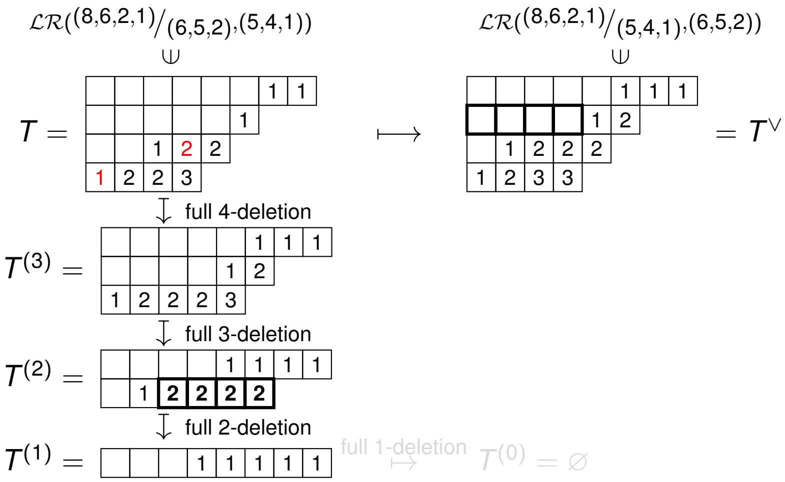
↧ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & & 1 & 1 & 1 & 1 \\ \hline
 & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6}
\end{array}$$

↧ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & & & 1 & 1 & 1 & 1 & 1 \\ \hline
\end{array} \quad \overset{\text{full 1-deletion}}{\longmapsto} \quad T^{(0)} = \emptyset$$

Azenhas' procedure



$\mathcal{LR}({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1))$ $\Cup$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 \\ \hline & & & & & 1 \\ \cline{1-6} & & 1 & 2 & 2 \\ \cline{1-5} 1 & 2 & 2 & 3 \\ \cline{1-4}\end{array} \longmapsto \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 1 \\ \hline & & & & 1 & 2 \\ \cline{1-6} & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4}\end{array} = T^{\vee}$$

$\mathcal{LR}({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2))$ $\Cup$

$\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & & 1 & 1 & 1 \\ \hline & & & & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5}\end{array}$$

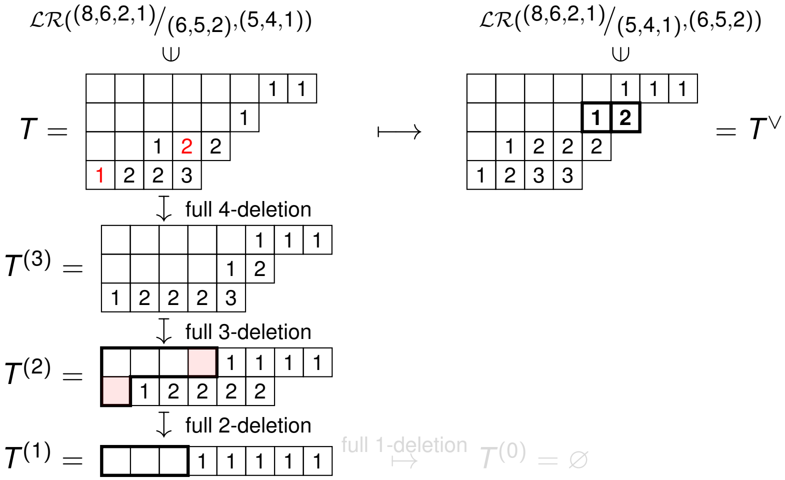
$\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & 1 & 1 & 1 & 1 \\ \hline & 1 & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \cline{1-6}\end{array}$$

$\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array} \xmapsto{\text{full 1-deletion}} T^{(0)} = \varnothing$$

Azenhas' procedure



$$\mathcal{LR}\big({}^{(8,6,2,1)}/_{(6,5,2)},(5,4,1)\big) \qquad \mathcal{LR}\big({}^{(8,6,2,1)}/_{(5,4,1)},(6,5,2)\big)$$

$$\Cup \qquad\qquad \Cup$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & {\color{red}2} & 2 \\ \cline{1-5} {\color{red}1} & 2 & 2 & 3 \\ \cline{1-4}\end{array} \quad \longmapsto \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & \mathbf{1} & \mathbf{2} \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4}\end{array} = T^{\vee}$$

$\big\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5}\end{array}$$

$\big\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6}\end{array}$$

$\big\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array} \quad \overset{\text{full 1-deletion}}{\longmapsto} \quad T^{(0)} = \varnothing$$

Azenhas' procedure

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$$T = \begin{ytableau} \; & \; & \; & \; & \; & \; & 1 & 1 \\ \; & \; & \; & \; & \; & 1 \\ \; & \; & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \end{ytableau} \quad \longmapsto \quad \begin{ytableau} \; & \; & \; & \; & \; & 1 & 1 & 1 \\ \; & \; & \; & \; & 1 & 2 \\ \; & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{ytableau} = T^\vee$$

$$\downarrow \text{ full 4-deletion}$$

$$T^{(3)} = \begin{ytableau} \; & \; & \; & \; & \; & 1 & 1 & 1 \\ \; & \; & \; & \; & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \end{ytableau}$$

$$\downarrow \text{ full 3-deletion}$$

$$T^{(2)} = \begin{ytableau} \; & \; & \; & \; & 1 & 1 & 1 & 1 \\ \; & 1 & 2 & 2 & 2 & 2 \end{ytableau}$$

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Azenhas' procedure

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$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & 1 & 1 \\ \hline \ & \ & \ & \ & \ & 1 \\ \cline{1-6} \ & \ & 1 & {\color{red}2} & 2 \\ \cline{1-5} {\color{red}1} & 2 & 2 & 3 \\ \cline{1-4} \end{array} \quad \longmapsto \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} \ & 1 & 2 & 2 & 2 \\ \cline{1-5} 1 & 2 & 3 & 3 \\ \cline{1-4} \end{array} = T^{\vee}$$

$\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \hline \ & \ & \ & \ & 1 & 2 \\ \cline{1-6} 1 & 2 & 2 & 2 & 3 \\ \cline{1-5} \end{array}$$

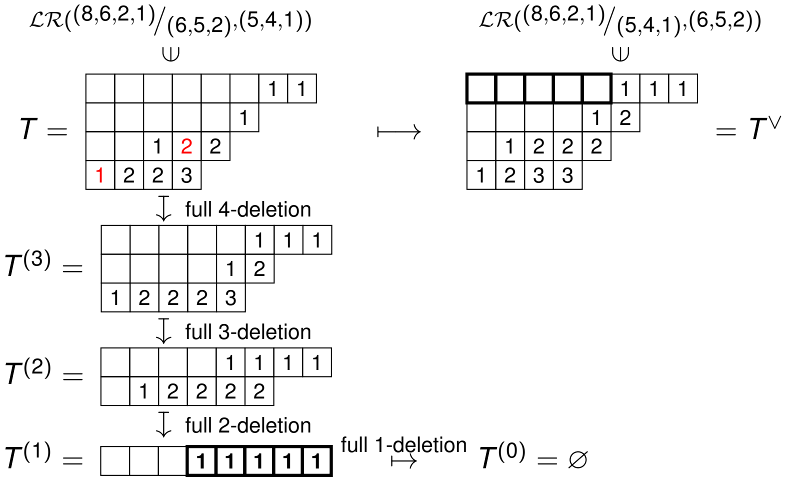
$\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 & 2 & 2 \\ \cline{1-6} \end{array}$$

$\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \overset{\text{full 1-deletion}}{\longmapsto} \quad T^{(0)} = \varnothing$$

Azenhas' procedure



$$\mathcal{LR}((8,6,2,1)/(6,5,2),(5,4,1)) \qquad\qquad \mathcal{LR}((8,6,2,1)/(5,4,1),(6,5,2))$$

$$T = \begin{array}{cccccccc} \ & \ & \ & \ & \ & \ & 1 & 1 \\ \ & \ & \ & \ & \ & 1 & & \\ \ & \ & 1 & \color{red}{2} & 2 & & & \\ \color{red}{1} & 2 & 2 & 3 & & & & \end{array} \longmapsto \begin{array}{cccccccc} \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \ & \ & \ & \ & 1 & 2 & & \\ \ & 1 & 2 & 2 & 2 & & & \\ 1 & 2 & 3 & 3 & & & & \end{array} = T^{\vee}$$

$\downarrow$ full 4-deletion

$$T^{(3)} = \begin{array}{cccccccc} \ & \ & \ & \ & \ & 1 & 1 & 1 \\ \ & \ & \ & \ & 1 & 2 & & \\ 1 & 2 & 2 & 2 & 3 & & & \end{array}$$

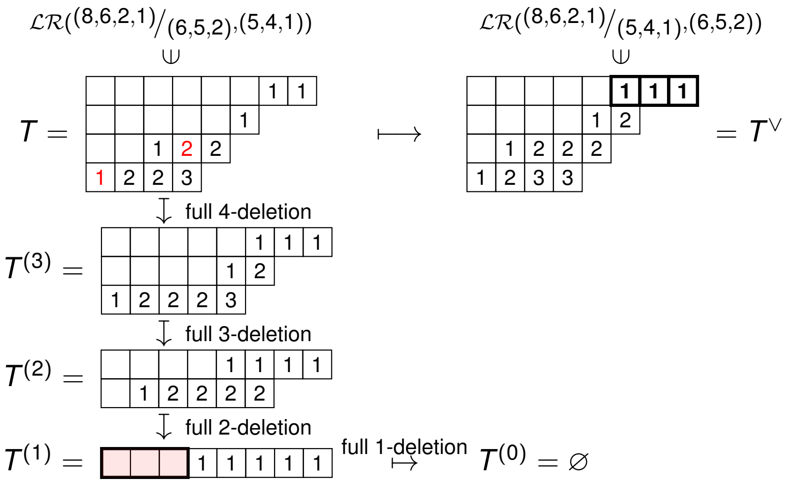
$\downarrow$ full 3-deletion

$$T^{(2)} = \begin{array}{cccccccc} \ & \ & \ & \ & 1 & 1 & 1 & 1 \\ \ & 1 & 2 & 2 & 2 & 2 & & \end{array}$$

$\downarrow$ full 2-deletion

$$T^{(1)} = \begin{array}{cccccccc} \ & \ & \ & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \xmapsto{\text{full 1-deletion}} T^{(0)} = \varnothing$$

Azenhas' procedure



Hall varieties

- Let $\mathbb{C}[t]$ be the polynomial ring in t over $\mathbb{C}$.
- Consider $\mathbb{C}[t]$-modules only of the form $M = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \mathbb{C}[t]/(t^{\lambda_2}) \oplus \cdots \oplus \mathbb{C}[t]/(t^{\lambda_l})$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ being a partition.
- Call it a (nilpotent) $\mathbb{C}[t]$-module of type λ, write type $M = \lambda$.
- $\dim_{\mathbb{C}} M = |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l$.
- A submodule or a quotient of M is also of that kind.
- Fix partitions λ, μ, ν with $|\lambda| = |\mu| + |\nu|$, and M of type λ.
- Tentatively call $\mathcal{G}^M_{\mu\nu} := \{\, N \subset M \text{ submodule} \mid \text{type } {}^{M}\!/_{N} = \mu,\ \text{type } N = \nu \,\}$ a Hall variety.
- It is a locally closed subvariety of a Grassmannian.
- If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then $\#\mathcal{G}^M_{\mu\nu} = g^{\lambda}_{\mu\nu}(q)$, the Hall polynomial evaluated at q. (P. Hall, T. Klein, I. G. Macdonald)

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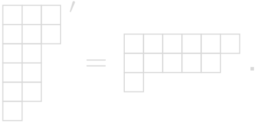
Green-Klein tableaux, Green-Klein varieties

Let $'$ denote the conjugate partition, e.g.



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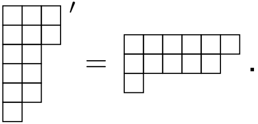

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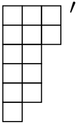
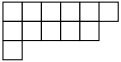
Let $'$ denote the conjugate partition, e.g.  $=$  .

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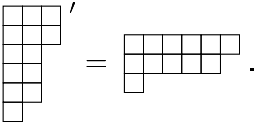
Let $'$ denote the conjugate partition, e.g.  .

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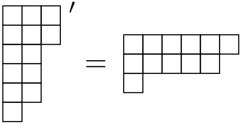
Let ′ denote the conjugate partition, e.g.  .

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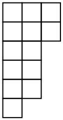
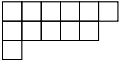
Let $'$ denote the conjugate partition, e.g.  $'$ $=$  .

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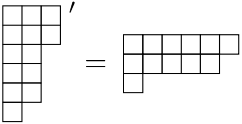
Let ′ denote the conjugate partition, e.g.  .

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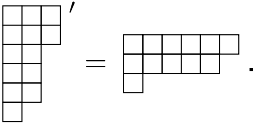
Let ′ denote the conjugate partition, e.g.  .

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More facts about the Green-Klein varieties

- Each $\mathcal{G}^M_T$ is irreducible, nonsingular, locally closed in $\mathcal{G}^M_{\mu\nu}$.
- $\dim \mathcal{G}^M_T = n(\lambda) - n(\mu) - n(\nu)$, where $n(\lambda) = \sum_{i=1}^{l} (i-1)\lambda_i$.
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- $\overline{\mathcal{G}^M_T}$, $T \in \mathcal{LR}({}^{\lambda'}/_{\mu'}, \nu')$, are the irreducible components of $\mathcal{G}^M_{\mu\nu}$.

The above is sufficient to state the main theorem, but here are some more facts useful for the proof.

- $N \mapsto (N, tN, t^2N, \dots)$ embeds $\mathcal{G}^M_T$ into a slightly larger variety $\widehat{\mathcal{G}^M_T} = \{ (N_0, N_1, \dots, N_u) \text{ submodules} \mid \ tN_{s-1} \subset N_s, \ \text{type } {}^M/_{N_s} = (\mu^{(s)})' \ ({}^\forall s) \}$ as an open subvariety.
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- If M is a nilpotent $\mathbb{C}[t]$-module of type λ, so is $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ ($t \curvearrowright M^*$ as the transpose of $t \curvearrowright M$).
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The isomorphism from an affine space to a piece of an open covering of $\widehat{\mathcal{G}_T^M}$ can be recursively given as follows.

- Consider $\pi\colon \widehat{\mathcal{G}_T^M} \to G_{\nu'_u}(\ker t)$, $(N_s)_{s=0}^u \mapsto N_{u-1}$.
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- The subvariety of $G_{\nu'_u}(\ker t)$ specified by such dimensions has an open covering by certain affine spaces $(U_\alpha)_\alpha$.
- For each U_α and $N_{n-1} \in U_\alpha$, the isomorphism $\mathbb{A}^d \to U_\alpha$ can be lifted to $\mathbb{A}^d \times \pi^{-1}(N_{u-1}) \xrightarrow{\sim} \pi^{-1}(U_\alpha)$.
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Coordinates, example

- The pieces of the open covering of $\widehat{\mathcal{G}_T^M}$ are parametrized by the fillings $\mathit{\Xi}$ of the Young diagram of λ' which are column increasing and rowwise permutations of T.
- If $T = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline \ & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$, then one such $\mathit{\Xi}$ is $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 1 \\ \hline 1 & 1 & \ & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$.
- In this example the dimension is 4, and $U_{\mathit{\Xi}} \cap \mathcal{G}_T^M = U_{\mathit{\Xi}}$.
- The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.

Coordinates, example

- The pieces of the open covering of $\widehat{\mathcal{G}_T^M}$ are parametrized by the fillings Ξ of the Young diagram of λ' which are column increasing and rowwise permutations of T.
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- The pieces of the open covering of $\widehat{\mathcal{G}_T^M}$ are parametrized by the fillings $\varXi$ of the Young diagram of λ' which are column increasing and rowwise permutations of T.
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- The submodule corresponding to $(a, x, y, z) \in \mathbb{A}^4$ is the one generated by the column vectors of the matrix product in the next slide.

Coordinate, example (continued)

$$\begin{pmatrix} t^3 & & & \\ & t^3 & & \\ & & t^2 & \\ & & & t^2 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & a \\ & & & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & & & \\ & t^{-1} & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ x & y & 1 & z \\ & & & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & & & \\ & t^{-1} & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} t & & & \\ & t & & \\ xt & yt & t^2 & zt+a \\ & & & 1 \end{pmatrix}, \widetilde{N_1} = \left\langle \begin{pmatrix} t \\ 0 \\ xt \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \\ yt \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zt+a \\ 1 \end{pmatrix} \right\rangle \subset \mathbb{C}[t]^4$$

$$N_1 = \widetilde{N_1} / \left\langle \begin{pmatrix} t^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \right\rangle \subset \mathbb{C}[t]^4 / \left\langle \begin{pmatrix} t^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \right\rangle$$

Remarks

- Even though Lemma is an essential ingredient, the proof of the Theorem still requires a technique similar to Steinberg's result on the Steinberg variety, showing that $T^{\perp} \prec T^{\vee}$ holds for a certain ordering $\prec$ on $\mathcal{LR}({}^{\lambda'}/_{\nu'}, \mu')$ and then using the fact that both $T \mapsto T^{\perp}$ and $T \mapsto T^{\vee}$ are bijections.
- The parametrization of the irreducible components of $\mathcal{G}^{M}_{\mu\nu}$ and their dimension were given by M. van Leeuwen (2000).
- The affine coordinates of the open covering of $\mathcal{G}^{M}_{T}$ was given in the form of “generic vectors” by T. Maeda (2003 and later).
- This also proves the involutiveness of Azenhas' procedure. (A combinatorial proof of the involutiveness has been given by Azenhas, R. C. King and T (2017).)

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