Hook formulas for skew shapes, day 2: product formulas II, asymptotics, lozenge tilings

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Algebraic and Enumerative Combinatorics in Okayama, 2018

Hook-Length formula for skew shapes

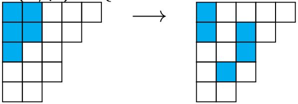
Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

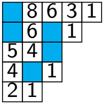
where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ*.*

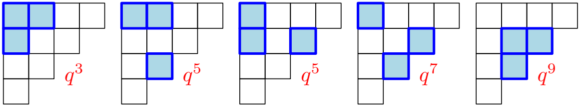
Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



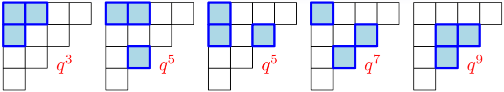
Hook lengths inside λ:





$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

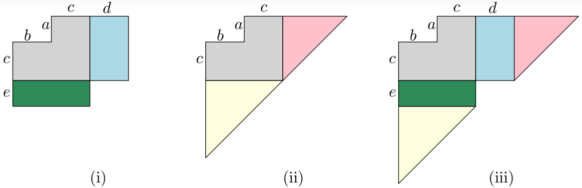
Theorem (Morales-Pak-P)

For (reverse) plane partitions of skew shape λ/μ we have that

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D,$ for some $D \in \mathcal{E}(\lambda/\mu)\}$ is the set of "pleasant diagrams".

Product formulas



(i) (ii) (iii)

$\Phi(n) := 1! \cdot 2! \cdots (n-1)!$, $\Psi(n) := 1!! \cdot 3!! \cdots (2n-3)!!$,
$\Psi(n;k) := (k+1)!! \cdot (k+3)!! \cdots (k+2n-3)!!$, $\Lambda(n) := (n-2)!(n-4)! \cdots$

Theorem (MPP)

For nonnegative integers a, b, c, d, e, let n be the size of the corresponding skew shape, then for the shapes in (i), (ii), (iii) we have the following product formulas for the number of skew SYTs:

$$f^{sh(i)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+e+d)}{\Phi(a+b)\Phi(e+d)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+e+d)},$$

$$f^{sh(ii)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)} \, \frac{\Psi(c)\Psi(a+b+c)}{\Psi(a+c)\Psi(b+c)\Psi(a+b+2c)},$$

$$f^{Sh(iii)} = \frac{n! \quad \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)\,\Psi(c;d+e)\Psi(a+b+c;d+e)\,\Lambda(2a+2c)\Lambda(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)\Psi(a+c)\Psi(b+c)\Psi(a+b+2c;d+e)\Lambda(2a+2c+d)\Lambda(2b+2c+e)}$$

Multivariate identities I

Set $z_{\lambda_i+d-i+1}(\lambda) = x_i$ and $z_{\lambda'_j+n-d-j+1}(\lambda) = y_j$.

Theorem (Ikeda-Naruse)

$$\sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (x_i - y_j) = s_\mu^{(d)}(\mathbf{x} \mid z(\lambda))$$

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Proposition (MPP)

Let $\lambda/\mu \subset d\times(n-d)$ with $\lambda_d \geq \mu_1 + d - 1$. Then:

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In particular, the LHS is symmetric in $(x_1,\ldots,x_d)$.

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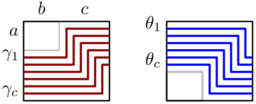
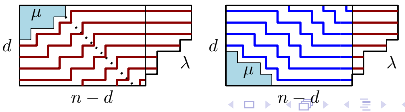
$$\sum_{\substack{\Gamma=(\gamma_1,\ldots,\gamma_c),NILP\\ \gamma_p:(a+p,1)\to(p,b+c)}} \prod_{(i,j)\in\Gamma} \frac{1}{x_i - y_j} \;=\; \sum_{\substack{\Theta=(\theta_1,\ldots,\theta_c),NILP\\ \theta_p:(p,1)\to(a+p,b+c)}} \prod_{(i,j)\in\Theta} \frac{1}{x_i - y_j}\,,$$





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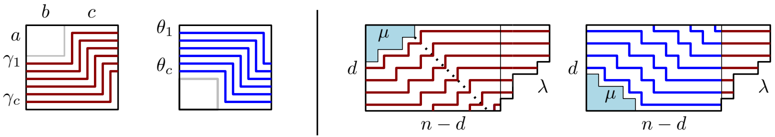

If $x_i = \lambda_i - i$ and $y_j = -\lambda_j + j - 1$, then $h_\lambda(i,j) = x_i - y_j$.
If λ is "nice", then any path θ : NW corner A $\to$ SE corner B has the same multiset of hooks $(h(\theta(1)), h(\theta(2)), \ldots)$

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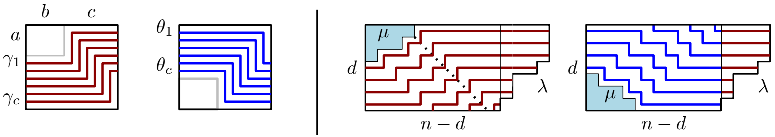

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NHLF: $$\frac{f^{\lambda/b^a}}{n!} = \left[\prod_{u\in[\lambda]\setminus R} \frac{1}{h_\lambda(i,j)}\right] \sum_{D\in\mathcal{E}(R/b^a)} \prod_{(i,j)\in R\setminus D} \frac{1}{h_\lambda(i,j)}$$

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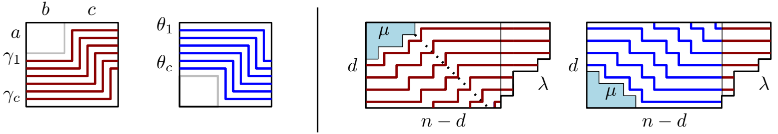

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$$\text{flip diagram/paths vertically} = (\textit{factor}) \prod_{(i,j)\in R/0^c b^a} \frac{1}{h_\lambda(i,j)} \times \#\,\mathcal{E}(R/b^a)$$

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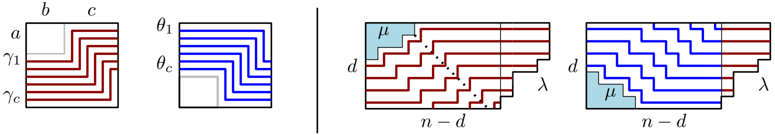

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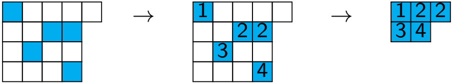
Excited diagrams $\leftrightarrow$ flagged tableaux of shape μ:



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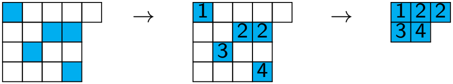
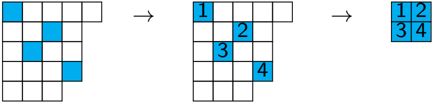


When $\mu = (b^a)$, then SSYTs with max entry $\leq \max\{k : \lambda_k \geq k+b-a\}$:



Product formulas for Schubert polynomials

Schubert polynomial for a permutation $w \in S_n$: $\mathfrak{S}_w(x_1, \ldots, x_n)$
$s_i := (i, i+1)$ – simple transposition in S_n.

Combinatorial/recursive definition:
$w_0 = n\, n-1\, \ldots 21 \to \mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$

$\mathfrak{S}_w = \partial_i\, \mathfrak{S}_{ws_i}$ if $\ell(ws_i) = \ell(w) + 1$,
where $\partial_i f = \frac{f(x) - s_i f(x)}{x_i - x_{i+1}}$ is the divided difference

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Macdonald's identity

$$\mathfrak{S}_w(1, 1, \ldots, 1) = \frac{1}{\ell!} \sum_{(r_1, \ldots, r_\ell) \in R(w)} r_1 r_2 \cdots r_\ell. \tag{1}$$

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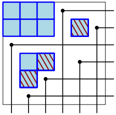
Question: Explicit formulas for $\mathfrak{S}_w(1^n)$? Asymptotics? Maximum?

Permutations and diagrams

$w = w_1 w_2 \dots w_n \in S_n$
Rothe diagram:

$$D(w) = \{(i, w_j) \mid i < j, w_j < w_i\}.$$

Essential set of w:
$Ess(w) = \{(i,j) \in D(w) \,|\, (i+1,j), (i,j+1), (i+1,j+1) \notin D(w)\}.$

$w = 461532 \quad \rightarrow$ 

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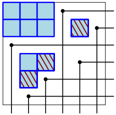
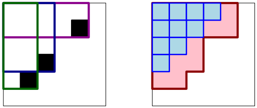


Vexillary permutation: if $D(w)$ is, up to permuting rows and columns, the Young diagram of a partition $\mu = \mu(w)$. Equivalently, 2143-avoiding permutations
Let $\lambda = \lambda(w)$ be the smallest partition containing the diagram $D(w)$– the **supershape** of w



[Knutson-Miller-Yong, Wachs]: The double Schubert for vexillary permutations:

$$\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (x_i - y_j).$$

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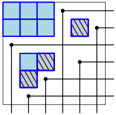


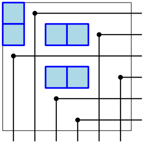
321-avoiding permutations: the diagram $D(w)$ of such a permutation is, up to removing rows and columns of the board not present in the diagram and flipping columns, the Young diagram of a skew shape that we denote $sh(w)$.

Theorem (Billey–Jockusch–Stanley)

For every skew shape λ/μ with $(n-1)$ diagonals, there is a 321-avoiding permutation $w \in S_n$, such that $sh(w) = \lambda/\mu$.

		1
4	3	2
5	4	

$s_1 s_4 s_3 s_2 s_5 s_4 = 251634$



Product formulas for Schubert polynomials at 1^n

Corollary: Let w be a vexillary permutation of shape μ and supershape λ. Then:

$$\mathfrak{S}_w(1^n) = \left|\mathcal{E}(\lambda/\mu)\right|.$$

Corollary[Fomin-Kirillov]: For a dominant permutation w with $D(w) = \mu$, we have:

$$\mathfrak{S}_{id_c,(w+c)}(1^{n+c}) = |RPP_\mu(c)| = \prod \text{..hook-content formula.}$$

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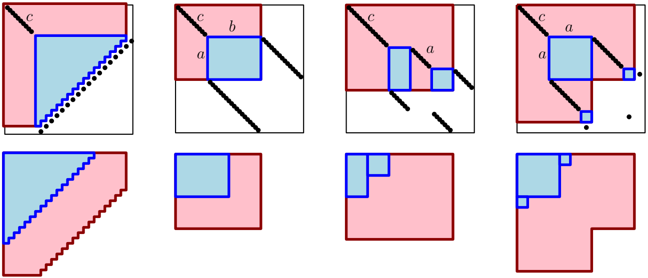

(a) (b) (c) (d)

Product formulas:[MPP] Case (c): $v(a) := 2413 \otimes 1^a$. Then, for all $c \geq a$, we have:

$$\mathfrak{S}_{(id_c, v(a)+c)}(1^n) = \frac{\Phi(4a+c)\Phi(c)\Phi(a)^4\Phi(3a)^2}{\Phi(3a+c)\Phi(a+c)\Phi(2a)^2\Phi(4a)}.$$

Case (d): $w(a) := (a+1, a+2, \ldots, 2a-1, \mathbf{2a+1}, 1, 2, \ldots, a-1, \mathbf{2a}, \mathbf{a})$.

$$\mathfrak{S}_{(id_c, w(a)+c)}(1^n) = \frac{\Phi(2a+c)\Phi(a)^2\Phi(c)}{\Phi(a+c)^2\Phi(2a-1)} \left[\frac{a(2a+c)(2ac+4a^2-1)}{2(4a^2-1)}\right].$$

Schubert polynomials at 1^n

Proposition (MPP)

Let w be a 321-avoiding permutation, then its diagram gives a skew shape λ/μ (and every skew shape gives a 321-avoiding w)

$$\mathfrak{S}_w(1^n) = \frac{1}{\ell!}\, r_1 \cdots r_\ell\, f^{\lambda/\mu},$$

where $\ell = |\lambda/\mu|$ and $(r_1, \ldots, r_\ell)$ is a reduced word of w.

Proof: Macdonalds identity, + all reduced words are a permutation of $(r_1, ..., r_\ell)$ + [Billey-Stanley-Jockush] $\#Red(w) = f^{\lambda/\mu}$.

Schubert polynomials at 1^n

Proposition (MPP)

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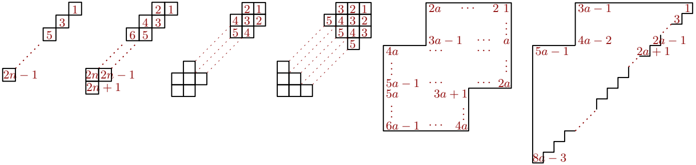

Example 1: Shape $(3a)^{2a}(2a)^a/a^a$ gives the permutation $s(a) := 351624 \otimes 1^a$, we have:

$$\mathfrak{S}_{s(a)}(1^{6a}) \;=\; \frac{\Phi(a)^5\Phi(3a)^2\Phi(5a)}{\Phi(2a)^4\Phi(4a)^2}$$

Example 2: Let $t(a)$ be the permutation of size $(8a-2)$ obtained from the reading word of the skew shape δ_{4a}/a^a:

$$\mathfrak{S}_{t(a)}(1^{8a-2}) \;=\; \frac{\Phi(a)^3\Phi(3a)\Phi(4a-1)\Phi(8a-2)\cdot\Psi(a)\Psi(3a)}{\Phi(2a)^2\Phi(3a-1)\Phi(5a-1)\cdot\Psi(2a)^2\Psi(4a)\cdot\Lambda(8a-2)}$$

Symmetric group S_n
Irreps $\mathbb{S}_\lambda$, $\lambda \vdash n$

$$Tr_{\mathbb{S}_\lambda}[\pi] = \chi^\lambda(\pi)$$

General linear group GL_N
V_λ, $\ell(\lambda) \leq N$

$$Tr_{V_\lambda}(diag(x_1, \ldots)) = s_\lambda(x_1, x_2, \ldots)$$

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Standard Young Tableaux (SYT)

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 7 & 9 \\ \hline 2 & 6 & 10 \\ \cline{1-3} 5 & 8 \\ \cline{1-2}\end{array}$$

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Semi-Standard Young Tableaux(SSYT)

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HLF: $$\dim \mathbb{S}_\lambda = f^\lambda = \frac{n!}{\prod_{\square \in \lambda} h_\square}$$

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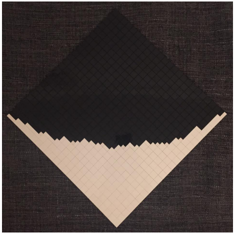

(Lego art by Dan Betea)

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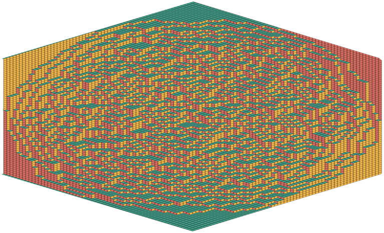
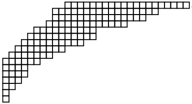
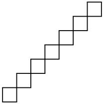
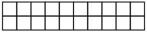


(Computer art by Leonid Petrov)

Asymptotics of the number of skew SYTs

$\lambda/\mu =$ , $|\lambda/\mu| = n \to \infty$

 $\to \; f^{\lambda/\mu} = n!$

 $\to \; f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

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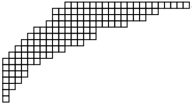
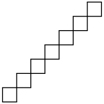
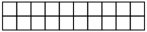
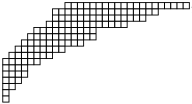
$\lambda/\mu =$ , $|\lambda/\mu| = n \to \infty$

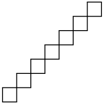
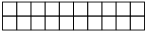
 $\to \; f^{\lambda/\mu} = n!$

 $\to \; f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

Asymptotics of the number of skew SYTs

$\lambda/\mu =$ , $\quad |\lambda/\mu| = n \to \infty$

 $\to \; f^{\lambda/\mu} = n!$  $\to \; f^{\lambda/\emptyset} = C_{n/2} \approx 2^n \frac{2\sqrt{2}}{n^{3/2}\sqrt{\pi}}$

Question: What is the asymptotic value of $f^{\lambda/\mu}$, $|\lambda/\mu| = n$ as $n \to \infty$ and λ, μ change under various regimes:

0. If $\mu = \emptyset$, then $f^{\lambda} \sim \sqrt{n!}(1 + O(1/n))$ for $\lambda \sim$ Plancherel.

1. [Stanley, 2001] (after [Vershik-Kerov]) when μ is fixed, $\lambda^n \to (a; b)$ (Frobenius limit):

$$f^{\lambda^n/\mu} \sim f^{\lambda^n} s_\mu(\rho_a^+; \rho_b^-)(1 + O(1/n)),$$

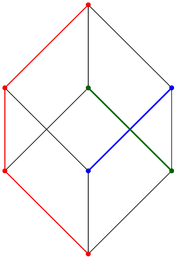
where ρ_a^+, ρ_b^- are the corresponding specializations.
Similar results in [Corteel-Goupil-Schaeffer]
[Okounkov-Olshanski]

General bounds for posets (folklore)

P – poset, $e(P)$ – number of linear extensions,
$P = A_1 \sqcup \ldots \sqcup A_\ell$ – antichains, $P = C_1 \sqcup \ldots \sqcup C_p$ – chains.

$$|A_1|!|A_2|!\cdots|A_\ell|! \le e(P) \le \frac{n!}{|C_1|!|C_2|!\cdots|C_p|!}$$

.



$36 = 1!3!3!1! \le 48 \le \frac{8!}{4!2!2!} = 420$

[Brightwell-Tetali]: Boolean lattice
$\frac{\log_2 e(B_n)}{2^n} = \binom{n}{n/2} - \frac{3}{2}\log_2(e) + o(1)$

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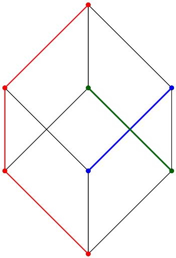

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In our case:
Theorem[MPP]: When $P = \lambda/\mu$ and A_i – ith antidiagonal, then

$$|A_1|!\cdots|A_\ell|! \leq F(\lambda/\mu)$$

if $|A_i| \leq |A_{i+1}|$.

Tool

Naruse Hook-Length formula:

$$f^{\lambda/\mu} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} \frac{1}{h_u}.$$

Define the "naive" hook-length formula:

$$F(\lambda/\mu) := \prod_{u \in \lambda/\mu} \frac{1}{h_u}.$$

$$\begin{array}{cccccc} & & & 5 & 4 & 1 \\ & & 5 & 3 & 2 & \\ & 7 & 4 & 2 & 1 & \\ & 4 & 1 & & & \\ 4 & 2 & & & & \\ 3 & 1 & & & & \\ 1 & & & & & \end{array}$$

$$F((6,5,5,3,2,2,1)/(3,2,1,1)) = \frac{1}{5\cdot 4\cdot 1\cdot 5\cdot 3\cdot 2\cdot 7\cdot 4\cdot 2\cdot 1\cdot 4\cdot 1\cdot 4\cdot 2\cdot 3\cdot 1\cdot 1}$$

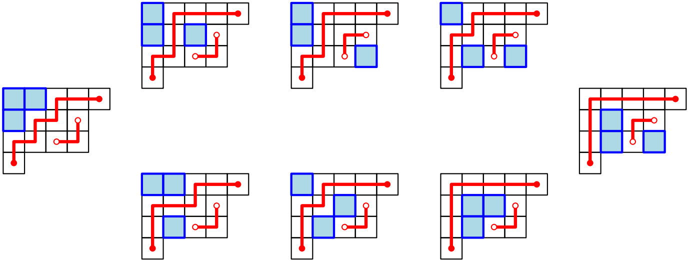
Corollary

$$F(\lambda/\mu) \le f^{\lambda/\mu} \le |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

General bounds: size of $\mathcal{E}(\lambda/\mu)$

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| F(\lambda/\mu)$$

$\mathcal{E}(\lambda/\mu) = \{$ Non-intersecting Lattice Paths in λ/μ $\}$



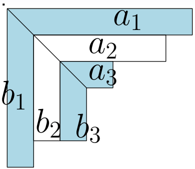
Lemma (MPP)

If $|\lambda/\mu| = n$ *then* $\mathcal{E}(\lambda/\mu) \leq 2^n$.

Lemma (MPP)

If d *is the Durfee square size of* λ, *then* $\mathcal{E}(\lambda/\mu) \leq n^{2d^2}$.

The "linear" regime



$a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ – Frobenius coordinates of λ. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta := (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$.

Thoma–Vershik–Kerov (TVK) *limit* if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \le i \le k$.

Theorem (MPP)

Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with a TVK limit, i.e. suppose $\lambda^{(n)} \to (\alpha, \beta)$, where $\alpha_1, \beta_1 > 0$, and $\mu^{(n)} \to (\pi, \tau)$ for some $\alpha, \beta, \pi, \tau \in \mathbb{R}_+^k$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} = cn + o(n) \quad \textit{as} \quad n \to \infty,$$

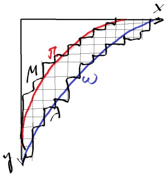
where

$$c = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i).$$

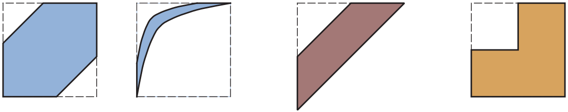
The stable shape: $\sqrt{n}$ scale



Theorem (MPP)

Let $\omega, \pi : [0,a] \to [0,b]$ be continuous non-increasing functions, and suppose that area$(\omega/\pi) = 1$. Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $[\lambda^{(n)}]/\sqrt{n} \to \omega$, $[\mu^{(n)}]/\sqrt{n} \to \pi$. Then

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n \quad \textit{as} \quad n \to \infty.$$



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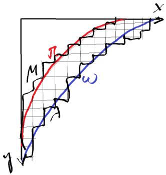


Theorem (MPP)

Suppose $(\sqrt{N}-L)\omega \subset [\lambda^{(n)}](\sqrt{N}+L)\omega$ *for some* $L>0$, *and similarly for* $\mu^{(n)}$ *wrt* π, *then*

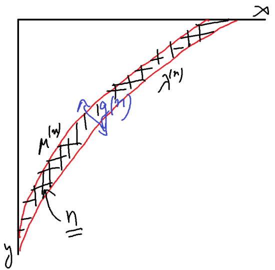
$$-\big(1+c(\omega/\pi)\big)n+o(n) \leq \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2}n\log n \leq -\big(1+c(\omega/\pi)\big)n+\log \mathcal{E}(\lambda^{(n)}/\mu^{(n)})+o(n),$$

as $n\to\infty$, *where*

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x,y)dxdy,$$

where $h(x,y)$ *is the hook length from* (x,y) *to* ω.

Subpolynomial depth, "thin" shapes



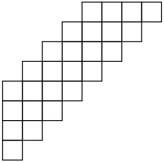
Suppose
depth:$= \max_{u\in\lambda/\mu} h_u =: g(n) = n^{o(1)}$
(subpolynomial growth).

Theorem (MPP)

Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew partitions with a subpolynomial depth shape associated with the function $g(n)$. Then

$$\log f^{\nu_n} = n\log n - \Theta\big(n\log g(n)\big) \quad \textit{as} \quad n\to\infty.$$

Thick ribbons



Theorem (MPP)

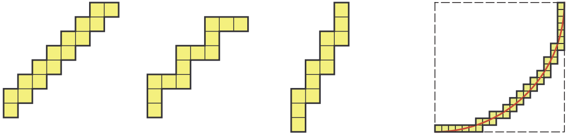
Let $\gamma_k := (\delta_{2k}/\delta_k)$, *where* $\delta_k = (k-1, k-2, \ldots, 2, 1)$. *Then*

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \le \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right) \le \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\gamma_k| = k(3k-1)/2$.

Question: Does there exist a c, s.t. $c = \lim_{n\to\infty} \frac{1}{n}\left(\log f^{\gamma_k} - \frac{1}{2} n \log n\right)$?
Answer: Yes (Martin Tassy's and others work in progress)
Jay Pantone's implementation (method of differential approximants) on 150+ terms of the sequence $\{\log f^{\gamma_k}\}$ to approximate $c \approx -0.1842$.

Thin ribbons



Zigzag: $\rho_k := \delta_{k+2}/\delta_k$, $E_n = |\{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}|$ – Euler numbers, alternating permutations.

$$f^{\rho_n} = E_{2n+1}; \qquad E_m \sim m!(2/\pi)^m 4/\pi(1 + o(1))$$

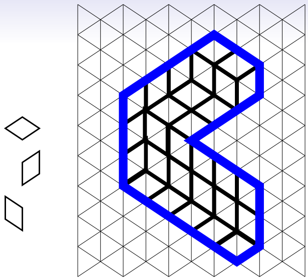
From theorem: $F(\rho_k) = n!/3^k$, $\mathcal{E}(\rho_k) = C_k$, so

$$\frac{(2k+1)!}{3^k} \leq E_{2k+1} \leq \frac{(2k+1)!C_k}{3^k}$$

Problem: If $\gamma_n := \lambda/\mu$ is a border strip (ribbon of thickness 1, n boxes) approaching a given curve γ under rescaling by n, what is $\log f^{\gamma_n} - n \log n$ in terms of γ? Is it true that $\frac{\log f^{\gamma_n} - n\log n}{n} \to c(\gamma)$ for some constant $c(\gamma)$? (Permutations with certain descent sequences)

Lozenge tilings

Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types (“lozenges”).



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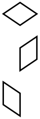
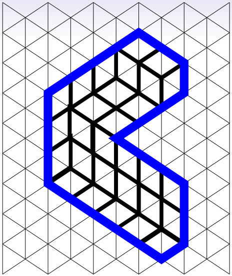
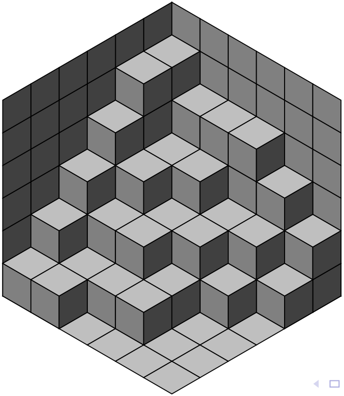




5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



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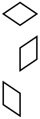

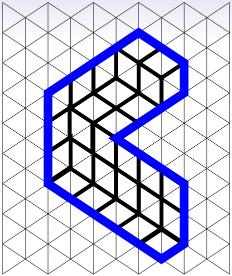

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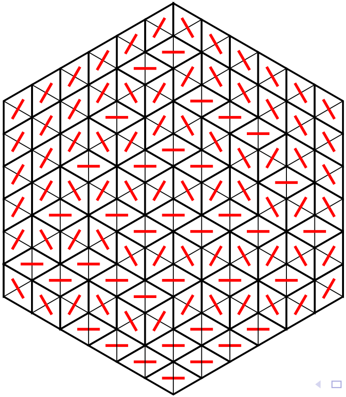
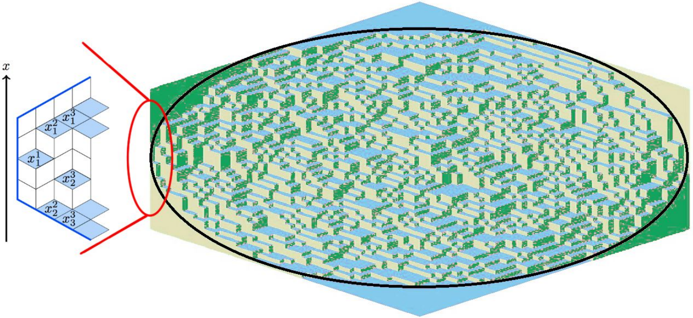


Classical probabilistic questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?



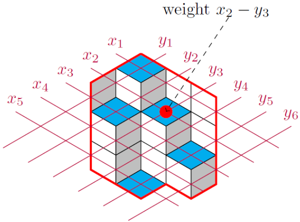
Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partition).

([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006] and newer via Schur generating functions [Borodin, Corwin, Bufetov–Gorin, Petrov, etc])

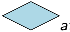
Behavior near boundary: Gaussian Unitary Ensemble eigenvalues,

conjectured by [Okounkov-Reshetikhin, 2006], proofs – hexagon [Johansson-Nordenstam, 2006], more general shapes [Gorin-Panova, 2012]

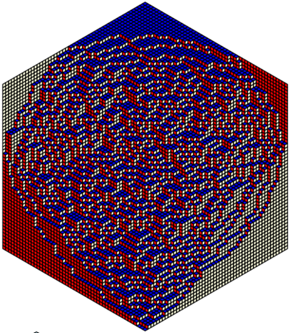
Multivariate local weights

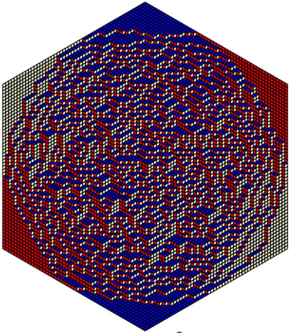


$$\text{Total weight} = \prod_{\text{at } (i,j)} (x_i - y_j)$$



$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$





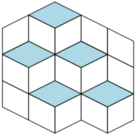
at $(i,j) = 2N - (i+j)$ $\qquad$ $= 1$

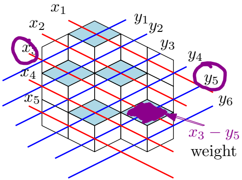
Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	





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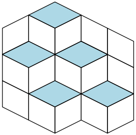
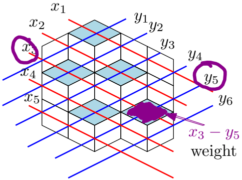




Theorem (Morales-Pak-P)

Consider tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j) \in T} (x_i - y_j) = \det[A_{i,j}(\mu, d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu,d) := \begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{d+\ell(\mu)-j})}{(x_i - x_{i+1})\cdots(x_i - x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu)+1, \ldots, \ell(\mu)+d, \\ \frac{(x_i - y_1)\cdots(x_i - y_{\mu_j+d})}{(x_i - x_{i+1})\cdots(x_i - x_{d+j})}, & \text{when } j = i-d, \ldots, \ell(\mu), \\ 0, & \text{when } j < i-d. \end{cases}$$

Corollary (Krattenthaler, Stanley etc)

Consider the set $PP(\mu, d)$ of plane partitions of base μ and entries less than or equal to d. Then their volume generating function is given by the following determinantal formula

$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

where

$$C_{i,j} = \begin{cases} \dfrac{(-1)^{d+\ell-i} q^{(d-i)(d+\ell-j) - \frac{(d-i+\ell)(d-i-\ell-1)}{2}}}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \ldots, \ell+d, \\ \dfrac{(-1)^{d+j-i} q^{(d-i)(\mu_j+d) - \frac{(d+j-i)(d-i-j-1)}{2}}}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \ldots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

where $(q;q)_m = (1-q)\cdots(1-q^m)$ is the $q-$Pochhammer symbol.

Theorem (Morales-Pak-P)

Consider tilings of the $a \times b \times c \times a \times b \times c$ (base $a \times b$, height c) hexagon with horizontal lozenges having weights $x_i - y_j$, i.e. tilings $\Omega_{a,b,c}$ with rectangular base $\mu = a \times b$ and height c. The partition function is given by

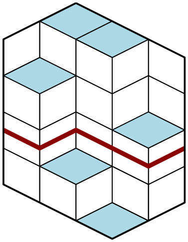
$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j) \in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{c+a-j})}{(x_i - x_{i+1})\cdots(x_i - x_{c+a})} & \text{if } j > a \\ \frac{(x_i - y_1)\cdots(x_i - y_{b+c})}{(x_i - x_{i+1})\ldots(x_i - x_{c+j})} & \text{if } j = i-c, \ldots, a \\ 0, & j < i - c \end{cases} \right]_{i,j=1}^{a+c}$$

Consider a path $P(d_1, \ldots)$ consisting of vertical lozenges (i.e. not the horizontal lozenges) passing through the points (i, d_i) (ith vertical line, distance of the midpoint $d_i + 1/2$ from the top axes) (necessarily $|d_i - d_{i+1}| \leq 1$, $d_i \leq d_{i+1}$ if $i \leq b$ and $d_i \geq d_{i+1}$ if $i > b$, and $d_1 = d_{a+b}$).

The probability that such path exists is given by

$$\mathrm{Prob(path)} = \frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\bar{\mu}, c-d-1)]}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and μ is given by its diagonals – $(d_1 - d, d_2 - d, \ldots)$, and $\bar{\mu}$ is the complement of μ in $a \times b$. The matrix $\bar{A}$ is defined as in previous Theorem with the substitution of x_i by $x_{a+c+1-i}$ and y_j by $y_{b+c+1-j}$.



$\mu = 31$

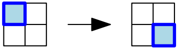
$\mu^* = 20$

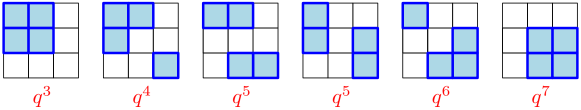
Origins: Excited diagrams and factorial Schur functions

Factorial Schur functions.

$$s_\mu^{(d)}(x|a) \; := \; \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

where $x = (x_1, x_2, \ldots, x_d)$ and $a = (a_1, a_2, \ldots)$ is a sequence of parameters.
Excited diagrams $\mathcal{E}(\lambda/\mu)$: Start with λ/μ. Move cells of μ inside λ via:





Origins: Excited diagrams and factorial Schur functions

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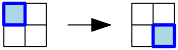

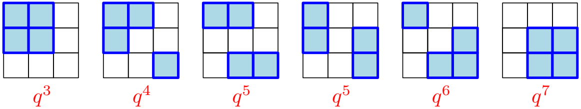

Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

Let $\mu \subset \lambda \subset d \times (n-d)$. Let v be the Grassmannian permutation with unique descent at position d corresponding to λ, i.e. $v(d'+1-i) = \lambda_i + (d'+1-i)$ and $v(j) = d' + j - \lambda'_j$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

Origins: Excited diagrams and factorial Schur functions

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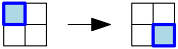

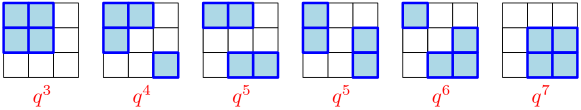

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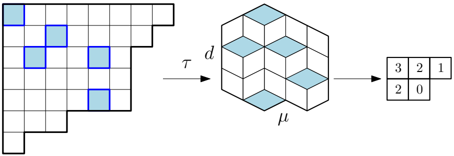
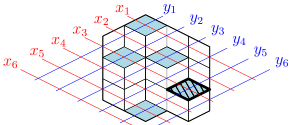
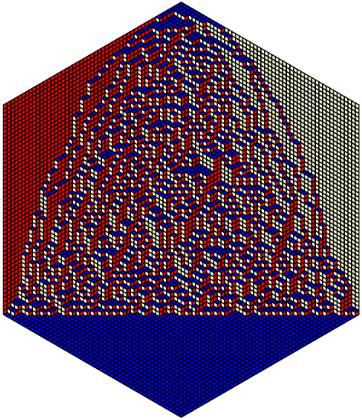
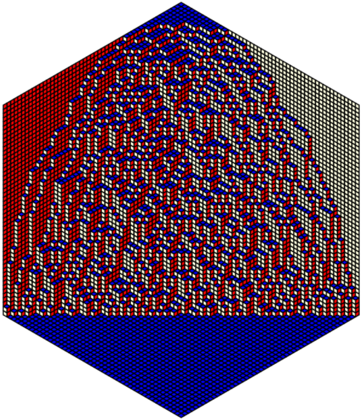




Simulation 2: base $= \delta_n$

Weights: "hook" weights $(4n - i - j)$ versus uniform (i.e. 1).





Schur functions in statistical mechanics

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph

1	1	1	2	2	...
2	2	3	...		
...					

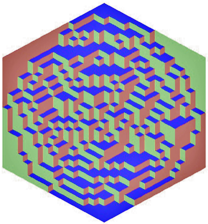
Alternating Sign Matrices (ASM)/ 6-Vertex model:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

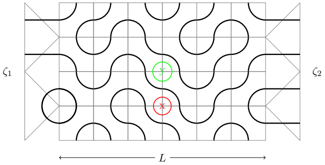
Normalized Schur functions:

$$S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1^{N-k})}{s_\lambda(1^N)}$$

Lozenge tilings:

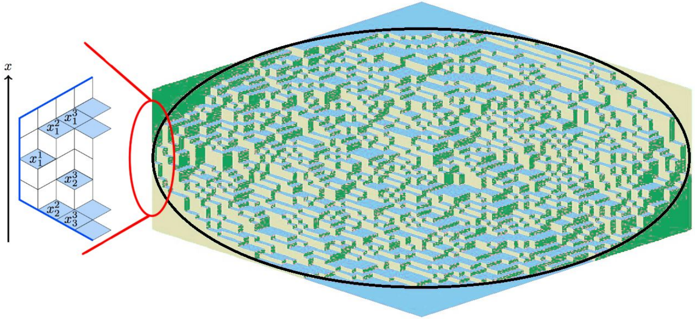


Dense loop model:



Classical questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?

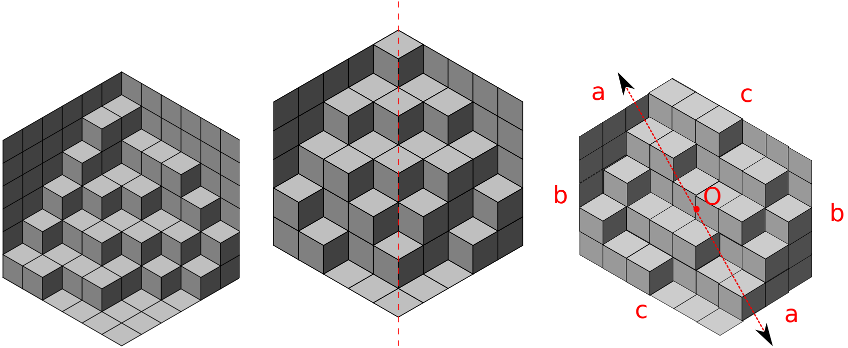


Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partition).
([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006])
Behavior near boundary: Gaussian Unitary Ensemble eigenvalues,
conjectured by [Okounkov-Reshetikhin, 2006], proofs – hexagon [Johansson-Nordenstam, 2006], [Gorin-Panova, 2013]

Uniform vs symmetric

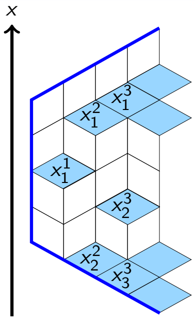
Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

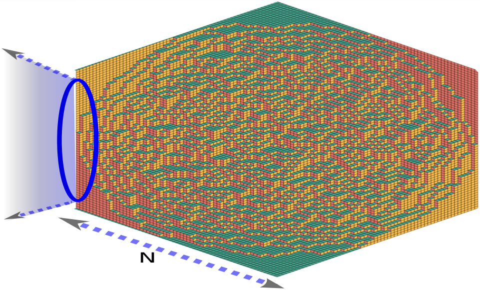
Unrestricted | Vertically symmetric | Centrally symmetric



Limit behavior: fluctuations near the boundary, limit surface, CLT?

Behavior near the flat boundary:





Horizontal lozenges near a flat boundary:

$$\begin{array}{ccccc} & & x_1^1 & & \\ & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & \\ & x_2^2 & & x_1^2 & \\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$}\ \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \\ x_3^3 & & x_2^3 & & x_1^3 \end{array}$$

Question: Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

Conjecture [Okounkov–Reshetikhin, 2006]:
Fixed boundary: The joint distribution converges to a *GUE*-corners (aka *GUE*-minors) process: eigenvalues of GUE matrices.

Proofs: hexagonal domain [Johansson-Nordenstam, 2006], more general domains [Gorin-P,2012], [Novak, 2014], unbounded [Mkrtchyan, 2013], symmetric tilings [P, 2014, 2015]

Behavior near the flat boundary:GUE

GUE: matrices $A = [A_{ij}]_{i,j}$: $A = \overline{A^T}$

$\mathrm{Re} A_{ij}, \mathrm{Im} A_{ij}$ – i.i.d. $\sim \mathcal{N}(0, 1/2)$, $i \neq j$

A_{ii} – i.i.d. $\sim \mathcal{N}(0, 1)$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$(x_1^k \leq x_2^k \leq \cdots \leq x_k^k)$ – eigenvalues of $[A_{i,j}]_{i,j=1}^k$

Interlacing condition: $x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$

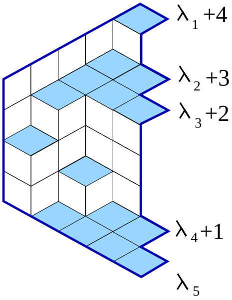
$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \leq & & x_1^2 & & x_2^2 & & \leq \\ & & & x_1^1 & & & \end{array}$$

The *joint distribution* of $\{x_i^j\}_{1 \leq i \leq j \leq k}$ is the
GUE–corners (also, GUE-minors) process, $=: \mathbb{GUE}_k$.

Tilings setup

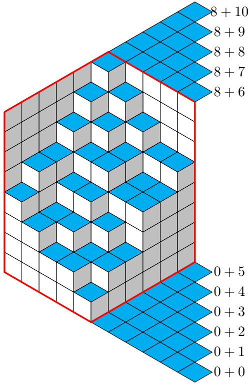
Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$



$\lambda(5) = (4, 3, 3, 0, 0)$

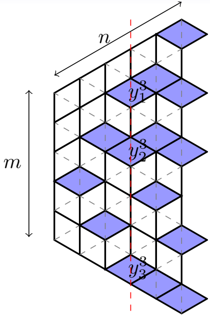
($\frac{1}{N}\Omega_{\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$, e.g. $\lambda(N) = (N, N-1, \dots, 2, 1)$)



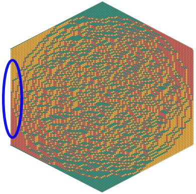
$\lambda = (\underbrace{a, \dots, a}_{c}, \underbrace{0, \dots, 0}_{b})$

$\leftrightarrow$ $a \times b \times c$... hexagon.

Behavior near the flat left boundary



Line $k = 3$



Theorem

Let $Y_n^k = (y_1^k, \ldots, y_k^k)$ – horizontal lozenges on kth line of a uniformly random tiling $T \in \mathcal{T}_n$. As $n \to \infty$ the collection

$$\left\{ \frac{Y_n^j - \mu_n}{\sqrt{n\sigma_n}} \right\}_{j=1}^{k} \to \mathbb{GUE}_k$$

weakly as RVs, where

- $\mathcal{T}_n$ *– all tilings of a hexagon [Johansson-Nordenstam].*
- $\mathcal{T}_n = \Omega_{\lambda(n)}$ – $\mu_n = E(f)$, $\sigma_n = S(f)$, "$f(t) = \lim_{n\to\infty} \frac{\lambda(n)_{nt}}{n}$" *[Gorin-P, 2013].*
- $\mathcal{T}_n$ *– vertically symmetric lozenge tilings of a $n \times m \times n$.. hexagon, $a = \lim_{n\to\infty} m/n$, $\mu_n = m/2$, $\sigma_n = \frac{a^2+2a}{8}$ [P, 2014].*
- $\mathcal{T}_n$ *– centrally-symmetric tilings of a $a \times b \times c$... hexagon with $a = 2qn$, $b = 2pn$, $c = 2(1-q)n$: $\mu_n = 2pqn$ and $\sigma_n = 2pq(1-q)(1+p)$ [P, 2015+].*

Limit shape (surface)

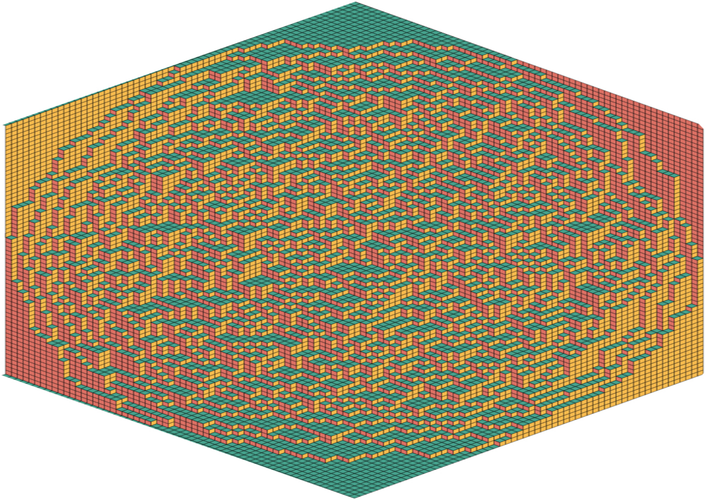


Image: Leonid Petrov

Limit shape (surface)

Theorem (P)

Let $H_n(u,v)$ – height function of a uniformly random tiling from a set $\mathcal{T}_n$, i.e.

$$H_n(u,v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v,$$

where y_i^k is the vertical height of the ith horizontal lozenge on the kth vertical line (left to right). For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u,v)$ converges uniformly in probability to a deterministic function $L(u,v)$ ("the limit shape"), which can be computed explicitly... when $\mathcal{T}_n$ is

- *$\mathcal{T}_n$ – polygonal domain [Cohn, Kenyon, Larsen, Propp, Okounkov]*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ for "nice" family $\lambda(n)$ [Bufetov-Gorin].*
- *$\mathcal{T}_n$ – symmetric tilings [P, 2014].*
- *$\mathcal{T}_n$ – centrally symmetric tilings [P, 2015+].*

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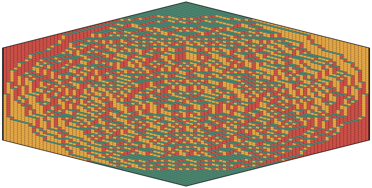
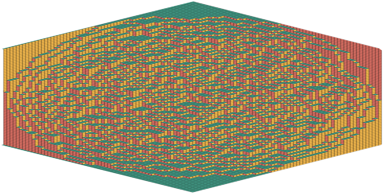
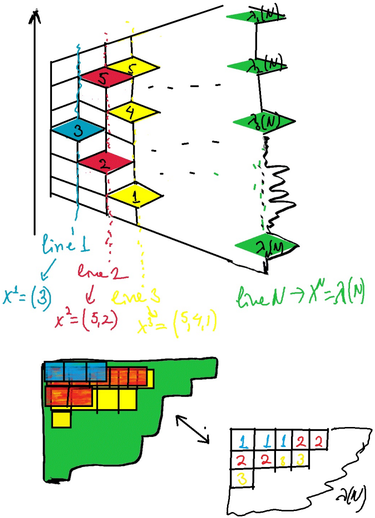
Symmetric:



General:





Lozenge tilings with right boundary $\lambda(N)$

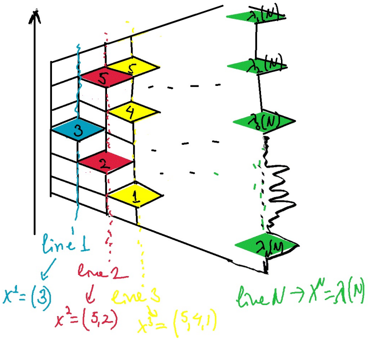
$\Longleftrightarrow$

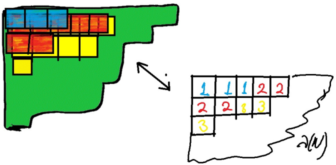
Semi-Standard Young Tableaux T of shape $\lambda(N)$ and entries $1, \dots, N$.

Tilings with horizontal lozenges on vertical line k at positions $x^k = \eta_1, \dots, \eta_k$

$\Longleftrightarrow$

SSYTs T whose entries $1..k$ have shape η





Lozenge tilings with right boundary $\lambda(N)$

$\Longleftrightarrow$

Semi-Standard Young Tableaux T of shape $\lambda(N)$ and entries $1, \dots, N$.

Tilings with horizontal lozenges on vertical line k at positions $x^k = \eta_1, \dots, \eta_k$

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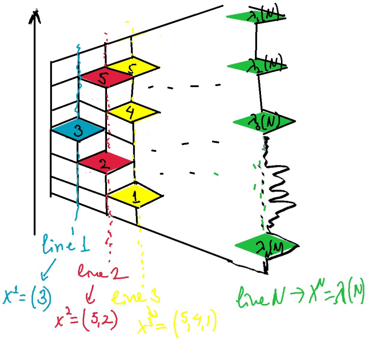
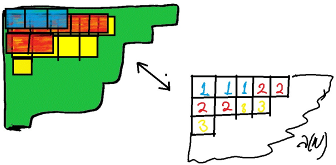
SSYTs T whose entries $1..k$ have shape η

Number of SSYTs of shape ν, entries $1...\ell$ = $s_\nu\underbrace{(1,\dots,1)}_{\ell}$.

$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$





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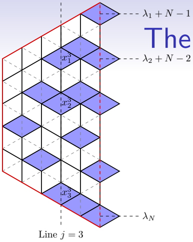
Proposition[Gorin-P] For any variables $y_1, \ldots, y_k$, the **Schur Generating Function** of x^k is

$$\mathbb{E}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} =: S_\lambda(y_1,\ldots,y_k).$$



The explicit Schur Generating Functions[1]

$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

MGF: $$\mathbb{E}\left[\frac{s_{x^k(T)}(y_1,\ldots,y_k)}{s_{x^k(T)}(\underbrace{1,\ldots,1}_{k})} \,\middle|\, T \sim Unif(\mathcal{T}_n) \right] = \sum_{\nu} \frac{s_\nu(y_1,\ldots,y_k)}{s_\nu(1^k)} \Pr(x^k(T)=\nu) = \ldots$$

- $= S_{\lambda(n)}(y_1,\ldots,y_k) = \frac{s_{\lambda(n)}(y_1,\ldots,y_k,1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{\left(\frac{m}{2}\right)^n}(y_1,\ldots,y_k,1^{n-k})}{\mathfrak{so}_{\left(\frac{m}{2}\right)^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{\left(\frac{b}{2}\right)^{a/2}}(y_1,\ldots,y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

[1] from [Gorin-Panova, *Ann. Prob.*], [Panova, *Comm. Math. Phys*], [Panova, in prep]

Tilings probability III: MGF asymptotics

Proposition (Gorin-P)

$$\mathbb{E}\left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})} \quad \nu \sim \mathbb{GUE}_k\right] = \exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2)\right),$$

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Compare:

$$S_\lambda(y_1,\ldots,y_k) = \mathbb{E}_{tiling}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right)$$

Proposition (Gorin-P)

For any k real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N \to f$ we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right) e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} \quad = \quad \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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Theorem. Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then $\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$ (GUE-corners process of rank k).

The limit surface

Counting measure for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_L)$

$$m[\mu] := \frac{1}{L} \sum_{i=1}^{L} \delta\left(\frac{\mu_i + L - i}{L}\right),$$

Random measure on partitions $\rho^n(\mu)$ (e.g. $= \mathrm{Prob}\{x^k(T) = \mu\}$ in tilings of size n), $m[\rho]$ – pushforward.

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Theorem[Bufetov-Gorin,2014] Suppose that ρ^N is a sequence of measures on partitions, s.t. for every r

$$\lim_{N\to\infty} \frac{1}{N} \ln\left(S_{\rho^N}(u_1, \ldots, u_r, 1^{N-r})\right) = Q(u_1) + \cdots + Q(u_r),$$

uniformly in a $\mathbb{C}$ nbhd of (1^r), Q – analytic. Then the random measures $m[\rho^N]$ converge, as $N \to \infty$, in probability to a deterministic measure M on $\mathbb{R}$ with moments

$$\int_{\mathbb{R}} t^p M(dt) = \sum_{\ell=0}^{p} \binom{p}{\ell} \frac{1}{(\ell+1)!} \frac{\partial^\ell}{\partial u^\ell} u^p Q'(u)^{p-\ell}\Big|_{u=1}$$

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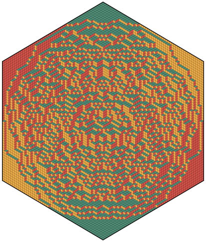
Our cases: MGF = normalized Schur $S_{\lambda(n)}$, SO characters, etc.
Asymptotics using [Gorin-P, 2013] for fixed r:

$$\lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_1, \ldots, u_r) = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_i) = \sum_{i=1}^{r} \Phi(u_i)$$

Limit surface



Theorem (P, 2014)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u, v)$ – height function of a symmetric tiling of $n \times m \times n$... hexagon,i.e.

$$H_n(u, v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit surface").

For any fixed $u \in (0, 1)$, $L(u, v)$ is the distribution function of the measure **m**, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi'_a(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y; a) - 2$ and...

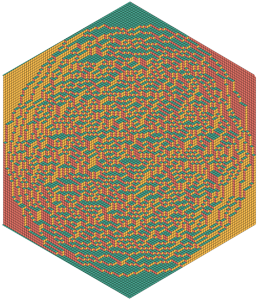
$$h(y) = \frac{1}{4}\left((e^y + 1) + \sqrt{(e^y+1)^2 + 4(a^2+a)(e^y-1)^2} \right)$$

$$\phi(y; a) = (\frac{a}{2}+1)\ln\left(h(y) - (\frac{a}{2}+1)(e^y-1)\right) - (\frac{a}{2}+\frac{1}{2})\ln\left(h(y) - (\frac{a}{2}+\frac{1}{2})(e^y-1)\right)$$
$$+ \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y-1)\right) - (\frac{a}{2}-\frac{1}{2})\ln\left(h(y) + (\frac{a}{2}-\frac{1}{2})(e^y-1)\right)$$

Limit surface

Theorem (P, 2015+)

The scaled height function $H_n(u,v)$ of a centrally symmetric tiling of an $a \times b \times c \ldots$ hexagon converges uniformly in probability to a deterministic function $L(u,v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.
The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).



Behind the scene: asymptotics of symmetric functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

Theorem (Gorin-P)

For any partition λ and any $x \in \mathbb{C}\setminus\{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))}dz,$$

where the contour C includes all the poles of the integrand. Similar formulas hold for the other normalized Lie group characters.

Theorem (Gorin-P)

If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ [under certain convergence conditions], for all fixed $y \neq 0$:

$$\lim_{N\to\infty}\frac{1}{N}\ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w-f(t)-1+t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$. If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ ["other" conv. cond.], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

$$\text{where } E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2 dt - 1/6 - E(f)^2.$$

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Theorem (Gorin-P)

Let $D_{i,1} = x_i \frac{\partial}{\partial x_i}$ *,* $\Delta-$ *Vandermonde det. Then* $\forall\ \lambda$*,* $k \leq N$*, we have*

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

Corollary (Gorin-P)

Suppose that the sequence $\lambda(N)$ *is such that, as* $N \to \infty$*,*

$$\frac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x) \quad \textit{uniformly on a compact } M \subset \mathbb{C}. \textit{ Then for any } k$$

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1) + \cdots + \Psi(x_k)$$

uniformly on M^k*. More informally, under various regimes of convergence for* $\lambda(N)$ *we have* $S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k)$.

More problems?

- More precise asymptotics of $f^{\lambda/\mu}$ in various regimes.
- Asymptotics of lozenge tilings using the multivariate weights, new regimes?
- Asymptotics of $\frac{s_{\lambda/\mu}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda/\mu}(1^n)}$ (Schur generating functions of tilings of arbitrary domains)
- Asymptotics of Littlewood-Richardson coefficients, $c^{\lambda}_{\mu,\nu}$... (e.g. if $\lambda \vdash 2n$, $\mu, \nu \vdash n$, when is it maximal)
- Maximal $f^{\lambda/\mu}$ under constraints...

Thank you