Hook length property of d-complete posets via q-integrals

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(Joint work with Meesue Yoo)

Outline

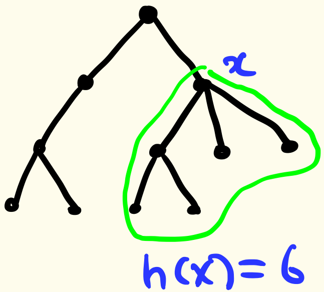
① Hook length formula for trees, shapes, and shifted shapes

② d-complete posets.

③ q-integral

④ Proof of HLF for d-complete posets.

Hook length formula for trees

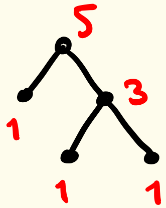


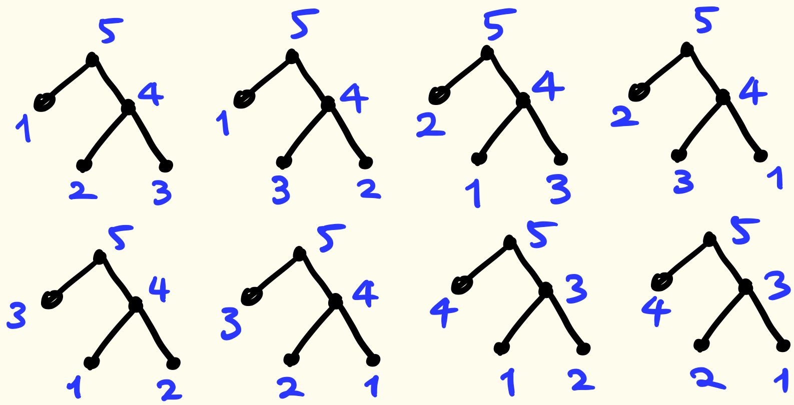
P : tree poset.

The **hook length** of $x \in P$
is $\#\{y \in P : y \leq x\}$

Thm # linear extensions of P is $\dfrac{n!}{\prod_{x \in P} h(x)}$.

ex

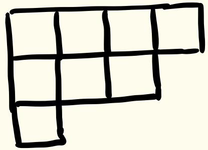




$$\frac{5!}{1\cdot 1\cdot 1\cdot 3\cdot 5} = 8$$

Hook length formula for shapes

- $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition of n if $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$
- Young diagram of $\lambda = (4,3,1)$ is

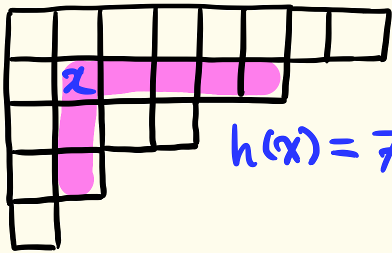


- A standard Young tableau of shape $\lambda = (4,3,1)$ is

1	2	3	8
4	5	7	
6			

.

- hook length of $x \in \lambda$:



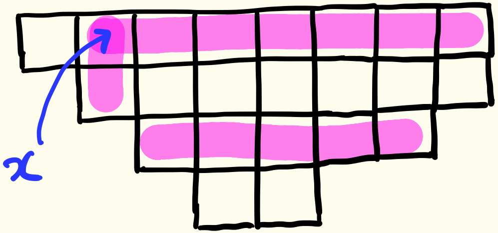
$h(x) = 7$

Thm

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

Hook length formula for shifted shapes

- $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a strict partition of n if $\lambda_1 > \cdots > \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$
- shifted Young diagram of $\lambda = (8, 7, 5, 2)$



$h(x) = 13$

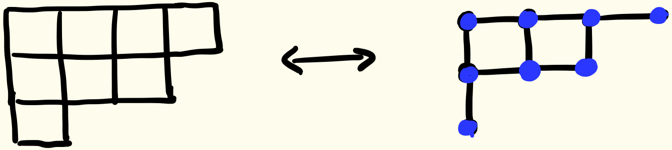
1	2	3	6	7	12	15	16
	4	5	8	13	14	20	21
		9	10	17	19	22	
			11	18			

SYT

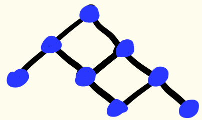
<u>Thm</u> λ is a strict partition of n

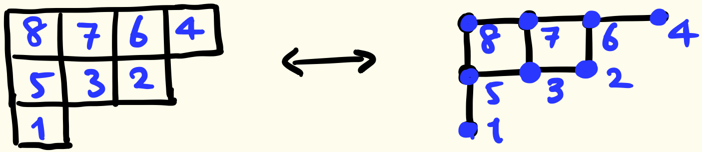
$$\Rightarrow g^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

- reverse SYTs are linear extensions.



poset rotated 45°





<u>Thm</u> P : poset (shape, shifted shape or tree)

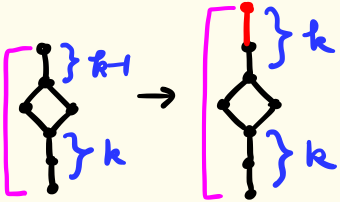
$\Rightarrow$ # linear extensions of P

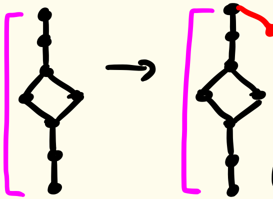
$$= \frac{n!}{\prod_{x \in P} h(x)}$$

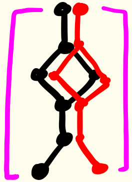
Proctor generalized this to d-complete posets.

d-complete poset

Def P is d-complete if $\qquad \left[{}^{y}_{x}\right.$ means $[x,y]$

① 

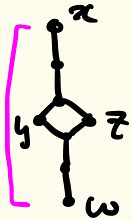
②  (x)

③  (x)

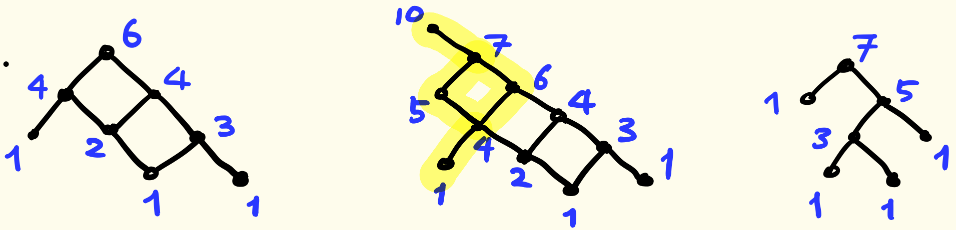
Def P : d-complete.

The hook length of $x \in P$ is

$$h(x) = \begin{cases} h(y)+h(z)-h(w) & \text{if} \\ \# \, y \le x & \text{otherwise} \end{cases}$$



ex).



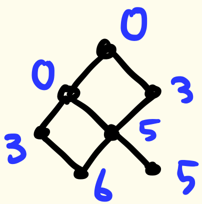
P-partitions

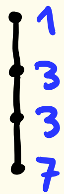
Def P : poset.

A P-partition is an order-reversing map.

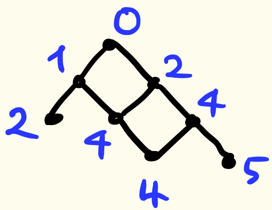
$(\sigma : P \to \mathbb{N}$ s.t $\sigma(x) \geq \sigma(y)$ if $x \leq_P y)$

ex





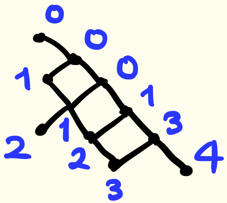
(partition)



$\Longleftrightarrow$

0	2	4	5
1	4	4	
2			

(reverse plane partition)



$\Longleftrightarrow$

0	0	0	1	3	4
	1	1	2	3	
		2			

(shifted RPP)

- The size of $\sigma : P \to \mathbb{N}$ is $|\sigma| = \sum_{x \in P} \sigma(x)$.
- $GF_q(P) = \sum_{\sigma : P \to \mathbb{N}} q^{|\sigma|}$

q-Hook length formula

Thm P: shape, shifted shape or tree

$$GF_q(P) = \sum_{\sigma: P \to \mathbb{N}} q^{|\sigma|} = \prod_{x \in P} \frac{1}{1 - q^{h(x)}}$$

FACT: $$\sum_{\sigma: P \to \mathbb{N}} q^{|\sigma|} = \frac{\sum_{\pi \in \mathcal{L}(P)} q^{\mathrm{maj}(\pi)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

Thm (Peterson-Proctor)

P : d-complete poset.

$$GF_q(P) = \sum_{\sigma: P \to \mathbb{N}} q^{|\sigma|} = \prod_{x \in P} \frac{1}{1 - q^{h(x)}}$$

Goal: Prove this theorem.

Note

Thm has been proved by Proctor (with Peterson), Ishikawa and Tagawa, Nakada.

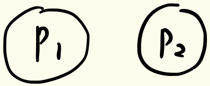
Proof of HLF for trees.

Thm P : tree

$$GF_q(P) = \prod_{x\in P} \frac{1}{1-q^{h(x)}}$$

Lem 1 $P = P_1 \uplus P_2$.

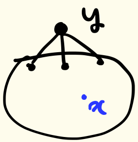
$$GF_q(P) = GF_q(P_1)\, GF_q(P_2)$$



Lem 2 P : poset with maximum y., $|P| = n$.

$$GF_q(P) = \frac{1}{1-q^n} GF_q(P'), \qquad P' = P - \{y\}$$

Pf)



$\sigma : P \to \mathbb{N}$. $\sigma(y) = k$

$\forall x \in P'$, $\sigma(x) \geq k$.

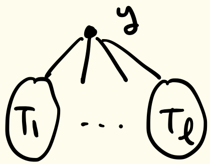
Let $\tau : P - \{y\} \to \mathbb{N}$, $\tau(x) = \sigma(x) - k$.

Then $|\sigma| = k\cdot n + |\tau|$

$$\sum_{\sigma : P \to \mathbb{N}} q^{|\sigma|} = \sum_{k=0}^{\infty} \sum_{\tau : P' \to \mathbb{N}} q^{kn + |\tau|}$$

Pf of Thm)

Induction on $n = |P|$.

$P =$ 

By Lem 2,

$$GF_q(P) = \frac{1}{1-q^n} GF_q(P')$$

By Lem 1,

$$GF_q(P') = GF_q(T_1) \cdots GF_q(T_\ell)$$

$h(y) = n$.

By ind hyp.

$$GF_q(P) = \prod_{x\in P} \frac{1}{1-q^{h(x)}}$$

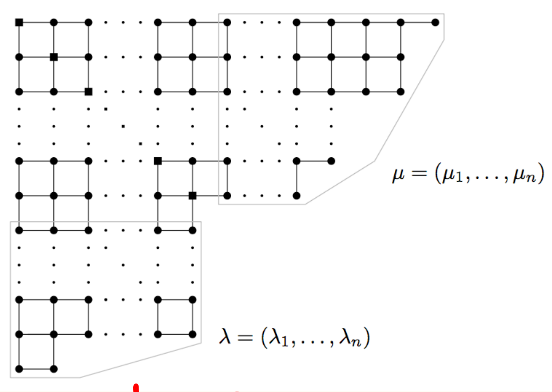
Outline of Proof

- (Proctor) Every d-complete poset is a slant-sum of irreducible d-complete posets.
- (Proctor) There are 15 classes of irr. d.c.p.
- (Kim, Stanton) $GF_q(P)$ can be written as q-integral

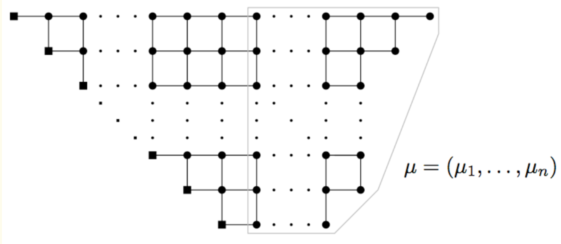
1. Show that semi-irr. d.c.p are enough to consider.
2. Express $GF_q(P)$ for each semi-irr. d.c.p P as q-integral.
3. Evaluate the q-integrals

   (Among 19 integrals, 2 of them are known,
   15 of them can be evaluated by computer.
   Evaluate the remaining 2 integrals by hand.)

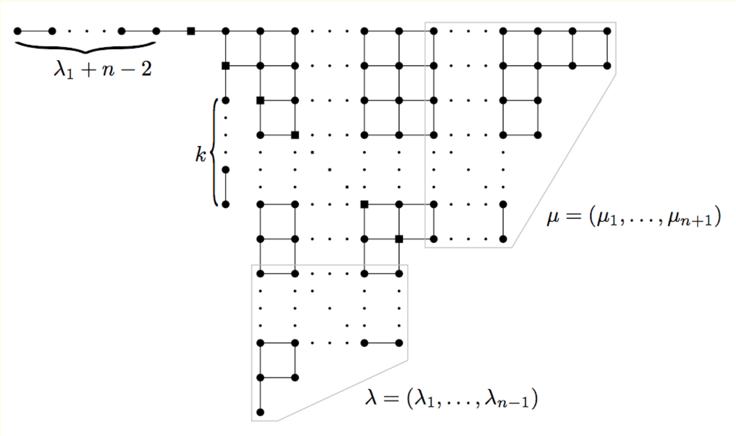
Semi-Irreducible d-complete posets

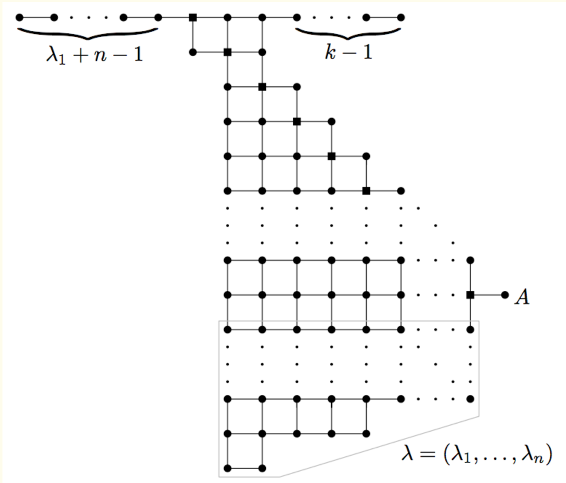


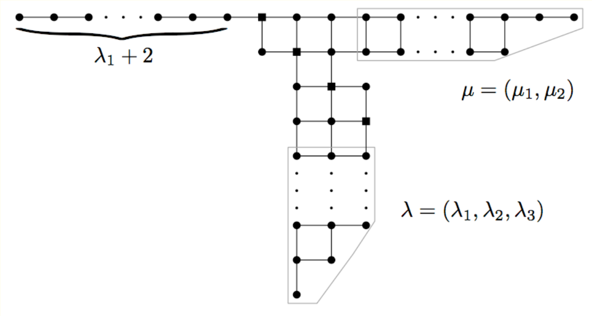
shapes

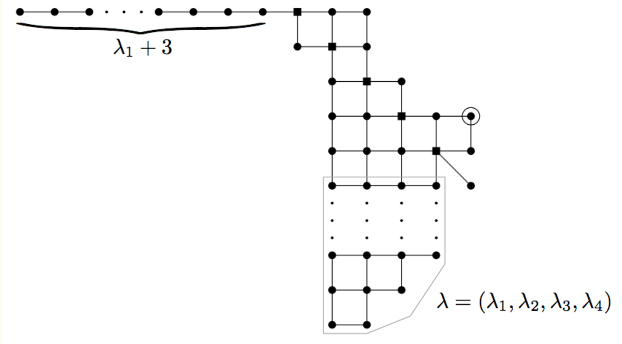


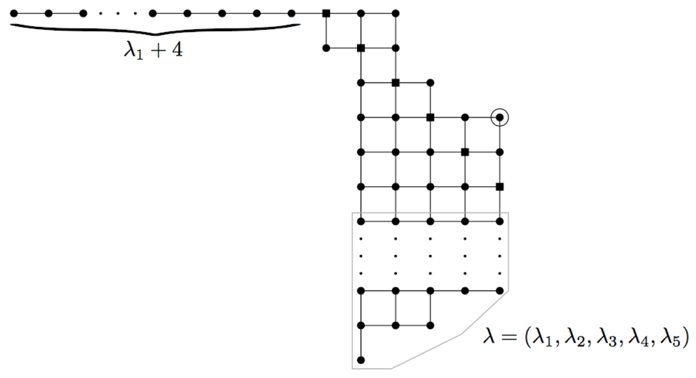
shifted shapes.

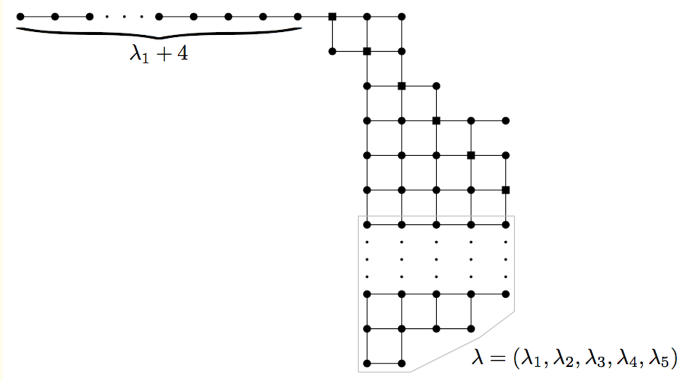


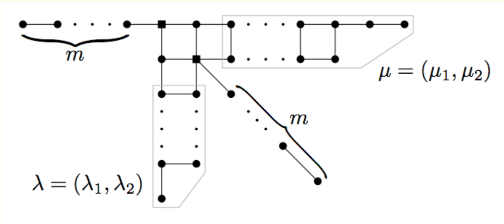


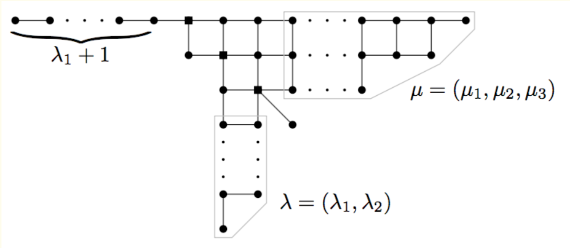

$\lambda_1 + 2$
$\mu = (\mu_1, \mu_2)$
$\lambda = (\lambda_1, \lambda_2, \lambda_3)$


$\lambda_1 + 3$
$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$


$\lambda_1 + 4$
$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$


$\lambda_1 + 4$
$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$


m
$\mu = (\mu_1, \mu_2)$
m
$\lambda = (\lambda_1, \lambda_2)$


$\lambda_1 + 1$
$\mu = (\mu_1, \mu_2, \mu_3)$
$\lambda = (\lambda_1, \lambda_2)$

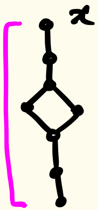
Slant sums

Def P : d-complete.

$x \in P$: **acyclic** if

① x is covered by at most one element,

②

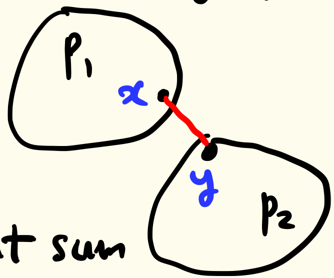


(X)

(Every irr. d.c.p has at most 2 acyclic elements.)

Def P_1, P_2 : d-complete. $x \in P_1$, $y \in P_2$, x : acyclic, y : max of P_2

Slant sum $P_1 {}^{x}\backslash_{y} P_2$ is



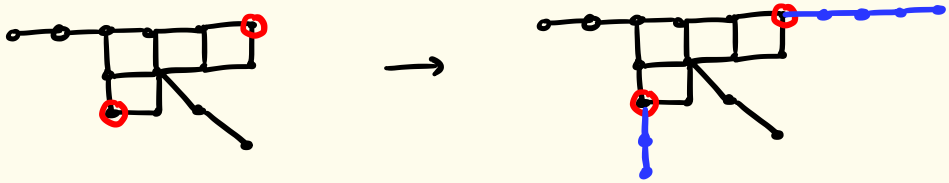
Thm (Proctor)

Every d-complete poset is a slant sum of irr. d-complete posets.

Semi-irreducible d-complete posets

Def A semi-irreducible d-complete poset is a poset obtained from an irr. d.c.p. by adding a chain below each acyclic element.

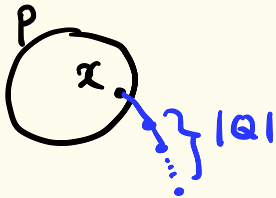
ex).



irr. semi-irr.

Lem P, Q : d-complete. $x \in P$: acyclic, $y \in Q$: max.

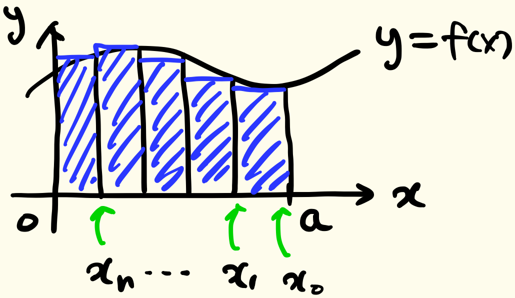
$$GF_q(P^x \setminus_y Q) = GF_q(P')\, GF_q(Q) \cdot \prod_{i=1}^{|Q|} (1-q^i)$$

$P' =$ 

Cor It is enough to consider semi-irr. d.c.p.

q-integrals

- $\int_0^a f(x)\,dx =$ area of



$$= \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)(x_i - x_{i+1})$$

$x_i = aq^i \quad (0<q<1)$

Def q-integral

$$\int_0^a f(x)\,d_qx = \sum_{i=0}^{\infty} f(aq^i)\,(aq^i - aq^{i+1})$$

$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx$$

Note

$$\lim_{q\to 1^-} \int_a^b f(x)\,d_qx = \int_a^b f(x)\,dx.$$

FACT

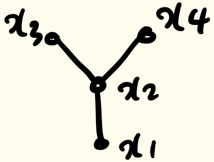
$$\int_a^b x^n\,d_qx = \frac{b^{n+1} - a^{n+1}}{1+q+\cdots+q^{n+1}}$$

Order polytopes

Def P : poset on $\{x_1, \dots, x_n\}$. (naturally labeled)

The order polytope of P is

$$\mathcal{O}(P) = \{(x_1, \dots, x_n) \in [0,1]^n : x_i \le x_j \text{ if } x_i \le_P x_j\}$$

ex) $P =$ 

$$\mathcal{O}(P) = \{(x_1, x_2, x_3, x_4) \in [0,1]^4 \mid x_1 \le x_2,\ x_2 \le x_3,\ x_2 \le x_4\}$$

- We can consider q-integrals over $\mathcal{O}(P)$.

$$\int_{\mathcal{O}(P)} f(x_1, x_2, x_3, x_4)\, d_q x_1\, d_q x_2\, d_q x_3\, d_q x_4$$

$$= \int_0^1 \int_0^1 \int_0^{\min(x_3, x_4)} \int_0^{x_2} f(x_1, x_2, x_3, x_4)\, d_q x_1\, d_q x_2\, d_q x_3\, d_q x_4$$

q-integral over order polytope

Thm (Kim, Stanton, 2016)

P : poset on $\{x_1, \dots, x_n\}$ (naturally labeled)

$$\int_{O(P)} f(x_1, \dots, x_n)\, d_q x_1 \cdots d_q x_n$$

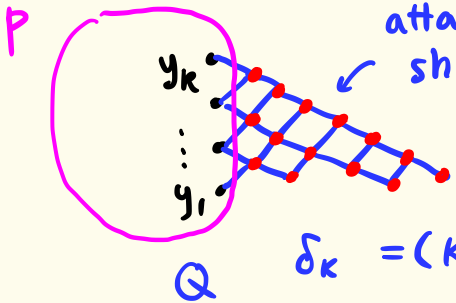
$$= (1-q)^n \sum_{\sigma : P \to \mathbb{N}} f(q^{\sigma(1)}, \dots, q^{\sigma(n)})\, q^{|\sigma|}$$

In particular,

$$\int_{O(P)} d_q x_1 \cdots d_q x_n = (1-q)^n \, GF_q(P)$$

Note : If P is not naturally labeled, we have (P, ω)-partitions. (The label ω is related to order of integration.)

Attaching shifted shapes



attach shifted shape $\lambda+\delta_k$

$\delta_k = (k-1, \dots, 1, 0)$

Lem (Kim, Stanton, 2016)

P : poset on $\{u_1, \dots, u_n\}$

Q : poset on $\{v_1, \dots, v_N\}$

$y_1 < \dots < y_k$: chain in P

(with certain conditions)

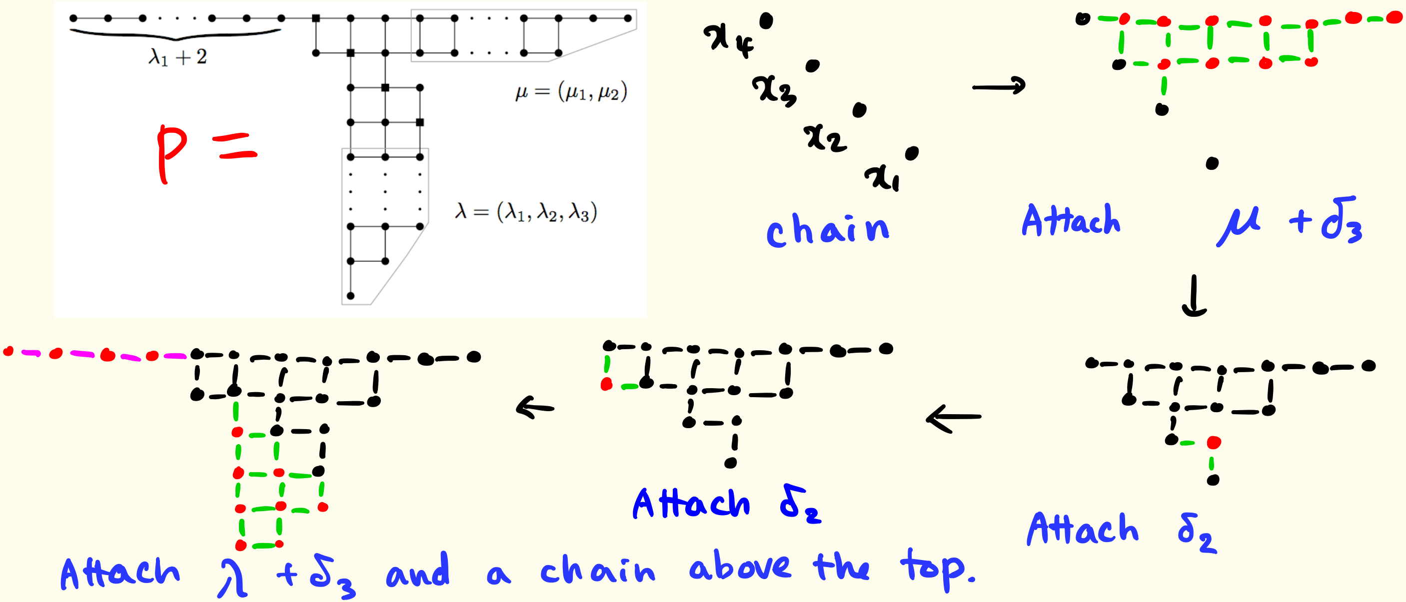
$$\int_{O(Q)} f(x_1, \dots, x_n)\, d_q v_1 \cdots d_q v_N$$

$$= (\text{const}) \cdot \int_{O(P)} f(x_1, \dots, x_n)\, a_{\lambda+\delta_k}(y_1, \dots, y_k)\, d_q u_1 \cdots d_q u_n$$

$$a_\lambda(x_1, \dots, x_n) = \det\left(x_i^{\lambda_j}\right)_{i,j=1}^{n}$$

Expressing $GF_q(P)$ as q-integral.

Observation: Every semi-irreducible d.e.p is obtained as follows.



$$GF_q(P) = \int_{O(P)} d_q y_1 \cdots d_q y_N \sim \int_{0 \le x_1 \le \cdots \le x_4 \le 1} a_{\mu+\delta_3}(x_2, x_3, x_4)\, a_{\delta_2}(x_1, x_2)\, a_{\delta_2}(x_3, x_4) \cdot a_{\lambda+\delta_3}(x_1, x_2, x_3)\, d_q x_1 \cdots d_q x_4$$

Evaluation of q-integral

FACT

$$\int_{0\le x_1\le\cdots\le x_n\le 1} x_1^{a_1}\cdots x_n^{a_n}\, d_q x_1\cdots d_q x_n = \prod_{i=1}^{n} \frac{1-q}{1-q^{a_1+\cdots+a_i+i}}$$

Thus,

$$\int_{0\le x_1\le\cdots\le x_4\le 1} a_{\mu+\delta_3}(x_2,x_3,x_4)\, a_{\delta_2}(x_1,x_2)\, a_{\delta_2}(x_3,x_4)\, a_{\lambda+\delta_3}(x_1,x_2,x_3)\, d_q x_1\cdots d_q x_4$$

can be evaluated by computer!

There are 19 classes of semi-irr. d.c.p.

$\Rightarrow$ " 19 q-integrals to evaluate.

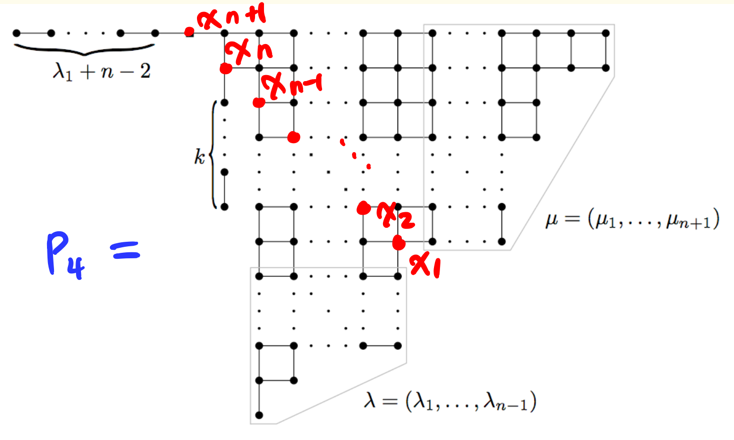
15 of them can be done by computer. (Took 11 hrs.)

2 of the remaing are known.

$\Rightarrow$ we need to evaluate 2 q-integrals.

* Semi-irr. d.c.p. of class 4.

$P_4 =$



$$\hat{\mu}^{(l)}_i = \begin{cases} \mu_i + 1 & \text{if } i < l \\ \mu_{i+1} & \text{if } i \geq l \end{cases}$$

$$GF_q(P_4) \sim \int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} x_n^k \, a_{\lambda+\delta_{n-1}}(x_1, \ldots, x_{n-1}) \, a_{\mu+\delta_{n+1}}(x_1, \ldots, x_{n+1}) \, d_q x_1 \cdots d_q x_{n+1}$$

$$a_{\mu+\delta_{n+1}}(x_1, \ldots, x_{n+1}) = \sum_{l=1}^{n+1} (-1)^{n+1-l} \, x_{n+1}^{\mu_l+n+1-l} \, a_{\hat{\mu}^{(l)}+\delta_n}(x_1, \ldots, x_n)$$

$$= \sum_{l=1}^{n+1} \int_0^1 (-1)^{n+1-l} x_{n+1}^{\mu_l+n+1-l} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1}} x_n^k \, a_{\lambda+\delta_{n-1}}(x_1, \ldots, x_{n-1}) \, a_{\hat{\mu}^{(l)}+\delta_n}(x_1, \ldots, x_n) \, d_q x_1 \cdots d_q x_n \; d_q x_{n+1}$$

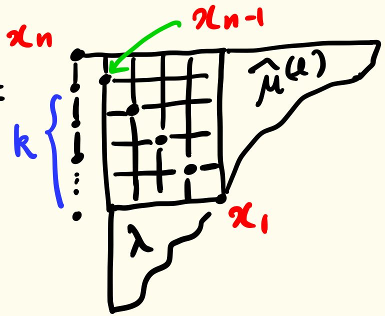
$$= \sum_{l=1}^{n+1} \underbrace{\int_0^1 (-1)^{n+1-l} x_{n+1}^{\mu_l+n+1-l+k+|\lambda|+|\hat{\mu}^{(l)}|+\binom{n-1}{2}+\binom{n}{2}} d_q x_{n+1}}_{A_l} \cdot Y_l .$$

$$GF_q(P_4) \sim \sum_{\ell=1}^{n+1} A_\ell Y_\ell, \qquad (A_\ell : \text{easy constant})$$

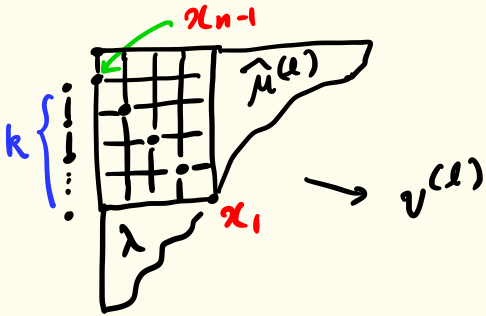
$$Y_\ell = \int_{0\le x_1\le\cdots\le x_n\le 1} x_n^k \, a_{\lambda+\delta_{n-1}}(x_1,\dots,x_{n-1}) \, a_{\hat{\mu}^{(\ell)}+\delta_n}(x_1,\dots,x_n) \, d_qx_1\cdots d_qx_n$$

$$Y_\ell \sim GF_q(Q_\ell) = \frac{1}{1-q^N} GF_q(Q_\ell^-) = \frac{1}{1-q^N}\cdot\frac{1}{(q;q)_k}\cdot\prod_{x\in\nu^{(\ell)}}\frac{1}{1-q^{h(x)}}$$

$Q_\ell =$



$Q_\ell^- =$



$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

The hook length formula is equivalent to the following identity.

$$\frac{\prod_{j=1}^{n-1}(1-q^{|\lambda|+|\mu|+\lambda_j+n^2+n-j+k+1})}{\prod_{i=1}^{n+1}(1-q^{|\lambda|+|\mu|-\mu_i+n^2-n+k+i})}=\sum_{\ell=1}^{n+1}\frac{q^{-|\lambda|-|\mu|+\mu_\ell-n^2+n-k-\ell}}{1-q^{|\lambda|+|\mu|-\mu_\ell+n^2-n+k+\ell}}\cdot\frac{\prod_{j=1}^{n-1}(1-q^{\mu_\ell+\lambda_j+2n-\ell-j+1})}{\prod_{j=1,j\neq\ell}^{n+1}(1-q^{\mu_\ell-\mu_j+j-\ell})}.$$

This can be obtained by the known partial fraction expansion.

$$\frac{\prod_{j=1}^{n+1}(1-b_j/t)}{\prod_{j=1}^{n}(1-a_j/t)}=\sum_{\ell=1}^{n}\frac{\prod_{j=1}^{n+1}(1-a_\ell/b_j)}{(1-a_\ell/t)\prod_{j=1,j\neq\ell}^{n}(1-a_\ell/a_j)},\qquad \text{for } b_1\cdots b_{n+1}=a_1\cdots a_n t.$$

The hook length formula for the remaining semi-irreducible d-complete poset can be proved similarly. In this case we need the following partial fraction expansion.

$$\prod_{i=1}^{n}\frac{1-tx_iy_i}{1-tx_i}=y_1\cdots y_n+\sum_{\ell=1}^{n}\frac{1-y_\ell}{1-tx_\ell}\prod_{i=1,i\neq\ell}^{n}\frac{1-x_iy_i/x_\ell}{1-x_i/x_\ell}.$$

THANK
YOU