Equivariant K-theory and hook formula for skew shape on d-complete set

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Idea of Proof

Equivariant K-theory

Kashiwara thick flag variety $\mathcal{G}/\mathcal{P}_-$

Equivariant K-theory $K_{\mathcal{T}}(\mathcal{G}/\mathcal{P}_-)$ and localization map

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Application to d-complete posets

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Main Theorem (N-Okada)

Theorem

P: a connected d-complete poset, $c : P \to I$ coloring.
F: an order filter of P,
$\mathcal{A}(P \setminus F) := \{\sigma : P \setminus F \to \mathbb{N} \mid p \le q \implies \sigma(p) \ge \sigma(q)\}$.
Then we have

$$\sum_{\sigma \in \mathcal{A}(P\setminus F)} \boldsymbol{z}^{\sigma} = \sum_{D \in \mathcal{E}_P(F)} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D}(1 - \boldsymbol{z}[H_P(v)])}, \tag{1}$$

where $\boldsymbol{z}^{\sigma} = \prod_{v \in P \setminus F} z_{c(v)}^{\sigma(v)}$,
$\boldsymbol{z}[H_P(v)]$ *is the hook monomial,*
$\mathcal{E}_P(F)$ *is the set of excited giagrams of F in P, and*
$B(D)$ *is the set of excited peaks of D in P.*

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Idea of Proof

Let $G_{P\setminus F}$ be the left hand side and $E_{P\setminus F}$ be the right hand side of the equation (1). Let $Z_{P\setminus F} := \left[\frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}\right]_{e^{\alpha_i}=z_i}$. For order filters, $F' \succ F$ means $F' \supset F$ and $F'\setminus F$ is non-empty antichain.

1) From Billey type formula we can see $Z_{P\setminus F} = E_{P\setminus F}$.

2) By Chevalley formula we have a recurrence relation on $Z_{P\setminus F}$.

$$Z_{P\setminus F} = \frac{1}{1-\boldsymbol{z}[P\setminus F]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} Z_{P\setminus F'}.$$

3) By inclusion-exclusion principle we can easily deduce the same recurrence relation on $G_{P\setminus F}$.

$$G_{P\setminus F} = \frac{1}{1-\boldsymbol{z}[P\setminus F]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} G_{P\setminus F'}.$$

Comparing initial values we get $\boxed{G_{P\setminus F} = Z_{P\setminus F}}$.

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Outline

Kashiwara thick flag variety $\mathcal{G}/\mathcal{P}_-$

$A = (a_{ij})_{i,j \in I}$ a symmetrizable generalized Cartan matrix,

Γ the corresponding Dynkin diagram with node set I.

Then the associated Kac–Moody group $\mathcal{G}$ over $\mathbb{C}$ is constructed from the following data:

the weight lattice $\mathbb{Z}$-module	$\Lambda \simeq \mathbb{Z}^m$,
the (linearly independent) simple roots	$\Pi = \{\alpha_i : i \in I\} \subset \Lambda$,
the simple coroots	$\Pi^\vee = \{\alpha_i^\vee : i \ni I\} \subset \Lambda^*$,
the fundamental weights	$\{\lambda_i : i \in I\} \subset \Lambda$

where $\Lambda^* = Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ is the dual lattice. We will write the canonical pairing $\langle \quad , \quad \rangle : \Lambda^* \times \Lambda \to \mathbb{Z}$. These satisfy

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}, \quad \langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}.$$

The Weyl group W is generated by simple reflections s_i ($i \in I$) and it is known to be a crystallographic Coxeter group.

Let $\mathcal{B}$ be a Borel subgroup and $\mathcal{B}_-$ the opposite Borel i.e. $\mathcal{B} \cap \mathcal{B}_- = \mathcal{T}$ is the maximal torus.

Let us fix a subset $J \subsetneq I$. Then we can consider the subgroup $W_J \subset W$ generated by $s_j, (j \in J)$ and the parabolic subgroup $\mathcal{P}_- \supset \mathcal{B}_-$ corresponding to J.

Let $W^J = W/W_J$ be the set of minimum length coset representatives.
Then we can consider

the Kashiwara thick partial flag variety $\mathcal{X}_{\mathcal{P}_-} = \mathcal{G}/\mathcal{P}_-$.

$\mathcal{X}_{\mathcal{P}_-} = \mathcal{G}/\mathcal{P}_-$ has cell decomposition.

$$\mathcal{X}_{\mathcal{P}_-} = \bigsqcup_{w \in W^J} \mathcal{X}_w^\circ$$

where $\mathcal{X}_w^\circ = \mathcal{B}w\mathcal{P}_-/\mathcal{P}_-$ is the $\mathcal{B}$-orbit of $\mathcal{T}$-fixed point corresponding to w, $e_w = w\mathcal{P}_-/\mathcal{P}_- \in \mathcal{X}_{\mathcal{P}_-}$.
The Zariski closure $\mathcal{X}_w = \overline{\mathcal{X}_w^\circ}$ of $\mathcal{X}_w^\circ$ is called the Schubert variety.
Using Bruhat order $\leq$ on W^J induced from W, we have cell decomposition

$$\mathcal{X}_w = \bigsqcup_{u \in W^J, u \geq w} \mathcal{X}_u^\circ.$$

$\mathcal{X}_w$ has codimension $\ell(w)$ in $\mathcal{X}_{\mathcal{P}_-}$ and it is infinite dimensional if $|W| = \infty$.

Outline

Equivariant K-theory $K_{\mathcal{T}}(\mathcal{G}/\mathcal{P}_-)$

We can consider $\mathcal{T}$-equivariant K-theory of $\mathcal{X}_{\mathcal{P}_-} = \mathcal{G}/\mathcal{P}_-$
$K_{\mathcal{T}}(\mathcal{G}/\mathcal{P}_-)$ is the Grothendieck group of coherent sheaves on $\mathcal{G}/\mathcal{P}_-$. It has ring structure and

$$K_{\mathcal{T}}(\mathcal{X}_{\mathcal{P}_-}) \cong \prod_{w \in W^J} K_{\mathcal{T}}(\mathrm{pt})[\mathcal{O}_w],$$

where $\mathcal{O}_w$ is the structure sheaf of the Schubert variety $\mathcal{X}_w$ and $[\mathcal{O}_w]$ is the corresponding class in $K_{\mathcal{T}}(\mathcal{X}_{\mathcal{P}_-})$.
$K_{\mathcal{T}}(\mathrm{pt})$ is the $\mathcal{T}$-equivariant K-theory of a point and it can be identified with the representation ring of $\mathcal{T}$.

$$K_{\mathcal{T}}(\mathrm{pt}) \simeq R(\mathcal{T}) \simeq \mathbb{Z}[\Lambda].$$

Each element in $\mathbb{Z}[\Lambda]$ is expressed as a ($\mathbb{Z}$-)linear combination of e^{λ} ($\lambda \in \Lambda$). e^{λ} corresponds to the class of line bundle L^{λ} with character λ.

Schubert class and Localization map

For $w \in W^J$, we will write $\xi^w = [\mathcal{O}_w] \in K_{\mathcal{T}}(\mathcal{X}_{\mathcal{P}_-})$ and call it $\mathcal{T}$-equivariant Schubert class.
Each $v \in W^J$ gives a $\mathcal{T}$-fixed point $e_v = v\mathcal{P}_-/\mathcal{P}_- \in \mathcal{X}$. Then the inclusion map $\iota_v : \{e_v\} \to \mathcal{X}$ induces the pull-back ring homomorphism, called the localization map at v,

$$\iota_v^* : K_{\mathcal{T}}(\mathcal{X}_{\mathcal{P}_-}) \to K_{\mathcal{T}}(e_v) \cong \mathbb{Z}[\Lambda].$$

For two elements $v, w \in W^J$, we denote by $\xi^w|_v$ the image of ξ^w under the localization map ι_v^*:

$$\xi^w|_v = \iota_v^*([\mathcal{O}_w]).$$

Billey type formula for $\xi^w|_v$

(Lam-Schilling-Shimozono 2010)

Proposition

Let $v, w \in W^J$, and fix a reduced expression $v = s_{i_1} s_{i_2} \dots s_{i_N}$ of v. Then we have

$$\xi^w|_v = \sum_{(k_1,\dots,k_r)} (-1)^{r-l(w)} \prod_{a=1}^{r} \left(1 - e^{\beta^{(k_a)}}\right), \tag{2}$$

where the summation is taken over all sequences $(k_1, \dots, k_r)$ such that $1 \le k_1 < k_2 < \cdots < k_r \le N$ and $s_{i_{k_1}} * \cdots * s_{i_{k_r}} = w$ (with respect to the Demazure product), and $\beta^{(k)}$ is given by $\beta^{(k)} = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$ for $1 \le k \le N$.

Outline

Littlewood-Richardson coefficients

We consider the structure constants for the multiplication in $K_{\mathcal{T}}(\mathcal{X})$ with respect to the Schubert classes.

$$[\mathcal{O}_u][\mathcal{O}_v] = \sum_{w \in W^J} c_{u,v}^w [\mathcal{O}_w],$$

where $u, v, w \in W^J$, $c_{u,v}^w \in K_{\mathcal{T}}(\mathrm{pt})$.

Lemma

If $c_{u,v}^w \neq 0$, then $u \leq w$ and $v \leq w$.

Lemma

For $u, w \in W^J$, we have

$$c_{u,w}^w = \xi^u|_w. \tag{3}$$

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Littlewood-Richardson coefficients

We consider the structure constants for the multiplication in $K_{\mathcal{T}}(\mathcal{X})$ with respect to the Schubert classes.

$$[\mathcal{O}_u][\mathcal{O}_v] = \sum_{w \in W^J} c^w_{u,v} [\mathcal{O}_w],$$

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For $u, w \in W^J$, we have

$$c^w_{u,w} = \xi^u|_w. \tag{3}$$

Lemma

Let $u, v, w \in W^J$ and $s \in W^J$ a simple reflection. If $c_{s,w}^{w} \neq c_{s,u}^{u}$, then we have

$$c_{u,v}^{w} = \frac{1}{c_{s,w}^{w} - c_{s,u}^{u}} \left(\sum_{u < x \leq w} c_{s,u}^{x} c_{x,v}^{w} - \sum_{u \leq y < w} c_{s,y}^{w} c_{u,v}^{y} \right).$$

In particular, we have

$$c_{u,w}^{w} = \frac{1}{c_{s,w}^{w} - c_{s,u}^{u}} \sum_{u < x \leq w} c_{s,u}^{x} c_{x,w}^{w}.$$

i.e.

$$\xi^{u}|_{w} = \frac{1}{c_{s,w}^{w} - c_{s,u}^{u}} \sum_{u < x \leq w} c_{s,u}^{x} \xi^{x}|_{w}. \qquad (4)$$

Proof.
Consider the associativity

$$([\mathcal{O}_s][\mathcal{O}_u])\,[\mathcal{O}_v] = [\mathcal{O}_s]\,([\mathcal{O}_u][\mathcal{O}_v])$$

and take the coefficients of $[\mathcal{O}_w]$ in the both hand sides.

$$c_{s,u}^{u} c_{u,v}^{w} + \sum_{u<x\leq w} c_{s,u}^{x} c_{x,v}^{w} = c_{s,w}^{w} c_{u,v}^{w} + \sum_{u\leq y<w} c_{s,y}^{w} c_{u,v}^{y},$$

then we get

$$c_{u,v}^{w} = \frac{1}{c_{s,w}^{w} - c_{s,u}^{u}} \left(\sum_{u<x\leq w} c_{s,u}^{x} c_{x,v}^{w} - \sum_{u\leq y<w} c_{s,y}^{w} c_{u,v}^{y} \right).$$

Outline

$\mathcal{T}$-quivariant Chevalley formula

W-orbit of the set of simple roots Π(resp. simple coroots $\Pi^\vee$) determine the set of real roots (resp. real coroots)

$$\Phi = W\Pi \text{ (resp. } \Phi^\vee = W\Pi^\vee)$$

and the decomposition of Φ (resp. $\Phi^\vee$) into the positive system Φ_+ (resp. $\Phi_+^\vee$) and the negative system Φ_- (resp. $\Phi_-^\vee$).
For a dominant weight $\lambda \in \Lambda$, we define

$$\mathbb{H}_\lambda = \{(\gamma^\vee, k) : \gamma^\vee \in \Phi_+^\vee, 0 \le k < \langle \gamma^\vee, \lambda \rangle, k \in \mathbb{N}\}.$$

Fix a total order on the Dynkin node I so that $I = \{i_1 < \cdots < i_r\}$.
define a map $\iota : \mathbb{H}_\lambda \to \mathbb{Q}^{r+1}$ by

$$\iota\left(\sum_{j=1}^{r} c_i \alpha_{i_j}^\vee, k\right) = \frac{1}{\langle \gamma^\vee, \lambda \rangle}(k, c_1, \cdots, c_r).$$

Then it is known that ι is injective.

We define a total ordering $<$ on $\mathbb{H}_\lambda$ by

$$h < h' \iff \iota(h) <_{\text{lex}} \iota(h'),$$

where $<_{\text{lex}}$ is the lexicographical ordering on $\mathbb{Q}^{r+1}$. For $h = (\gamma^\vee, k)$, we define affine transformations r_h and $\widetilde{r}_h$ on Λ by

$$\begin{aligned} r_h(\mu) &= \mu - \langle \gamma^\vee, \mu \rangle \gamma, \\ \widetilde{r}_h(\mu) &= r_h(\mu) + \left(\langle \gamma^\vee, \lambda \rangle - k \right) \gamma. \end{aligned}$$

Note that $r_h = s_\gamma$ is the reflection corresponding to the positive root γ.

Lenart-Shimozono (2014)

Proposition

Let $s = s_i \in W^J$ be a simple reflection. Then for $w, v \in W^J$ we have

$$c^w_{s,v} = \begin{cases} 1 - e^{\lambda_i - v\lambda_i} & \text{if } w = v, \\ \displaystyle\sum_{(h_1,\cdots,h_r)} (-1)^{r-1} e^{\lambda_i - v\widetilde{r}_{h_1}\cdots\widetilde{r}_{h_r}\lambda_i} & \text{if } w > v, \end{cases} \tag{5}$$

where the summation is taken over all sequences $(h_1, \cdots, h_r)$ of length $r \geq 1$ satisfying the following two conditions:

(H1) $h_1 > h_2 > \cdots > h_r$ *in* $\mathbb{H}_{\lambda_i}$,

(H2) $v \lessdot vr_{h_1} \lessdot vr_{h_1}r_{h_2} \lessdot \cdots \lessdot vr_{h_1}\cdots r_{h_r} = w$ *is a saturated chain in* W^J.

Outline

Application to d-complete posets

In what follows, let P be a connected d-complete poset with top tree Γ together with d-complete coloring $c : P \to I$.
α_P the simple root, λ_P the fundamental weight corresponding to the color i_P of the maximum element of P.

We apply the above argument to the Kashiwara thick partial flag variety $\mathcal{X}_{\mathcal{P}_-} = \mathcal{G}/\mathcal{P}_-$, where $\mathcal{P}_-$ is the maximal parabolic subgroup corresponding to $J = I \setminus \{i_P\}$. In this case, the parabolic subgroup W_J coincides with the stabilizer of λ_P in W, and the minimum length coset representatives W^J is denoted by W^{λ_P}.

For $p \in P$, we put

$$\alpha(p) = \alpha_{c(p)}, \quad \alpha^\vee(p) = \alpha^\vee_{c(p)}, \quad s(p) = s_{c(p)}.$$

Take a linear extension and label the elements of P as $p_1, \cdots, p_N$ ($N = \#P$) so that $p_i < p_j$ in P implies $i < j$. Then we constructs an element $w = w_P \in W$ by putting

$$w_P = s(p_1)s(p_2)\cdots s(p_N).$$

For an order filter $F = \{p_{i_1}, \cdots, p_{i_r}\}$ ($i_1 < \cdots < i_r$), we define

$$w_F = s(p_{i_1})\cdots s(p_{i_r}).$$

$$F \subset F' \iff w_F \leq w_{F'} \text{ in Bruhat order}$$

If $p = p_k \in P$, then we define

$$\beta(p_k) = s(p_1)\cdots s(p_{k-1})\alpha(p_k),$$

Proposition

(Proctor 2014)

$$\boldsymbol{z}[H_P(p)] = [e^{\beta(p)}]_{e^{\alpha_i}=z_i}.$$

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Given a subset $D = \{p_{i_1}, \cdots, p_{i_r}\}$ ($i_1 < \cdots < i_r$) of P, we define elements $w_D \in W$ and $w_D^* \in W$ by putting

$$w_D = s(p_{i_1})s(p_{i_2})\cdots s(p_{i_r}) \text{ and } w_D^* = s(p_{i_1}) * s(p_{i_2}) * \cdots * s(p_{i_r}),$$

where $*$ is the Demazure product.

Proposition

Let F be an order filter of P and $D \subset P$.
(1) $D \in \mathcal{E}_P(F) \iff w_D = w_F$ and $|E| = |F|$
(2) $D \in \mathcal{E}_P^*(F) \iff w_D^* = w_F$
For the case (2) D is uniquely expressed as $D = D' \sqcup S$ s.t. $D' \in \mathcal{E}_P(F), S \subset B(D')$.

Outline

Billey type formula

Proposition

We have

$$\xi^{w_F}|_{w_P} = \sum_{E \in \mathcal{E}_P^*(F)} (-1)^{\#E - \#F} \prod_{p \in E} (1 - \mathbf{z}[H_P(p)]) , \qquad (6)$$

under the identification $z_i = e^{\alpha_i}$ $(i \in I)$. We can rewrite the above expression as

$$\xi^{w_F}|_{w_P} = \sum_{D \in \mathcal{E}_P(F)} \prod_{p \in D} (1 - \mathbf{z}[H_P(p)]) \prod_{p \in B(D)} \mathbf{z}[H_P(p)].$$

$$\boxed{\frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}} = \sum_{D \in \mathcal{E}_P(F)} \frac{\prod_{v \in B(D)} \mathbf{z}[H_P(v)]}{\prod_{v \in P \setminus D} (1 - \mathbf{z}[H_P(v)])}}$$

Outline

Chevalley formula for d-complete poset

Proposition

Let P be a connected d-complete poset and $s = s_{i_P}$. For two order filters F and F' of P, we have

$$c_{s,w_F}^{w_{F'}} = \begin{cases} 1 - \boldsymbol{z}[F] & \text{if } F' = F, \\ (-1)^{\#(F' \setminus F) - 1} \boldsymbol{z}[F] & \text{if } F' \supsetneq F \text{ and } F' \setminus F \text{ is an antichain}, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

under the identification $z_i = e^{\alpha_i}$ ($i \in I$).

Lemma

Let F be an order filter of P and $h = (\gamma^\vee, k) \in \mathbb{H}_{\lambda_P}$.

If $w_F r_h \in W^{\lambda_P}$ and $w_F \lessdot w_F r_h \leq w_P$, then there exists $p \in P$ such that $F \sqcup \{p\}$ is an order filter of P, $w_F r_h = w_{F'}$ and $\gamma^\vee = \gamma^\vee(p)$.
In this case $k = 0$, because $\langle \gamma^\vee, \lambda_P \rangle = 1$, and $\widetilde{r}_h \lambda_P = \lambda_P$.

To deduce recurrence relation on $Z_{P\setminus F} := \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}$, recall

$$\xi^{u}|_{w} = \frac{1}{c_{s,w}^{w} - c_{s,u}^{u}} \sum_{u<x\leq w} c_{s,u}^{x} \xi^{x}|_{w}. \tag{8}$$

Change variables $u = w_F, w = w_P$, $x = w_{F'}$. Then we have

$$\xi^{w_F}|_{w_P} = \frac{\mathbf{z}[F]}{\mathbf{z}[F] - \mathbf{z}[P]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} \xi^{w_{F'}}|_{w_P}. \tag{9}$$

Note that $c_{s,w_F}^{w_F} = 1 - \mathbf{z}[F]$.

$$\boxed{Z_{P\setminus F} = \frac{1}{1 - \mathbf{z}[P\setminus F]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} Z_{P\setminus F'}}$$

To deduce recurrence relation on $Z_{P\setminus F} := \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}$, recall

$$\xi^u|_w = \frac{1}{c^w_{s,w} - c^u_{s,u}} \sum_{u<x\leq w} c^x_{s,u}\xi^x|_w. \tag{8}$$

Change variables $u = w_F, w = w_P$, $x = w_{F'}$. Then we have

$$\xi^{w_F}|_{w_P} = \frac{\boldsymbol{z}[F]}{\boldsymbol{z}[F] - \boldsymbol{z}[P]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} \xi^{w_{F'}}|_{w_P}. \tag{9}$$

Note that $c^{w_F}_{s,w_F} = 1 - \boldsymbol{z}[F]$.

$$\boxed{Z_{P\setminus F} = \frac{1}{1 - \boldsymbol{z}[P\setminus F]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} Z_{P\setminus F'}}$$

To deduce recurrence relation on $Z_{P\setminus F} := \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}$, recall

$$\xi^u|_w = \frac{1}{c^w_{s,w} - c^u_{s,u}} \sum_{u<x\leq w} c^x_{s,u} \xi^x|_w. \qquad (8)$$

Change variables $u = w_F, w = w_P$, $x = w_{F'}$. Then we have

$$\xi^{w_F}|_{w_P} = \frac{\boldsymbol{z}[F]}{\boldsymbol{z}[F] - \boldsymbol{z}[P]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} \xi^{w_{F'}}|_{w_P}. \qquad (9)$$

Note that $c^{w_F}_{s,w_F} = 1 - \boldsymbol{z}[F]$.

$$\boxed{Z_{P\setminus F} = \frac{1}{1 - \boldsymbol{z}[P\setminus F]} \sum_{F' \succ F} (-1)^{|F'\setminus F|-1} Z_{P\setminus F'}}$$

Remark

Stembridge (2001) classified dominant λ-minuscule element of Weyl groups and found another two families other than 15-classes of Proctor's. , which are the cases of non-simply laced Dynkin diagrams. All our arguments can be applied to these cases.
Replace colored d-complete posets by a heap $H(w)$ of a dominant λ-minuscule element w.
Definitions of $\mathcal{E}_P(F)$ and $B(D)$ should be slightly modified.
Then the same ED-hook type formula of the theorem holds.

Outline

Smallest Bird

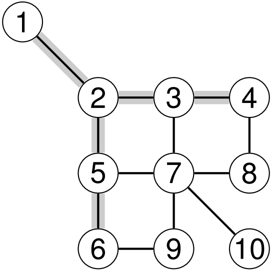
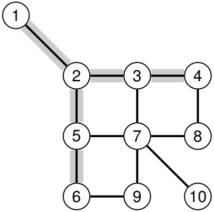
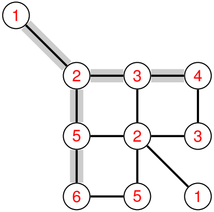


Figure: $e_6[1;2,2]$

Example





$$
\begin{aligned}
z[1] &= z_1 z_2^2 z_3^2 z_4 z_5^2 z_6\,,\\
z[2] &= z_1 z_2 z_3 z_4 z_5 z_6\,,\\
z[3] &= z_1 z_2 z_3 z_4 z_5\,,\\
z[4] &= z_3 z_4\,,\\
z[5] &= z_1 z_2 z_3 z_5 z_6\,,\\
z[6] &= z_5 z_6\,,\\
z[7] &= z_1 z_2 z_3 z_5\,,\\
z[8] &= z_3\,,\\
z[9] &= z_5\,,\\
z[10] &= z_1\,.\\
\Delta[P] &= \prod_{i=1}^{10}(1 - z[i])
\end{aligned}
$$

There are 29 order filters.

$F_7 = \{1, 2, 3, 5\}$

$\mathcal{E}_P(F_7) = \{\{1, 2, 3, 5\}, \{1, 2, 5, 8\}, \{1, 2, 3, 9\}, \{1, 2, 8, 9\}, \{1, 7, 8, 9\}\}$,

$$
\begin{aligned}
G_{P/F_7} &= \frac{(1-z[1])(1-z[2])(1-z[3])(1-z[5])}{\Delta(P)} + \frac{(1-z[1])(1-z[2])(1-z[5])(1-z[8])z[3]}{\Delta(P)} + \\
&\quad \frac{(1-z[1])(1-z[2])(1-z[3])(1-z[9])z[5]}{\Delta(P)} + \frac{(1-z[1])(1-z[2])(1-z[8])(1-z[9])z[3]z[5]}{\Delta(P)} + \\
&\quad \frac{(1-z[1])(1-z[7])(1-z[8])(1-z[9])z[2]}{\Delta(P)}\\
&= \frac{(1-z[1])F(z)}{\Delta(P)}
\end{aligned}
$$

where $F(z) = 1 - z_1 z_2 z_3^2 z_4 z_5 - z_1 z_2 z_3^2 z_4 z_5 z_6 - z_1 z_2 z_3 z_5^2 z_6 - z_1 z_2 z_3 z_4 z_5^2 z_6 + z_1 z_2 z_3^2 z_4 z_5^2 z_6 - z_1^2 z_2^2 z_3^2 z_4 z_5^2 z_6 + z_1^2 z_2^2 z_3^3 z_4 z_5^2 z_6 + z_1^2 z_2^2 z_3^3 z_4^2 z_5^2 z_6 + z_1^2 z_2^2 z_3^2 z_4 z_5^3 z_6 + z_1^2 z_2^2 z_3^2 z_4 z_5^3 z_6^2 - z_1^3 z_2^3 z_3^4 z_4^2 z_5^4 z_6^2$.

B_3 example

B_3 Dynkin diagram $s_0 = s_1 - s_2$

$w = s_0(s_1 s_0)(s_2 s_1 s_0)$ λ_0-minuscule

s_0	s_1	s_2
	s_0	s_1
		s_0

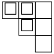
β_1	β_2	β_3
	β_4	β_5
		β_6

$\gamma_1^\vee$	$\gamma_2^\vee$	$\gamma_3^\vee$
	$\gamma_4^\vee$	$\gamma_5^\vee$
		$\gamma_6^\vee$

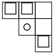
$\beta_1 = \alpha_0 + \alpha_1 + \alpha_2$, $\beta_2 = 2\alpha_0 + 2\alpha_1 + \alpha_2$, $\beta_3 = 2\alpha_0 + \alpha_1 + \alpha_2$
$\beta_4 = \alpha_0 + \alpha_1$, $\beta_5 = 2\alpha_0 + \alpha_1$, $\beta_6 = \alpha_0$,

$\gamma_1^\vee = \alpha_0^\vee$, $\gamma_2^\vee = \alpha_0^\vee + \alpha_1^\vee$, $\gamma_3^\vee = \alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee$,
$\gamma_4^\vee = \alpha_0^\vee + 2\alpha_1^\vee$, $\gamma_5^\vee = \alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee$, $\gamma_6^\vee = \alpha_0^\vee + 2\alpha_1^\vee + 2\alpha_2^\vee$

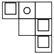
$$\lambda = 321, \mu = 21$$



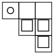
$$W_1 = \frac{1}{(1-z_0)(1-z_0^2z_1)(1-z_0^2z_1z_2)}$$



$$W_2 = \frac{z_0z_1}{(1-z_0z_1)(1-z_0^2z_1)(1-z_0^2z_1z_2)}$$



$$W_3 = \frac{z_0^2z_1^2z_2}{(1-z_0z_1)(1-z_0^2z_1z_2)(1-z_0^2z_1^2z_2)}$$



$$W_4 = \frac{z_0z_1z_2}{(1-z_0z_1z_2)(1-z_0^2z_1z_2)(1-z_0^2z_1^2z_2)}$$

ED-sum

$$W_1 + W_2 + W_3 + W_4 = \frac{1}{(1-z_0)(1-z_0z_1)(1-z_0z_1z_2)}$$

Thank you!