Nice formulas for plane partitions from an integrable system

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Contents

S. Kamioka, *Multiplicative partition functions for reverse plane partitions derived from an integrable dynamical system*, FPSAC (London, 2017), Article #29, 12 pp.

Nice formulas for plane partitions

Plane partitions

A plane partition of $a \times b$ rectangular shape is a 2D array $\pi = (\pi_{i,j})_{\substack{1 \le i \le a \\ 1 \le j \le b}}$ such that

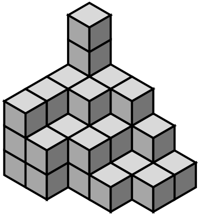
- entries $\pi_{i,j}$ are nonnegative integers;
- each of rows and columns are weakly decreasing.

$$\pi_{i,j} \ge \pi_{i+1,j}, \qquad \pi_{i,j} \ge \pi_{i,j+1}.$$

Example: A plane partition of 4×5 rectangular shape:

5	3	3	2	1
3	3	2	1	1
3	2	2	1	0
3	2	0	0	0

$=$



Reverse plane partitions

Let λ be a Young diagram.

A reverse plane partition of shape λ is a 2D array $\pi = (\pi_{i,j})_{(i,j)\in\lambda}$ such that

- entries $\pi_{i,j}$ are nonnegative integers;
- each of rows and columns are weakly increasing:

$$\pi_{i,j} \le \pi_{i+1,j}, \qquad \pi_{i,j} \le \pi_{i,j+1}.$$

Example: A reverse plane partition of shape $\lambda = (5,4,4,2)$:

0	0	1	2	4
0	1	2	3	
2	2	4	4	
3	4			

MacMahon's formula

Let $\mathrm{PP}(a,b)$ denote the set of plane partitions of $a \times b$ rectangular shape.

Theorem (MacMahon)

$$\sum_{\substack{\pi \in \mathrm{PP}(a,b) \\ \pi_{i,j} \leq c}} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

where $|\pi| := \sum_{i,j} \pi_{i,j}$.

P. A. MacMahon, Combinatory Analysis, Volumes 1–2, Cambridge, 1915–1916.

Trace generating function

Theorem (Stanley)

$$\sum_{\pi \in \mathrm{PP}(a,b)} y^{\mathsf{tr}(\pi)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1}{1 - y q^{i+j-1}}$$

where $\mathsf{tr}(\pi) := \sum_i \pi_{i,i}$, trace, and $|\pi| := \sum_{i,j} \pi_{i,j}$.

R. P. Stanley, *Theory and application of plane partitions, I–II*, Studies in Appl. Math. **50** (1971), 167–188, 259–279.

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Let $\mathrm{RPP}(\lambda)$ denote the set of reverse plane partitions of shape λ.

Theorem (Gansner)

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1-\prod_{(k,\ell)\in H_\lambda(i,j)} y_{\ell-k}}$$

where $H_\lambda(i,j)$ denotes the hook of cell $(i,j)\in\lambda$.

E. R. Gansner, *The Hillman–Grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory Ser. A **30** (1981), 71–89.

0	0	1	2	4
0	1	2	3	
2	2	4	4	
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$$\mapsto \quad \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = y_{-3}^{3} y_{-2}^{6} y_{-1}^{2} y_{0}^{5} y_{1}^{6} y_{2}^{4} y_{3}^{2} y_{4}^{4} y_{-1}^{2} y_{-2}^{6} y_{-3}^{3}$$

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$$\begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 4 \\ \hline 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 2 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} \boxed{3} & 4 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \quad \mapsto \quad \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = y_{-3}^{3} y_{-2}^{6} y_{-1}^{2} y_0^5 y_1^6 y_2^4 y_3^2 y_4^4 y_{-1}^{2} y_{-2}^{6} y_{-3}^{3}$$

How to prove/derive?

A method to prove/derive nice formulas for (reverse) plane partitions based on:

1. Non-intersecting lattice paths ($\Leftrightarrow$ determinants of Lindström–Gessel–Viennot type);
2. The discrete 2D Toda lattice:

$$q_n^{(s,t)} + e_{n+1}^{(s,t)} = q_n^{(s,t+1)} + e_n^{(s+1,t)}, \qquad q_{n+1}^{(s,t)} e_{n+1}^{(s,t)} = q_{n+1}^{(s,t+1)} e_n^{(s+1,t)}.$$

Remark: Viennot takes a similar approach to count non-intersecting Dyck paths by the quotient-difference (QD) formula (aka. discrete (1D) Toda lattice):

$$q_n^{(t)} + e_{n+1}^{(t)} = q_n^{(t+1)} + e_n^{(t+1)}, \qquad q_{n+1}^{(t)} e_{n+1}^{(t)} = q_{n+1}^{(t+1)} e_n^{(t+1)}.$$

X. G. Viennot, *A combinatorial interpretation of the quotient-difference algorithm*, FPSAC (Moscow, 2000), pp. 379–390.

Example 1: Yet another proof of MacMahon's formula

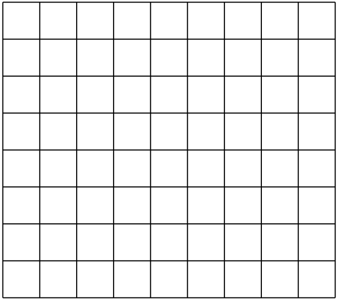
Non-intersecting lattice paths

A one-to-one correspondence between:

- Plane partitions $\pi \in \mathrm{PP}(a,b)$ with $\pi_{i,j} \le c$;
- Non-intersecting configurations $(P_1, \dots, P_c)$ of c lattice paths such that P_k goes from $S_k = (a+c-k, 0)$ to $T_k = (0, b+c-k)$.

5	3	3	2	1
3	3	2	1	1
3	2	2	1	0
3	2	0	0	0

$\overset{1:1}{\longleftrightarrow}$



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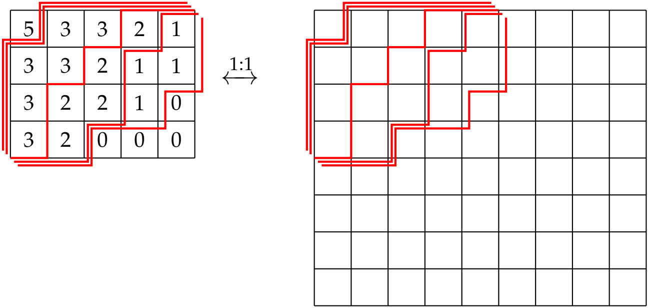

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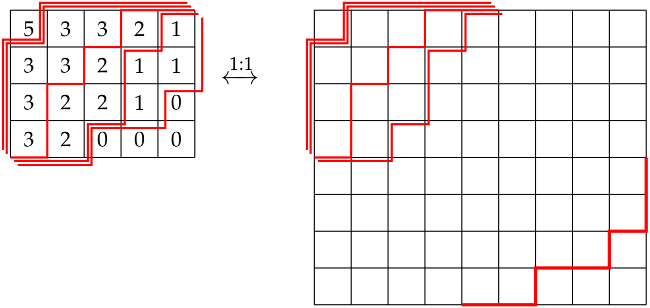

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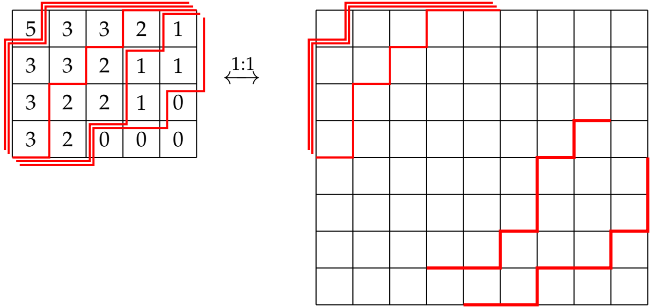

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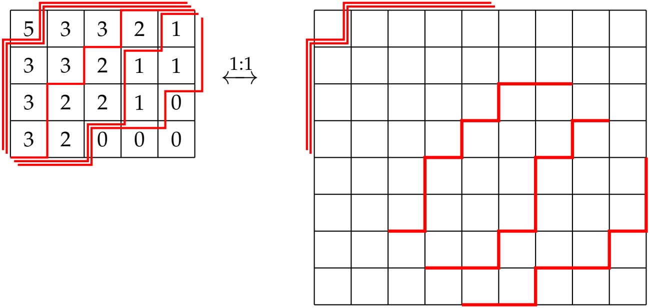

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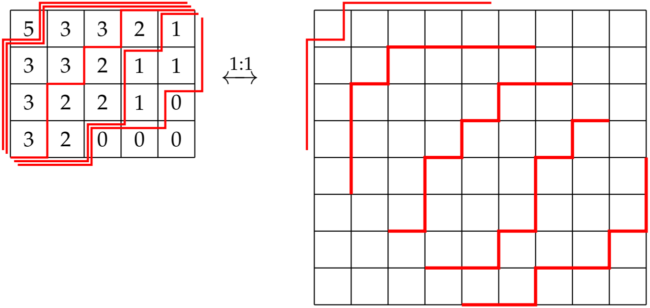

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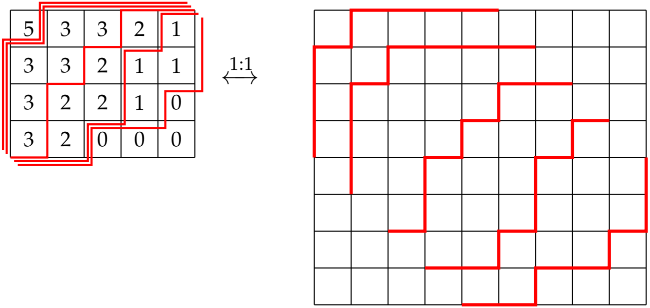

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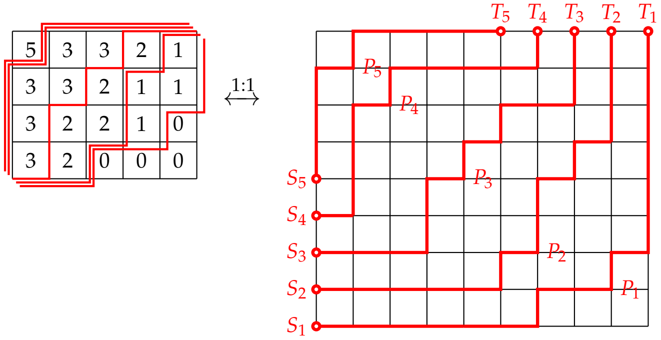

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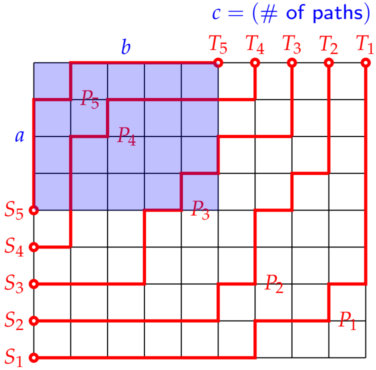


Sketch of yet another proof

MacMahon's formula:

$$\sum_{\substack{\pi \in \mathrm{PP}(a,b) \\ \pi_{i,j} \le c}} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

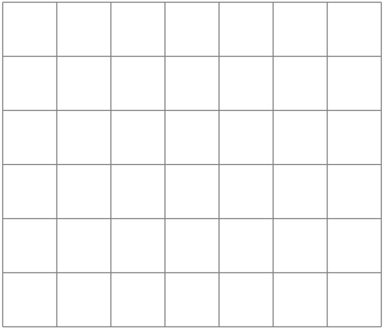
In view of the one-to-one correspondence with non-intersecting lattice paths:

Yet another proof of MacMahon's formula (sketch)

1. Construct a determinant of Lindström–Gessel–Viennot type which is equal to $\sum_{\pi} q^{|\pi|}$ (up to constant factor);
2. Evaluate the determinant to obtain the nice formula by:
   - Krattenthaler's determinants;
   - Jacobi's determinant identity ($\Leftrightarrow$ Dodgson condensation);
   - The method of corner deletion ($\Leftrightarrow$ the discrete 2D Toda lattice).

Weight for lattice paths

Edge-labels of the lattice:



For a lattice path P,

$$
\begin{aligned}
w(P) &:= \prod(\text{labels of the edges passed by } P) \\
&= q^{\mathsf{area}(P)}
\end{aligned}
$$

where area(P) denotes the area bordered by P and the lattice boundary.

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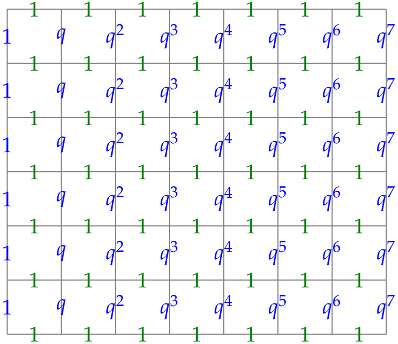

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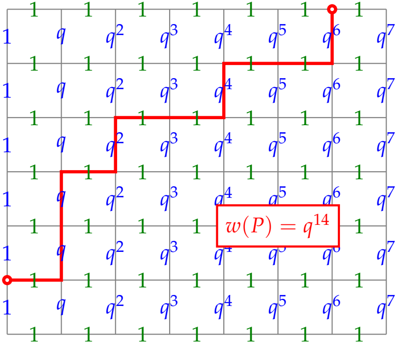

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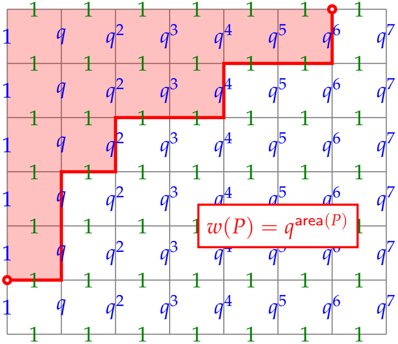

For a lattice path P,

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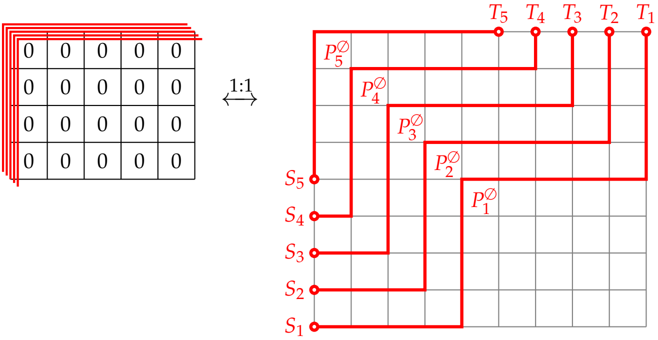
Let $(P_1^{\varnothing}, \dots, P_c^{\varnothing})$ denote the non-intersecting configuration of lattice paths that corresponds to the empty plane partition.



Lemma

If $\pi \overset{1:1}{\longleftrightarrow} (P_1, \dots, P_c)$ in the one-to-one correspondence then

$$q^{|\pi|} = \prod_{k=1}^{c} \frac{w(P_k)}{w(P_k^{\varnothing})}.$$

Proof: Suppose that $\pi \overset{1:1}{\longleftrightarrow} (P_1, \ldots, P_c)$ in the one-to-one correspondence. Then, the following are equivalent to each other:

1. Increase some entry $\pi_{i,j}$ of π by one;
2. Increase the area of some lattice path P_k in $(P_1, \ldots, P_c)$ by one.

Hence

$$
\begin{aligned}
|\pi| &= \sum_{i,j} \pi_{i,j} \\
&= \sum_{k=1}^{c} \mathsf{area}(P_k) - \sum_{k=1}^{c} \mathsf{area}(P_k^{\varnothing})
\end{aligned}
$$

and

$$
q^{|\pi|} = \prod_{k=1}^{c} \frac{q^{\mathsf{area}(P_k)}}{q^{\mathsf{area}(P_k^{\varnothing})}} = \prod_{k=1}^{c} \frac{w(P_k)}{w(P_k^{\varnothing})}.
$$

Let $\mathrm{NILP}(a,b,c)$ denote the set of non-interseting configurations $(P_1,\ldots,P_c)$ of lattice paths such that P_k goes from $S_k = (a+c-k, 0)$ to $T_k = (0, b+c-k)$.

From the one-to-one correspondence between $\{\pi \in \mathrm{PP}(a,b);\ \pi_{i,j} \le c\}$ and $\mathrm{NILP}(a,b,c)$:

Proposition

$$\sum_{\substack{\pi\in\mathrm{PP}(a,b)\\ \pi_{i,j}\le c}} q^{|\pi|} = \sum_{\substack{(P_1,\ldots,P_c)\\ \in\mathrm{NILP}(a,b,c)}} \prod_{k=1}^{c} w(P_k) \Big/ \prod_{k=1}^{c} w(P_k^{\emptyset})$$

where $w(P) = q^{\mathsf{area}(P)}$.

Lindström–Gessel–Viennot's lemma

Assumption:

- G: finite directed acyclic graph with edges labelled;
- $S_1, \ldots, S_n \in V(G)$;
- $T_1, \ldots, T_n \in V(G)$;
- Every path $P : S_i – T_j$ intersects with every path $P' : S_{i'} – T_{j'}$ if $i < i'$ and $j > j'$;

Lemma (Lindström–Gessel–Viennot)

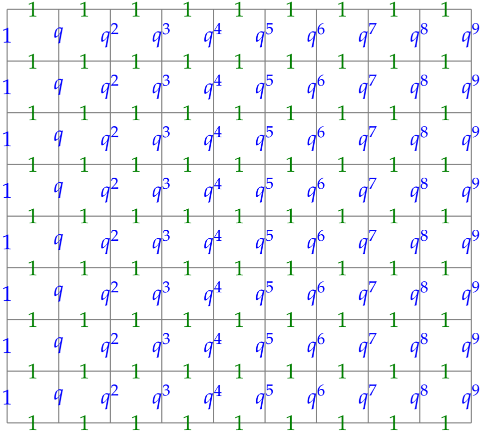
$$\sum_{(P_1,\ldots,P_n)} \prod_{k=1}^{n} w(P_k) = \det_{1 \le i,j \le n} (g_{i,j}) \qquad \text{with} \qquad g_{i,j} = \sum_{P : S_i – T_j} w(P)$$

where the first sum is over all the configurations $(P_1, \ldots, P_n)$ of paths on G such that

- P_k goes from S_k to T_k;
- $P_1, \ldots, P_n$ are non-intersecting.

Determinant of LGV type

$$\sum_{\substack{(P_1,\dots,P_c)\\ \in \mathrm{NILP}(a,b,c)}} \prod_{k=1}^{c} w(P_k) = \det_{1\le i,j\le c}(g_{i,j}) \qquad \text{with} \qquad g_{i,j} = \sum_{P:S_i - T_j} w(P)$$



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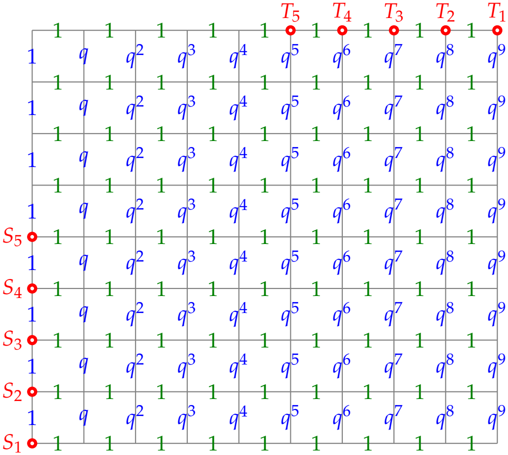

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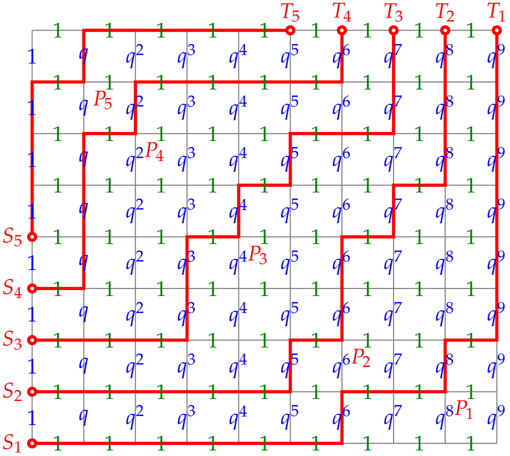


Remark: In the present case the determinant of LGV type is a q-binomial determinant:

$$\det_{1\le i,j\le c}(g_{i,j}) = \det_{1\le i,j\le c}\left(\begin{bmatrix} a+b+i+j-2 \\ a+i-1 \end{bmatrix}_q\right).$$

The q-binomial determinant can be directly evaluated by Krattenthaler's formula

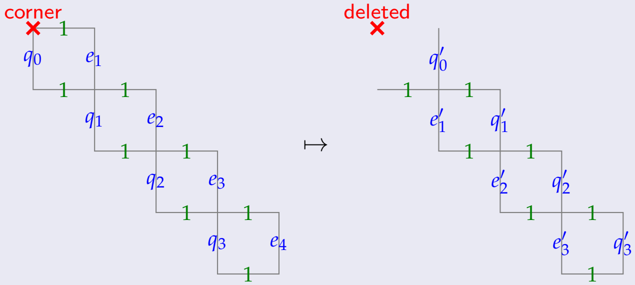
$$\det_{1\le i,j\le n}\left(\prod_{k=2}^{j}(x_i+b_k)\prod_{k=j+1}^{n}(x_i+a_k)\right) = \prod_{1\le i<j\le n}(x_i-x_j)\prod_{2\le i<j\le n}(b_i-a_j).$$

C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q.

Corner deletion

Reduce the lattice graph by:

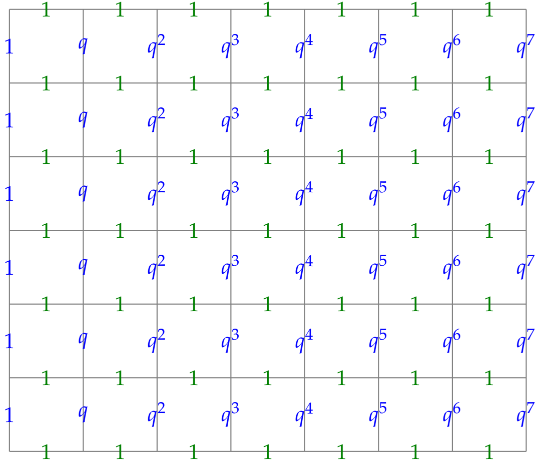
1. Delete an (upper left) corner;
2. Modify the edge-labels as:



where the edge-labels q_n, e_n and q'_n, e'_n satisfy

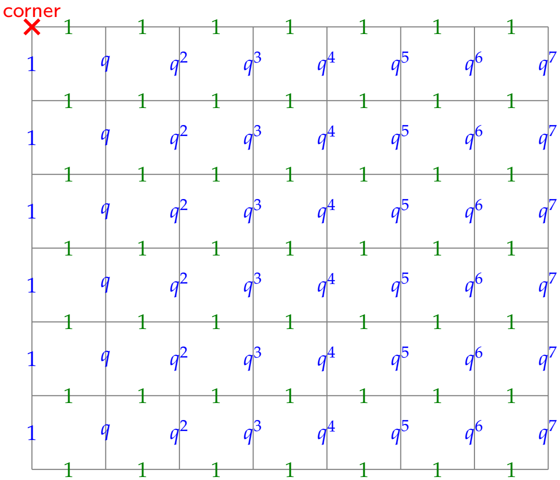
$$q_n + e_{n+1} = q'_n + e'_n, \qquad q_{n+1}e_{n+1} = q'_n e'_{n+1} \qquad \text{for } n \geq 0 \quad \text{with } e'_0 = 0.$$

In the present case:



- The corner deletion is succesively applicable (until no corners remain).
- The edge-labels on the last graph are independent of the order of corners deleted.

In the present case:



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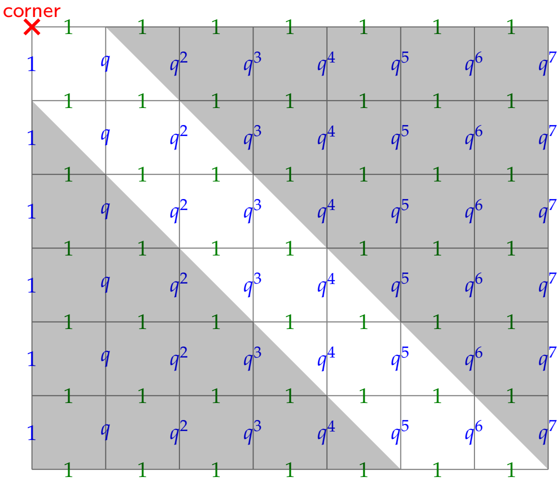

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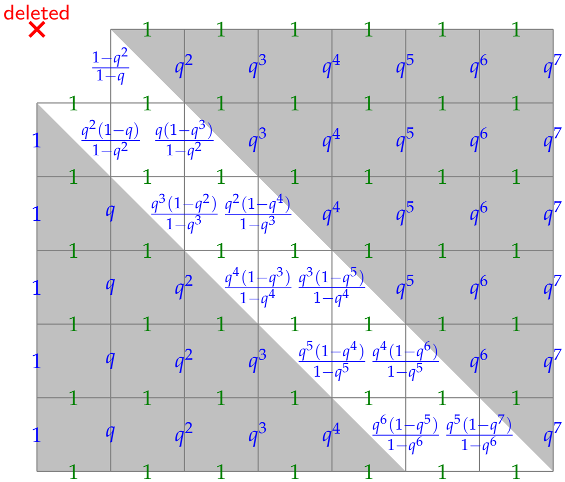

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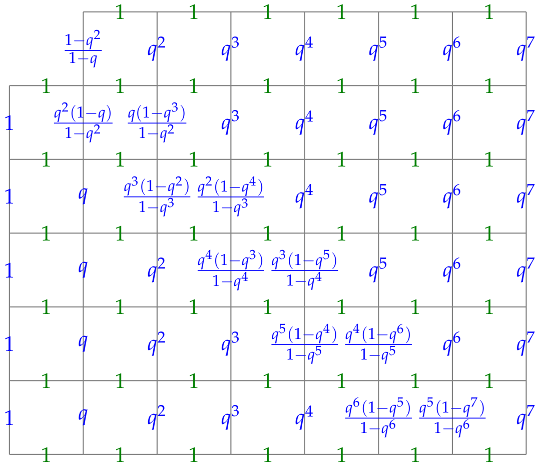

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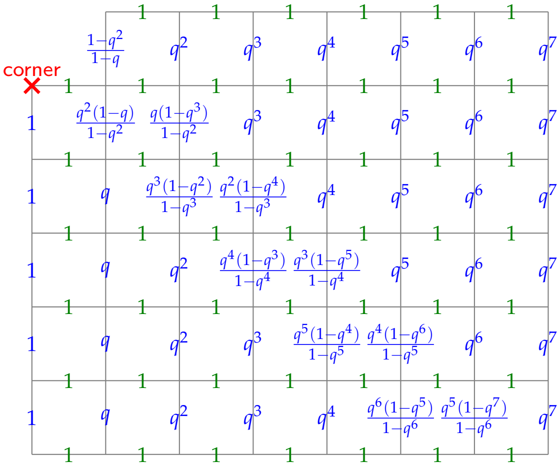

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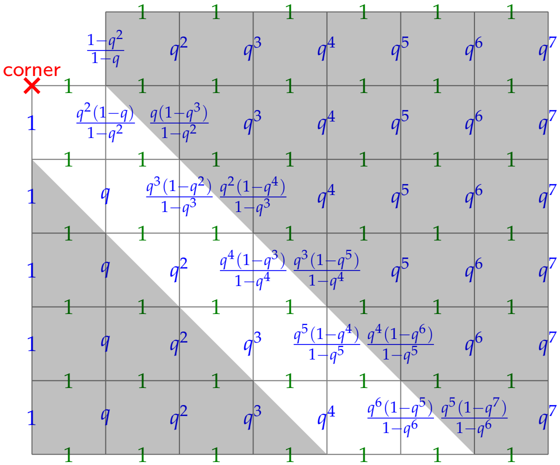

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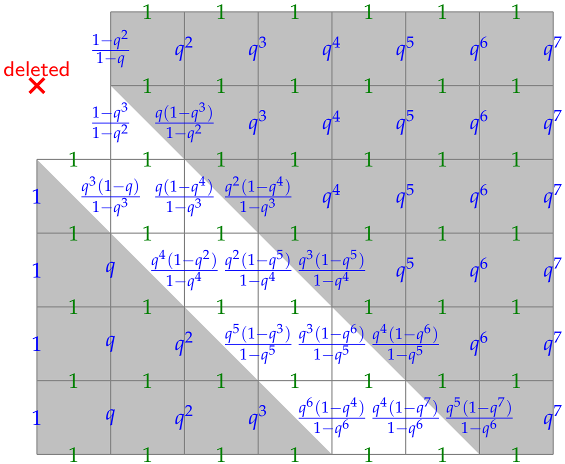

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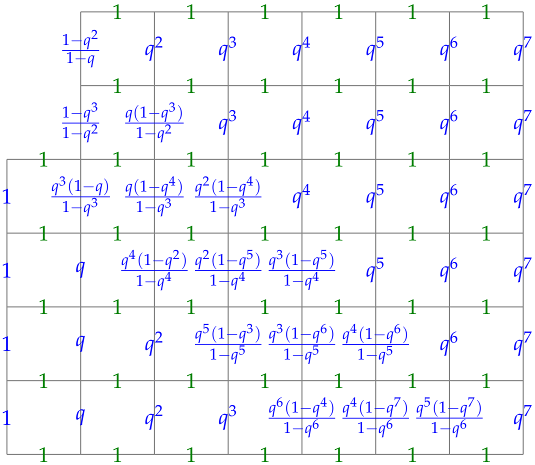

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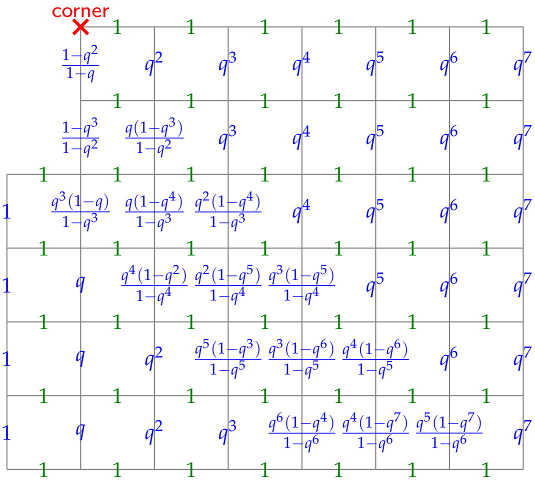

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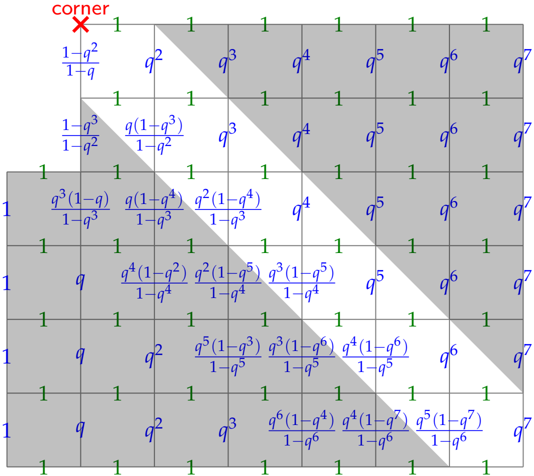

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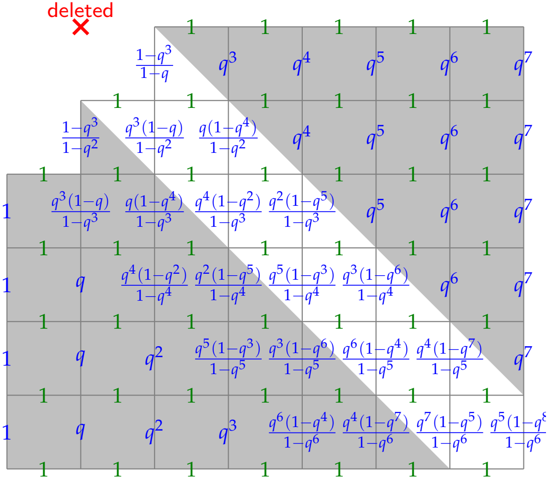

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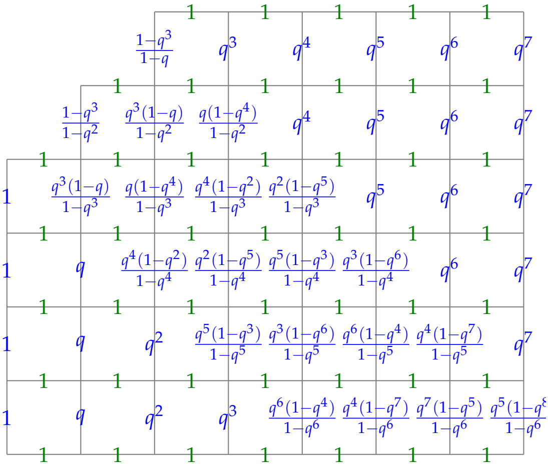

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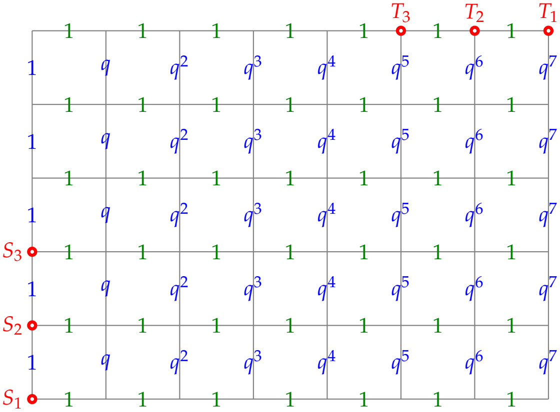
Theorem

The corner deletion unchanges the value of

$$g_{i,j} = \sum_{P:S_i \to T_j} w(P)$$

if neither S_i nor T_j is the deleted corner.

1. Perform corner deletion succesively until the (upper left) $a \times b$ rectangle is vacant.
2. $\det(g_{i,j})$ of LGV type on the last graph is (entrywise) equal to that on the original graph.
3. Evaluation of $\det(g_{i,j})$ is EASY on the last graph because the non-intersecting configuration $(P_1^*, \dots, P_c^*)$ on the last graph is unique!



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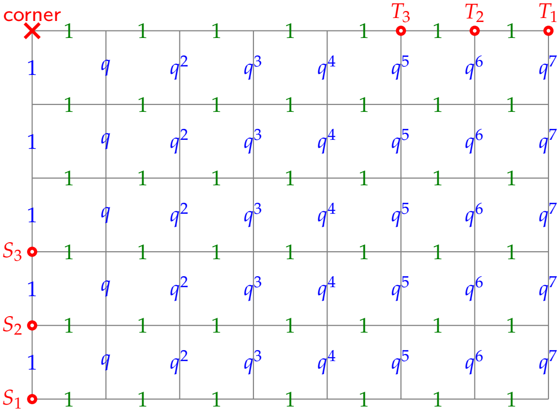

Determinant evaluation by corner deletion

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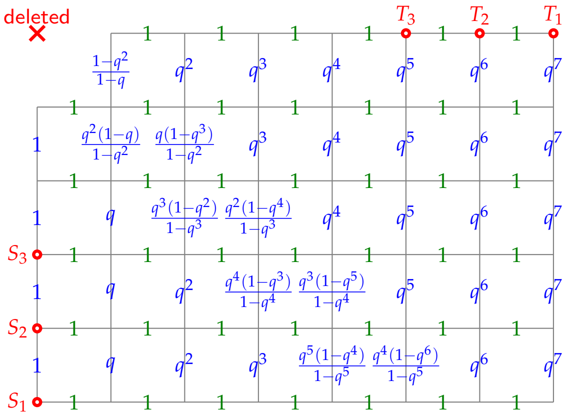

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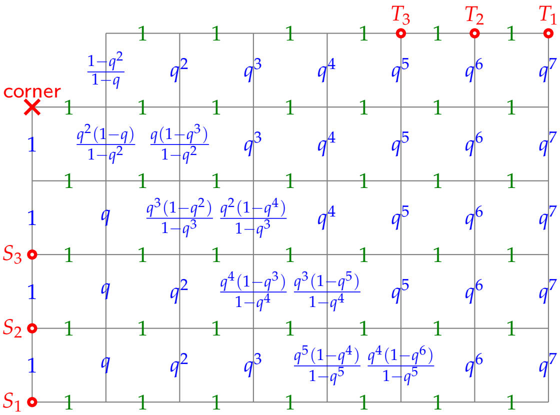

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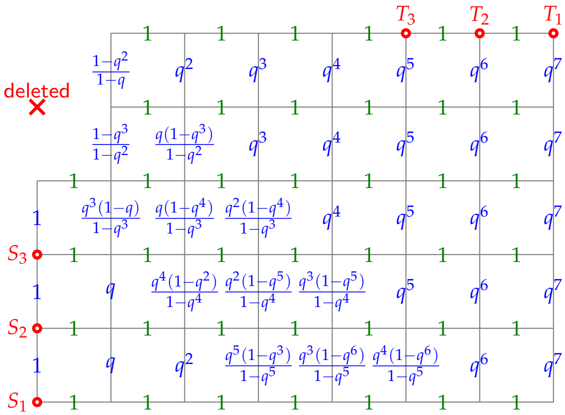

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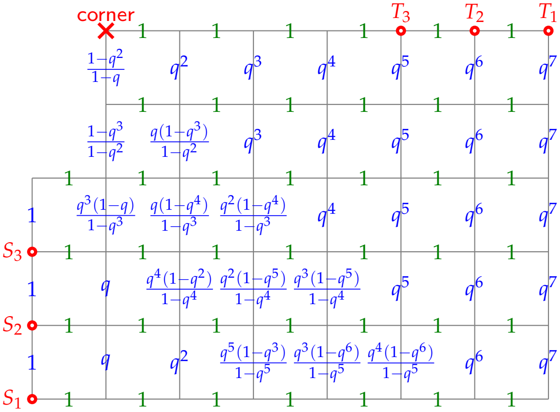

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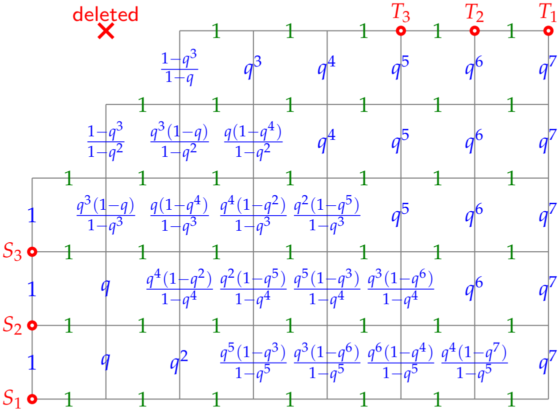

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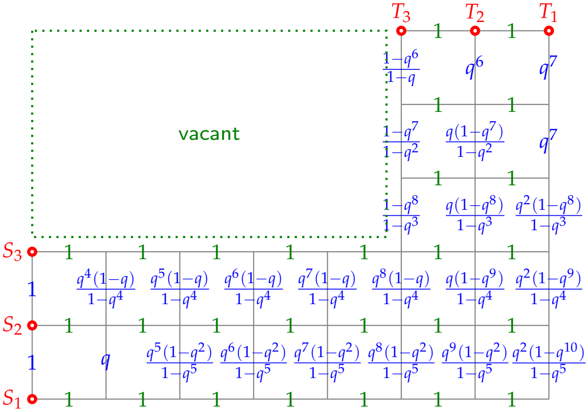

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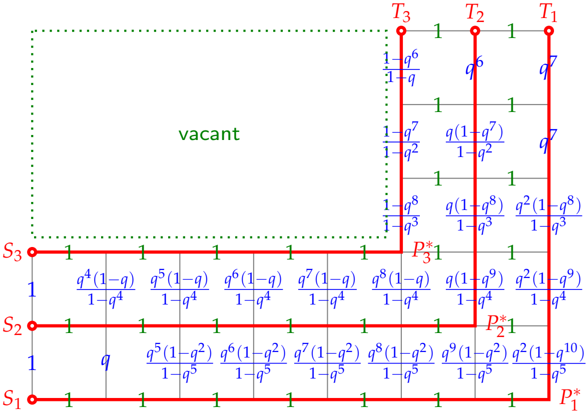
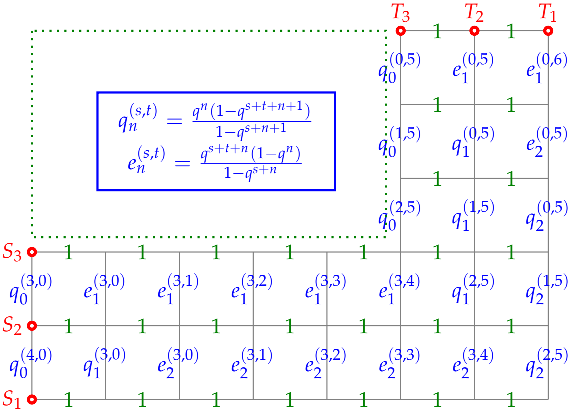


From Lindstöm–Gessel–Viennot's lemma

$$\det_{1\le i,j\le c}(g_{i,j}) = \prod_{k=1}^{c} w(P_k^*)$$

on the last graph. The edge-labels on the last graph are:



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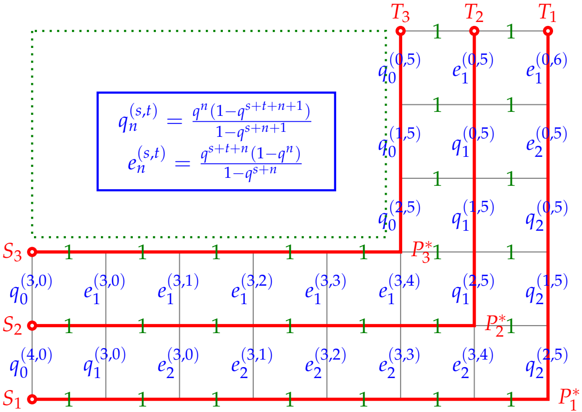

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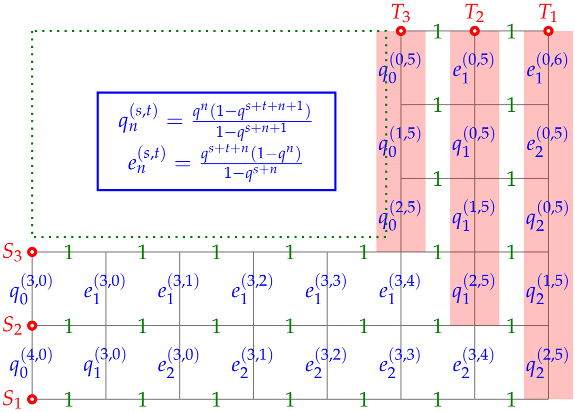

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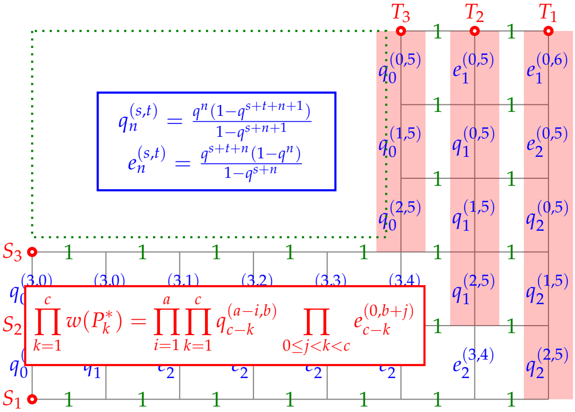


$$q_n^{(s,t)} = \frac{q^n(1-q^{s+t+n+1})}{1-q^{s+n+1}}$$
$$e_n^{(s,t)} = \frac{q^{s+t+n}(1-q^n)}{1-q^{s+n}}$$

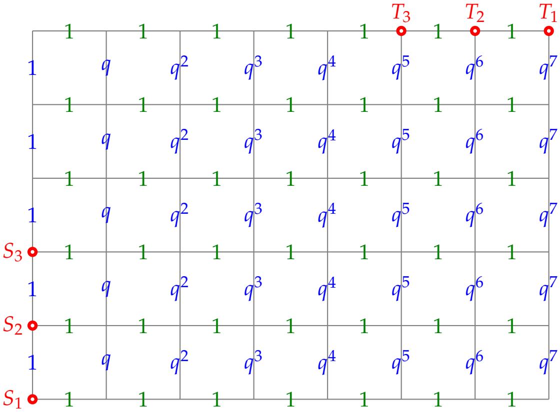
$$\prod_{k=1}^{c} w(P_k^*) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,b)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)}$$

Weight of $(P_1^{\emptyset}, \ldots, P_c^{\emptyset})$

We saw that MacMahon's generating function has the expression

$$\sum_{\substack{\pi \in \mathrm{PP}(a,b) \\ \pi_{i,j} \le c}} q^{|\pi|} = \sum_{\substack{(P_1,\ldots,P_c) \\ \in \mathrm{NILP}(a,b,c)}} \prod_{k=1}^{c} w(P_k) \Big/ \prod_{k=1}^{c} w(P_k^{\emptyset})$$

in terms of non-intersecting lattice paths. On the original graph



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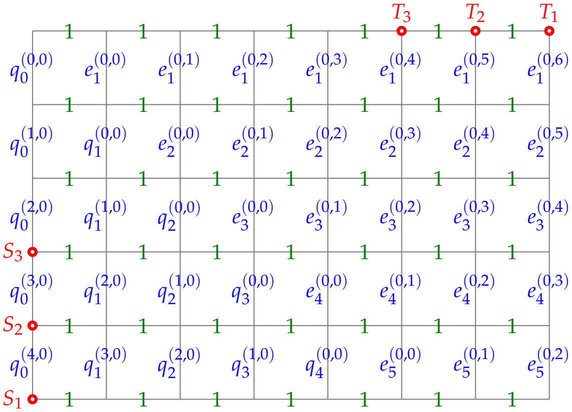

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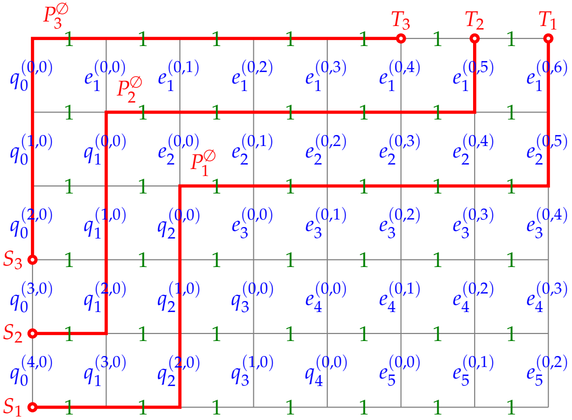

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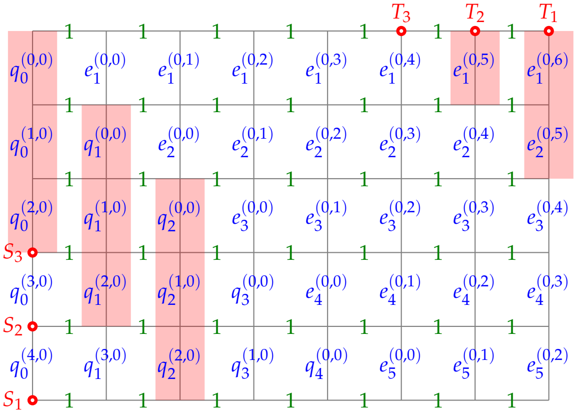

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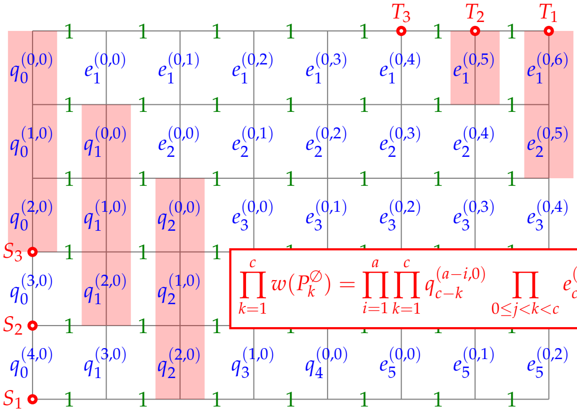


$$\boxed{\prod_{k=1}^{c} w(P_k^{\varnothing}) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,0)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)}}$$

Summary and proof

1. One-to-one correspondence with non-intersecting lattice paths:

$$\sum_{\substack{\pi\in \mathrm{PP}(a,b)\\ \pi_{i,j}\le c}} q^{|\pi|} = \sum_{\substack{(P_1,\dots,P_c)\\ \in \mathrm{NILP}(a,b,c)}} \prod_{k=1}^{c} w(P_k) \Big/ \prod_{k=1}^{c} w(P_k^{\varnothing});$$

2. Lindström–Gessel–Viennot's lemma:

$$\sum_{\substack{(P_1,\dots,P_c)\\ \in \mathrm{NILP}(a,b,c)}} \prod_{k=1}^{c} w(P_k) = \det_{1\le i,j\le c}(g_{i,j});$$

3. Corner deletion (with LGV's lemma):

$$\det_{1\le i,j\le c}(g_{i,j}) = \prod_{k=1}^{c} w(P_k^*) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,b)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)};$$

4. And:

$$\prod_{k=1}^{c} w(P_k^{\varnothing}) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,0)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)}.$$

where $q_n^{(s,t)}$, $e_n^{(s,t)}$ are the edge-labels on the original and last graphs given by

$$q_n^{(s,t)} = \frac{q^n(1-q^{s+t+n+1})}{1-q^{s+n+1}}, \qquad e_n^{(s,t)} = \frac{q^{s+t+n}(1-q^n)}{1-q^{s+n}}.$$

Therefore

$$\begin{aligned}\sum_{\substack{\pi\in\mathrm{PP}(a,b)\\ \pi_{i,j}\le c}} q^{|\pi|} &= \prod_{i=1}^{a}\prod_{k=1}^{c} \frac{q_{c-k}^{(a-i,b)}}{q_{c-k}^{(a-i,0)}} \\ &= \prod_{i=1}^{a}\prod_{k=1}^{c} \frac{1-q^{a+b+c-i-k+1}}{1-q^{a+c-i-k+1}} \\ &= \prod_{i=1}^{a}\prod_{k=1}^{c} \frac{1-q^{i+b+k-1}}{1-q^{i+k-1}} \\ &= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.\end{aligned}$$

That completes "yet another proof" of MacMahon's formula.

Example 2: A boxed version of hook-length formula

Gansner's formula:

$$\sum_{\pi\in \mathrm{RPP}(\lambda)} \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1-\prod_{(k,\ell)\in H_\lambda(i,j)} y_{\ell-k}}$$

reducing by $y_\ell = q$ into:

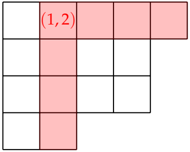
Theorem (Hook-length formula for reverse plane partitions)

For reverse plane partitions of shape λ,

$$\sum_{\pi\in \mathrm{RPP}(\lambda)} q^{|\pi|} = \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_\lambda(i,j)}}$$

where $h_\lambda(i,j)$ denotes the hook-length of the hook $H_\lambda(i,j)$.

$\lambda =$



$H_\lambda(1,2)$ with $h_\lambda(1,2) = 7$

Refine the hook-length formula

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} q^{|\pi|} = \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h_\lambda(i,j)}}$$

for a boxed reverse plane partitions like

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} q^{|\pi|}$$

Let λ be a Young diagram of a rows and b columns.

A one-to-one correspondence between:

- Reverse plane partitions $\pi \in \mathrm{RPP}(\lambda)$ with $\pi_{i,j} \leq c$;
- Non-intersecting configurations $(P_1, \dots, P_c)$ of c lattice paths such that P_k goes from $S_k = (a+c-k, 0)$ to $T_k = (0, b+c-k)$ where the upper-left corner of the lattice is trimmed in the form of λ (rotated 180°).

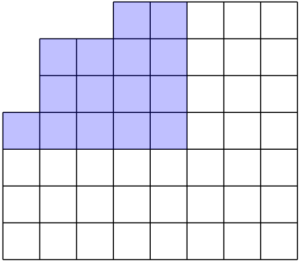
0	0	1	2	4
0	1	2	3	
2	2	4	4	
3	4			

$\overset{1:1}{\longleftrightarrow}$

			4	3
	4	4	2	2
	3	2	1	0
4	2	1	0	0

(rotated 180°)

$\overset{1:1}{\longleftrightarrow}$



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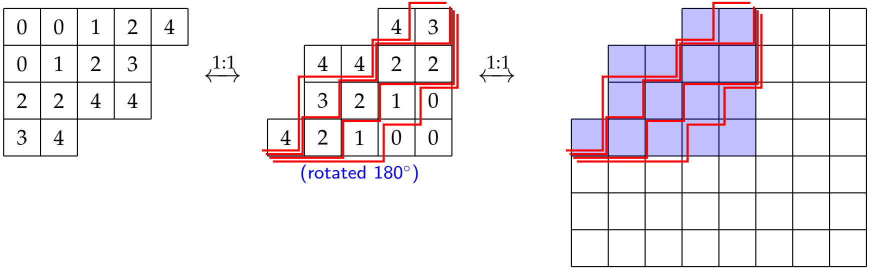

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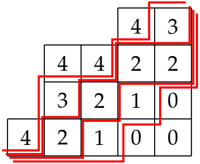

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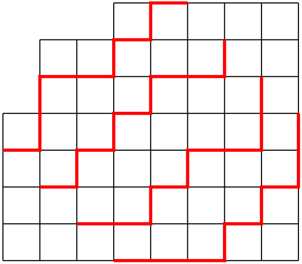

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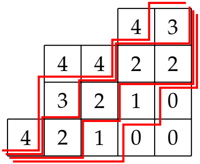

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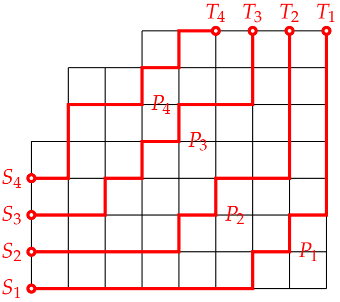
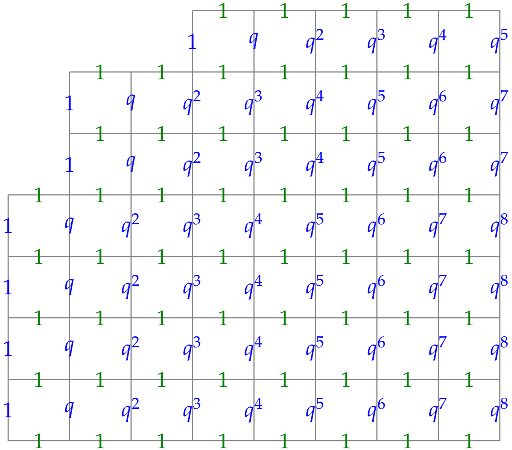


Weight for lattice paths

Let us realize the generating function $\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} q^{|\pi|}$ in terms of non-intersecting lattice paths.



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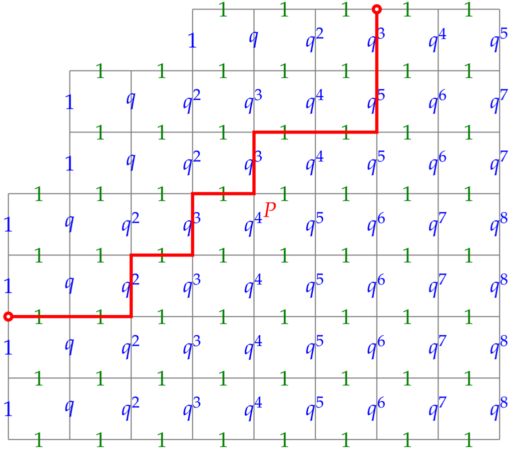

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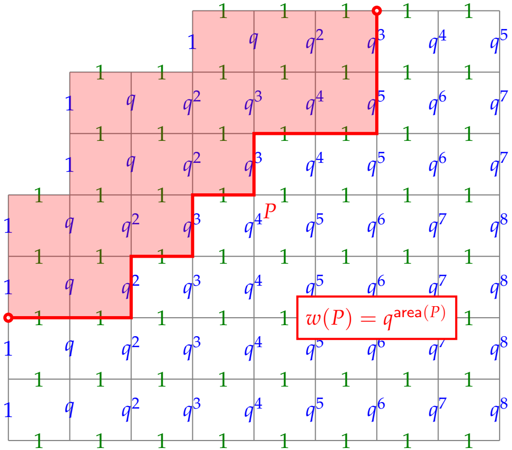
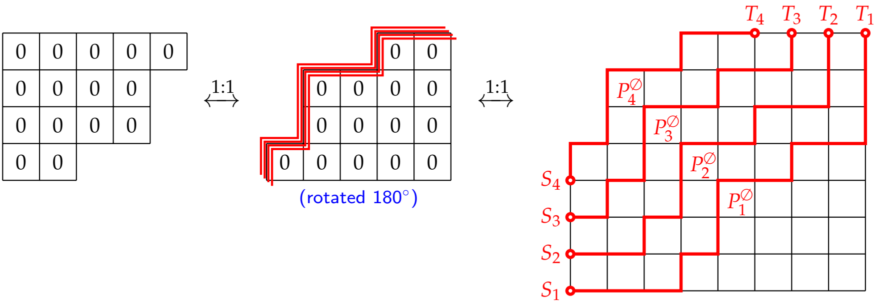


Let $(P_1^{\varnothing}, \ldots, P_c^{\varnothing})$ denote the non-intersecting configuration of lattice paths that corresponds to the empty reverse plane partition.



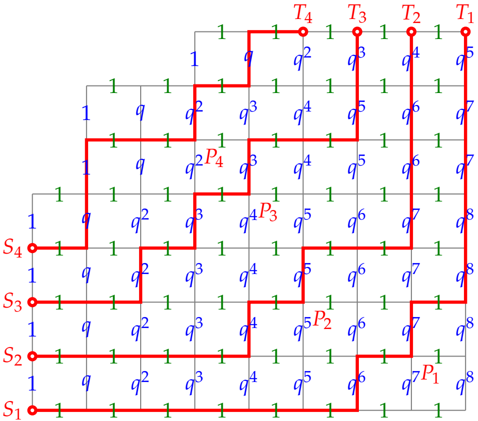
If $\pi \overset{1:1}{\longleftrightarrow} (P_1, \ldots, P_c)$ in the one-to-one correspondence then $q^{|\pi|} = \prod_{k=1}^{c} \frac{w(P_k)}{w(P_k^{\varnothing})}$. Hence

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \le c}} q^{|\pi|} = \sum_{\substack{(P_1, \ldots, P_c) \\ \in \mathrm{NILP}(\lambda, c)}} \prod_{k=1}^{c} w(P_k) \Big/ \prod_{k=1}^{c} w(P_k^{\varnothing})$$

where $\mathrm{NILP}(\lambda, c)$ denotes the set of non-intersecting configurations $(P_1, \ldots, P_c)$ of lattice paths on the λ-trimmed lattice such that P_k goes from S_k to T_k.

From Lindström–Gessel–Viennot's lemma:

$$\sum_{\substack{(P_1,\ldots,P_c)\\ \in \mathrm{NILP}(\lambda,c)}} \prod_{k=1}^{c} w(P_k) = \det_{1\le i,j\le c}(g_{i,j}) \qquad \text{with} \qquad g_{i,j} = \sum_{P:S_i - T_j} w(P).$$



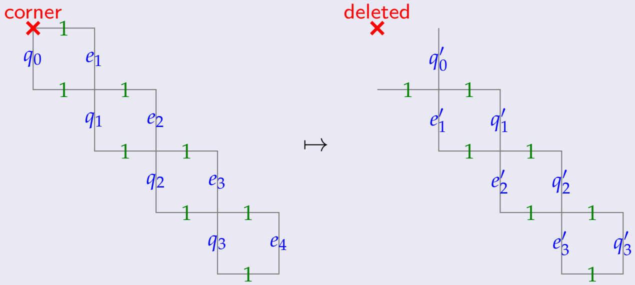
Try the corner deletion to evaluate the determinant of LGV type!

Corner deletion (rep.)

Corner deletion

Reduce the lattice graph by:

1. Delete an (upper left) corner;
2. Modify the edge-labels as:



where the edge-labels q_n, e_n and q'_n, e'_n satisfy

$$q_n + e_{n+1} = q'_n + e'_n, \qquad q_{n+1}e_{n+1} = q'_n e'_{n+1} \qquad \text{for } n \geq 0 \quad \text{with } e'_0 = 0.$$

Theorem

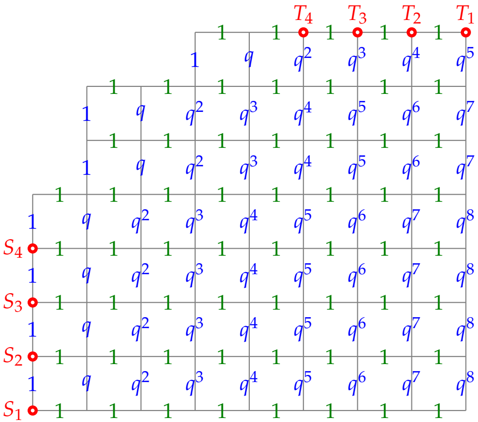
The corner deletion unchanges the value of

$$g_{i,j} = \sum_{P:S_i\text{–}T_j} w(P)$$

if neither S_i nor T_j is the deleted corner.

Idea of corner deletion

1. Perform corner deletion succesively until the (upper left) $a \times b$ rectangle is vacant.
2. $\det(g_{i,j})$ of LGV type on the last graph is (entrywise) equal to that on the original graph.
3. Evaluation of $\det(g_{i,j})$ is EASY on the last graph because the non-intersecting configuration $(P_1^*, \ldots, P_c^*)$ on the last graph is unique!



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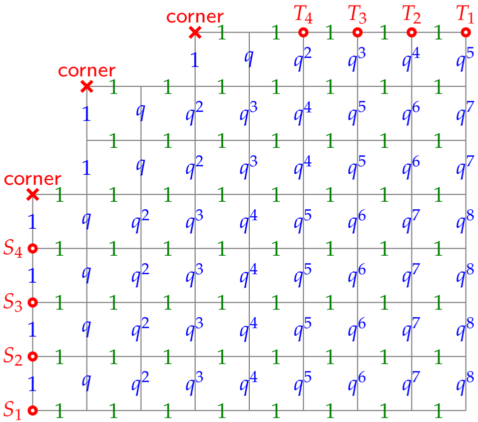

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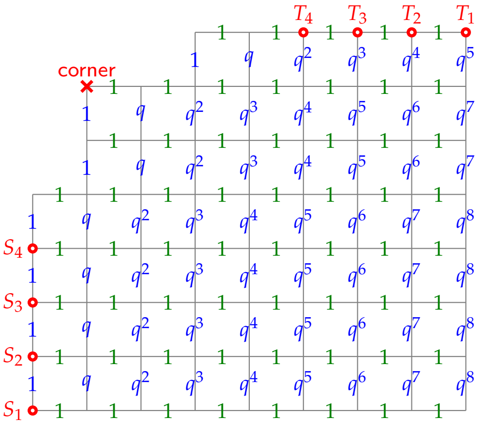

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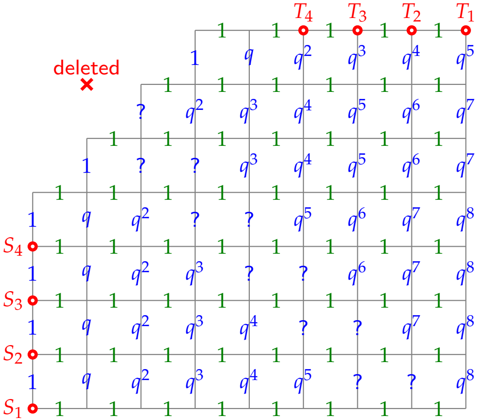

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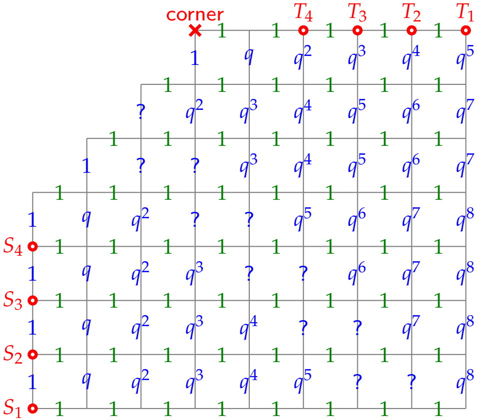

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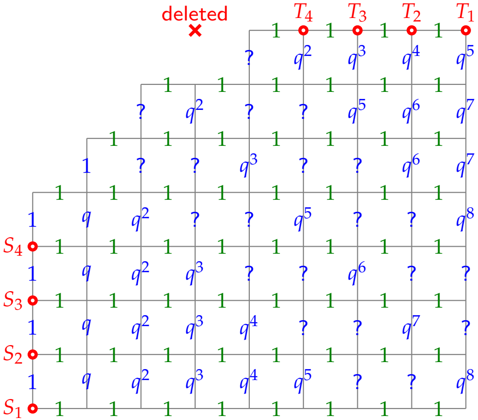

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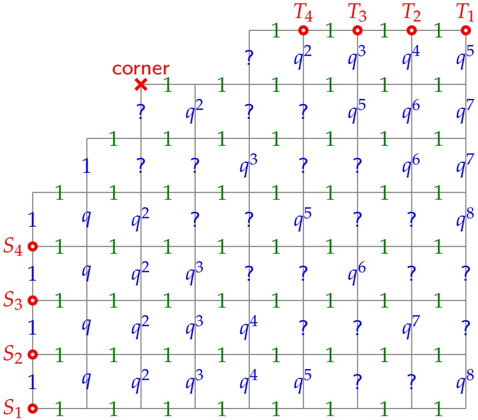

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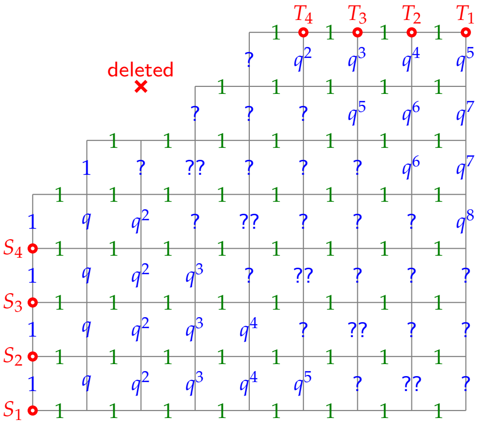

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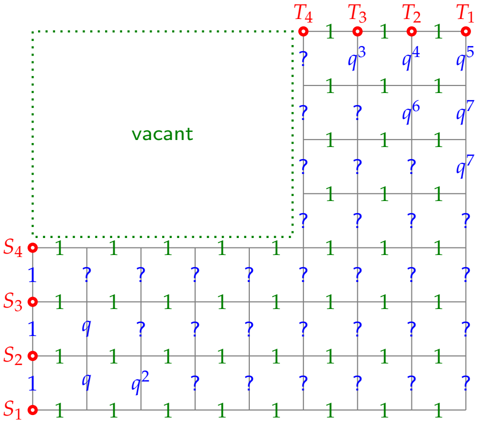

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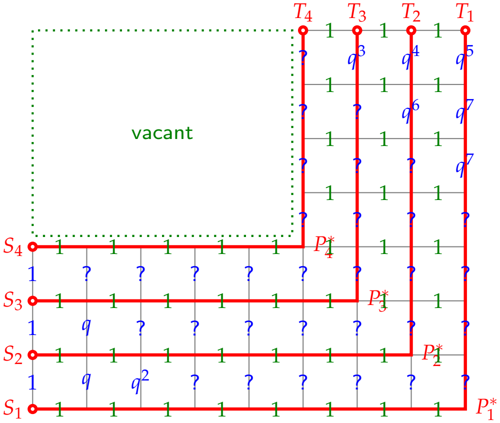

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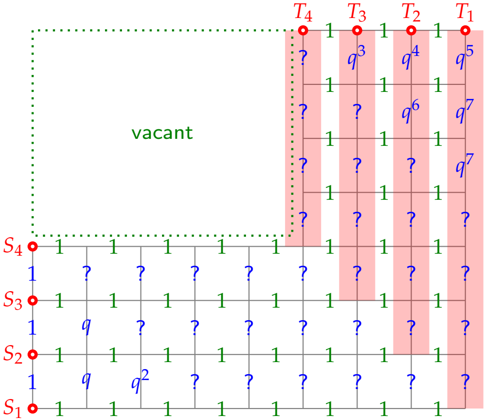

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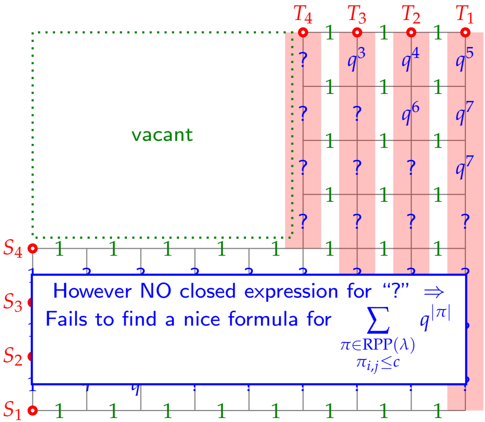


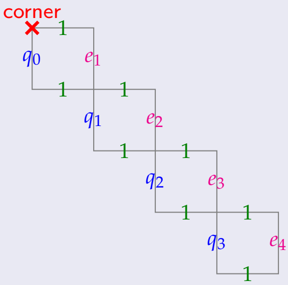
However NO closed expression for "?" $\Rightarrow$ Fails to find a nice formula for $\displaystyle\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \le c}} q^{|\pi|}$

Nice formulas from the discrete 2D Toda lattice

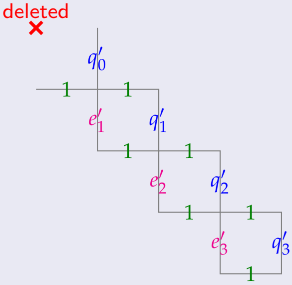
Corner deletion

Reduce the lattice graph by:

1. Delete an (upper left) corner;
2. Modify the edge-labels as:



$\mapsto$



where the edge-labels satisfy

$$q_n + e_{n+1} = q'_n + e'_n, \qquad q_{n+1}e_{n+1} = q'_n e'_{n+1}$$
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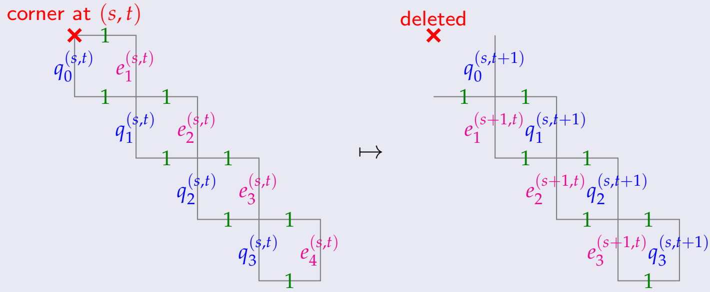


where the edge-labels satisfy

$$q_n^{(s,t)} + e_{n+1}^{(s,t)} = q_n^{(s,t+1)} + e_n^{(s+1,t)}, \qquad q_{n+1}^{(s,t)} e_{n+1}^{(s,t)} = q_n^{(s,t+1)} e_{n+1}^{(s+1,t)}$$

$$\text{for } n \geq 0 \quad \text{with } e_0^{(s+1,t)} = 0.$$

The discrete two-dimensional (2D) Toda lattice

$$q_n^{(s,t)} + e_{n+1}^{(s,t)} = q_n^{(s,t+1)} + e_n^{(s+1,t)}, \qquad q_{n+1}^{(s,t)} e_{n+1}^{(s,t)} = q_n^{(s,t+1)} e_{n+1}^{(s+1,t)},$$
$$s,t \in \mathbb{Z}, \quad n = 0,1,2,\ldots, \quad e_0^{(s,t)} = 0.$$

A discrete analogue of

- (continuous) Toda lattice

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1});$$

- (continuous) 2D Toda lattice

$$\frac{\partial^2 x_n}{\partial s \partial t} = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1}).$$

R. Hirota, S. Tsujimoto, T. Imai, *Difference scheme of soliton equations*, Sūrikaisekikenkyūsho Kōkyūroku, 822, pp. 144-152, 1993.

(Hirota) bilinear form of the discrete 2D Toda lattice

$$\tau_{n+1}^{(s,t)}\tau_{n-1}^{(s+1,t+1)} - \tau_n^{(s,t)}\tau_n^{(s+1,t+1)} + \tau_n^{(s+1,t)}\tau_n^{(s,t+1)} = 0,$$
$$s,t \in \mathbb{Z}, \quad n = 0,1,2,\ldots, \quad \tau_0^{(s,t)} = 1$$

by the transformation

$$q_n^{(s,t)} = \frac{\tau_n^{(s,t)}\tau_{n+1}^{(s+1,t)}}{\tau_{n+1}^{(s,t)}\tau_n^{(s+1,t)}}, \qquad e_n^{(s,t)} = \frac{\tau_{n+1}^{(s,t)}\tau_{n-1}^{(s,t+1)}}{\tau_n^{(s,t)}\tau_n^{(s,t+1)}}. \tag{*}$$

The general solution $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$ is given by (*) and

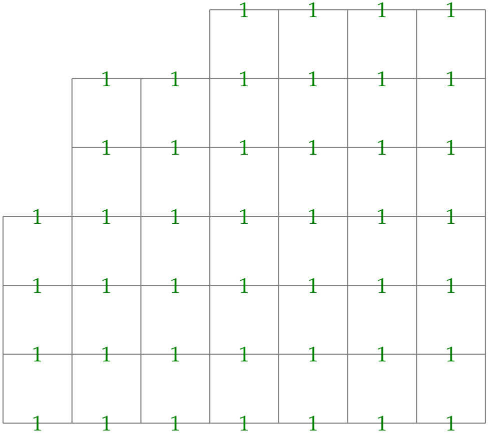
$$\tau_n^{(s,t)} = \det_{0\le i,j<n}\left(f_{s+i,t+j}\right)$$

with arbitrary $f_{i,j}$, $i,j \in \mathbb{Z}$, such that the determinant does not vanish.

Edge-labels

Let $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda lattice.

Assign the edge-labels with $q_n^{(s,t)}$, $e_n^{(s,t)}$ as:



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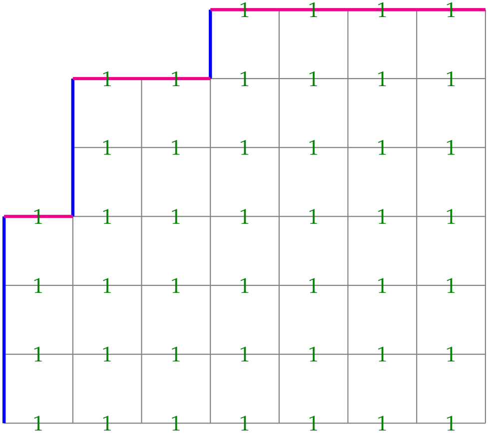
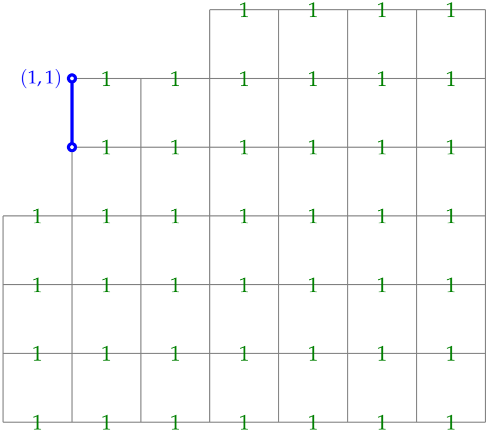


Edge-labels

Let $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda lattice.

Assign the edge-labels with $q_n^{(s,t)}$, $e_n^{(s,t)}$ as:



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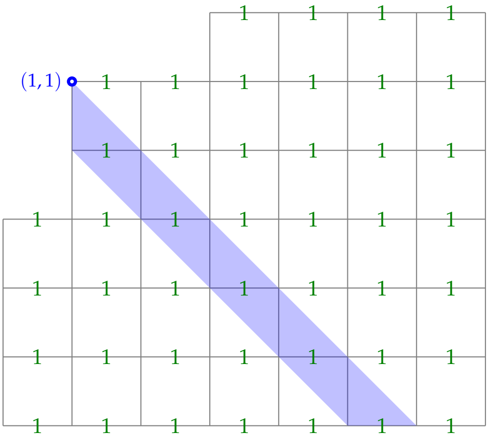

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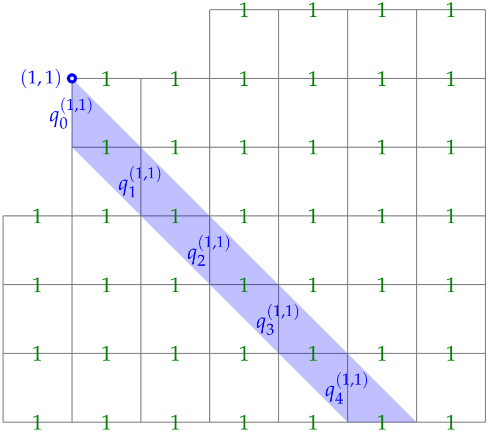

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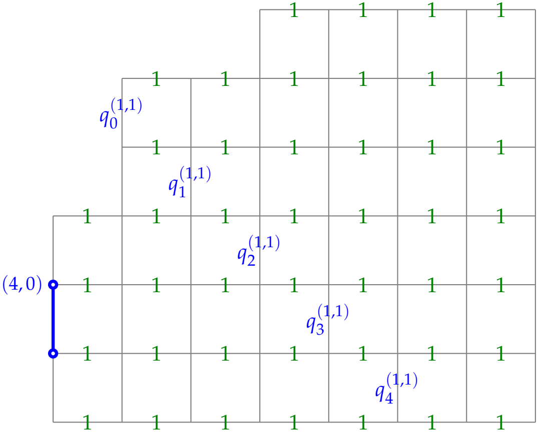

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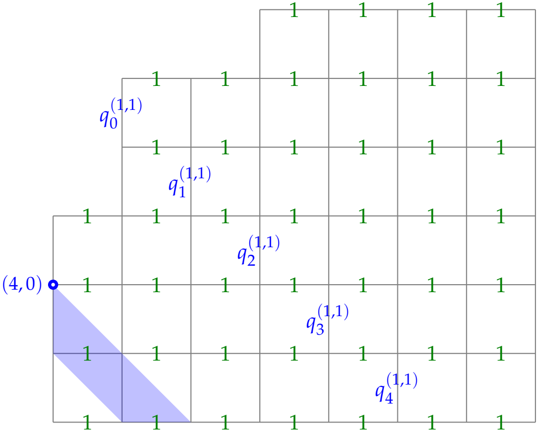

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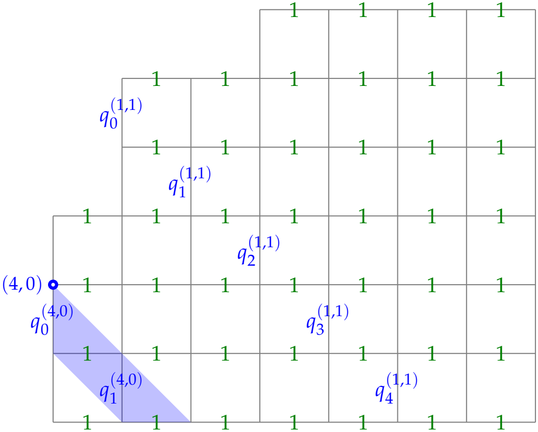

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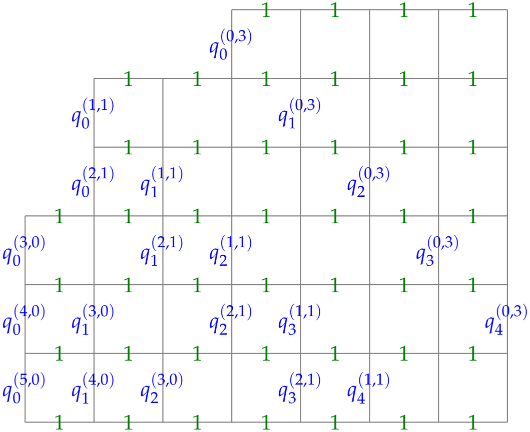

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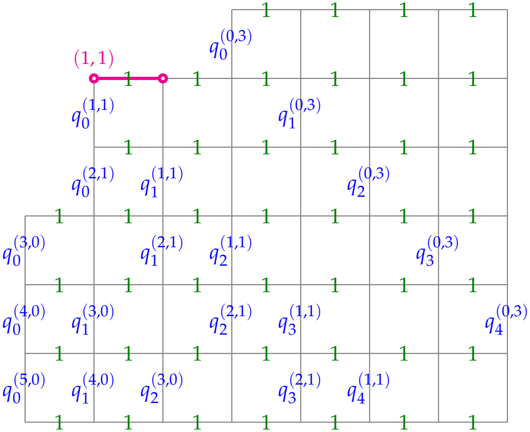

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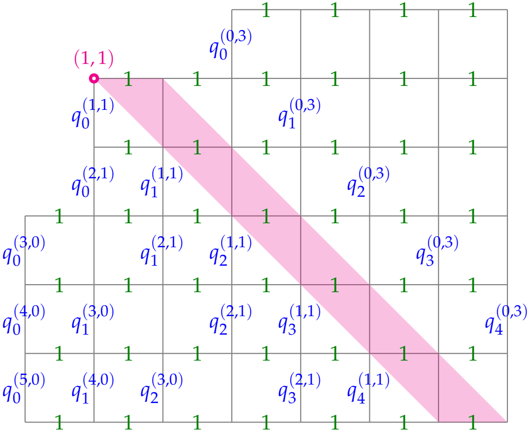

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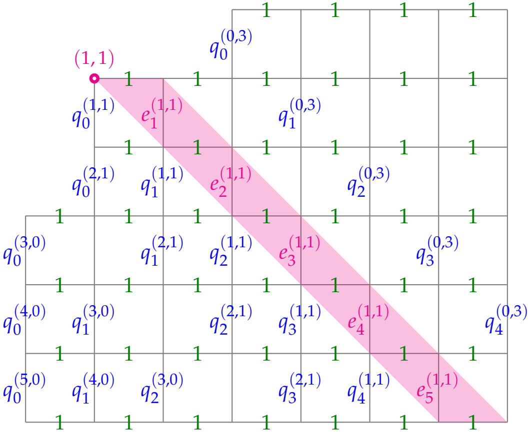

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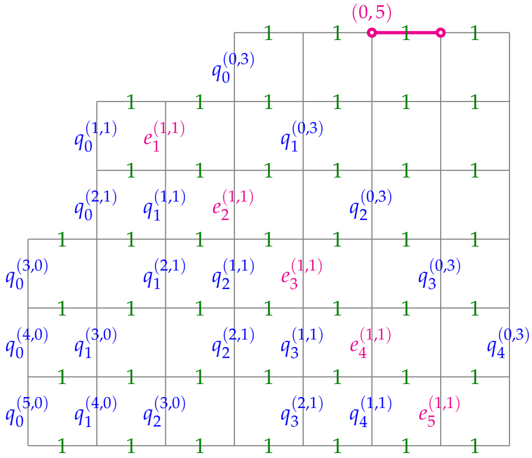

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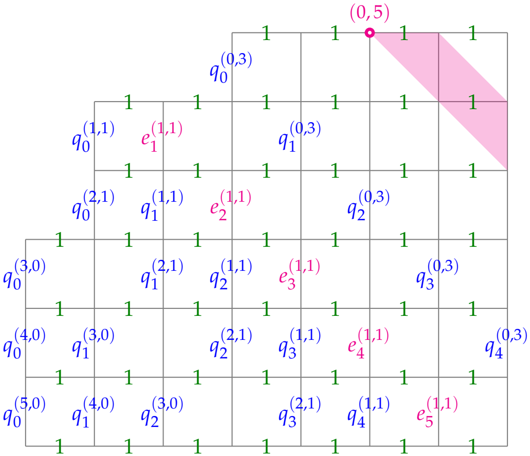

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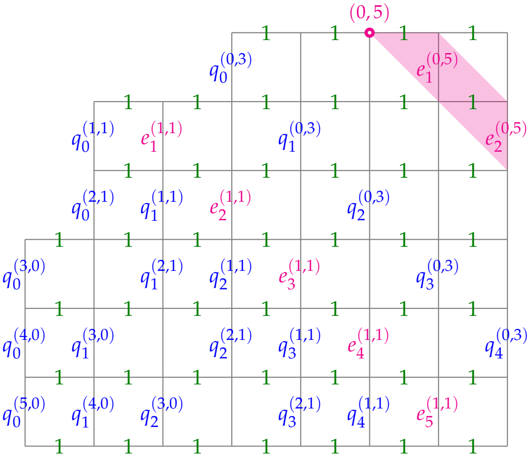

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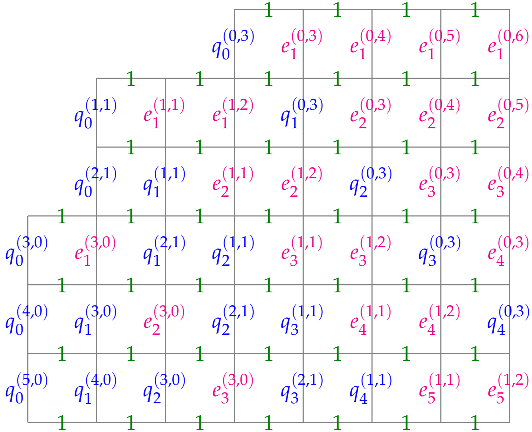
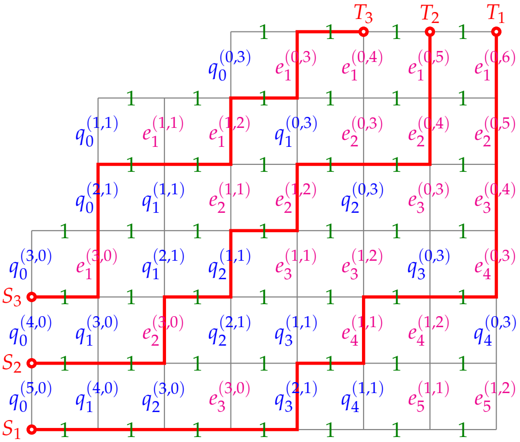


How to evaluate the determinant of LGV type

$$\sum_{\substack{(P_1,\dots,P_c)\\ \in \mathrm{NILP}(\lambda,c)}} \prod_{k=1}^{c} w(P_k) = \det_{1\le i,j\le c}(g_{i,j}) \quad \text{with} \quad g_{i,j} = \sum_{P:S_i - T_j} w(P).$$



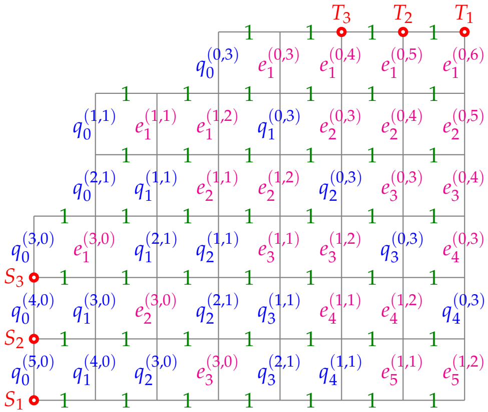
Nice formula for non-intersecting lattice paths

Theorem

Let $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda lattice. Assign the edge-labels on the λ-trimmed lattice with $q_n^{(s,t)}$, $e_n^{(s,t)}$ as above. Then

$$\sum_{\substack{(P_1,\dots,P_c)\\ \in \mathrm{NILP}(\lambda,c)}} \prod_{k=1}^{c} w(P_k) = \det_{1\le i,j\le c}(g_{i,j}) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,b)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)}.$$

Proof: By succesive corner deletion to the original graph we obtain:



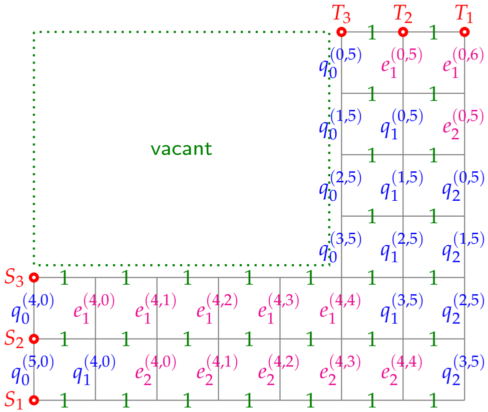
1. On the last graph, from LGV's lemma,

$$\det_{1\le i,j\le c}(g_{i,j}) = \prod_{k=1}^{c} w(P_k^*) = \prod_{i=1}^{a}\prod_{k=1}^{c} q_{c-k}^{(a-i,b)} \prod_{0\le j<k<c} e_{c-k}^{(0,b+j)}$$

where $(P_1^*,\ldots,P_c^*)$ is the unique non-intersecting configuration on the last graph.

2. Since corner deletion unchages $g_{i,j} = \sum_{P:S_i - T_j} w(P)$ then this determinant is equal to $\det(g_{i,j})$ on the original graph.

Proof: By succesive corner deletion to the original graph we obtain:



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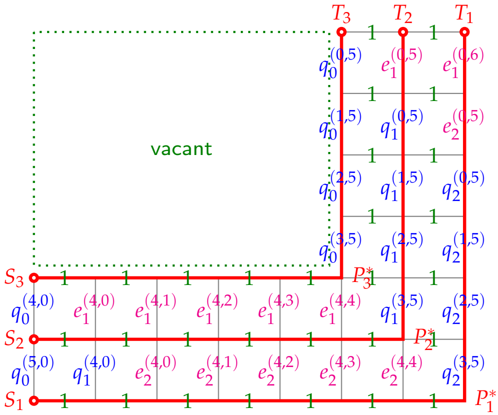

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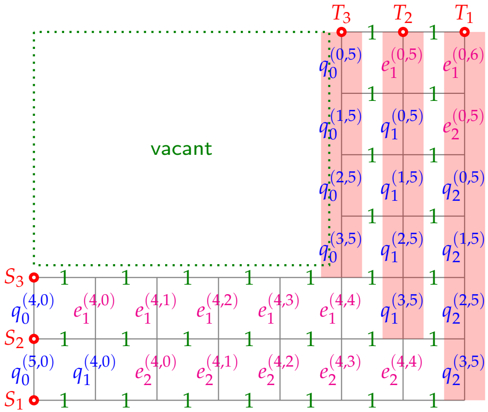

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2. Since corner deletion unchages $g_{i,j} = \sum_{P:S_i - T_j} w(P)$ then this determinant is equal to $\det(g_{i,j})$ on the original graph.

Nice formula for reverse plane partitions

For a reverse plane partition $\pi \in \mathrm{RPP}(\lambda)$ with $\pi_{i,j} \leq c$,

$$w(\pi) := \prod_{k=1}^{c} \frac{w(P_k)}{w(P_k^{\varnothing})}$$

where $(P_1, \ldots, P_c) \overset{1:1}{\longleftrightarrow} \pi$ and $(P_1^{\varnothing}, \ldots, P_c^{\varnothing}) \overset{1:1}{\longleftrightarrow} \pi^{\varnothing}$, the empty reverse plane partition, in the one-to-one correspondence. (Normalized so that $w(\pi^{\varnothing}) = 1$.)

Theorem

Let

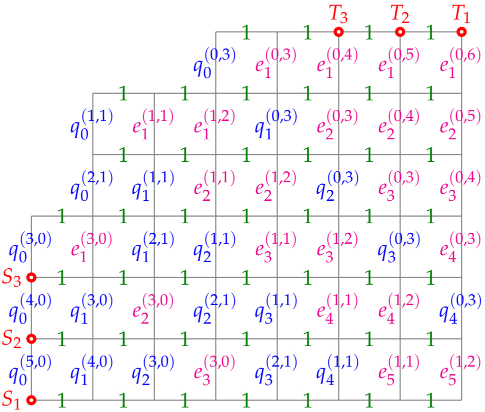
- $\lambda = (\lambda_1 \geq \cdots \geq \lambda_a)$: Young diagram of a rows and b columns;
- $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$: solution to the discrete 2D Toda lattice;
- $w(\pi)$: defined as above.

Then

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} w(\pi) = \prod_{i=1}^{a} \prod_{k=1}^{c} \frac{q_{c-k}^{(a-i,b)}}{q_{c-k}^{(a-i,b-\lambda_i)}}.$$

Proof: On $(P_1^{\varnothing}, \dots, P_c^{\varnothing})$,

$$\prod_{k=1}^{c} w(P_k^{\varnothing}) = \prod_{i=1}^{a} \prod_{k=1}^{c} q_{c-k}^{(a-i, b-\lambda_i)} \prod_{0 \le j < k < c} e_{c-k}^{(0, b+j)}.$$



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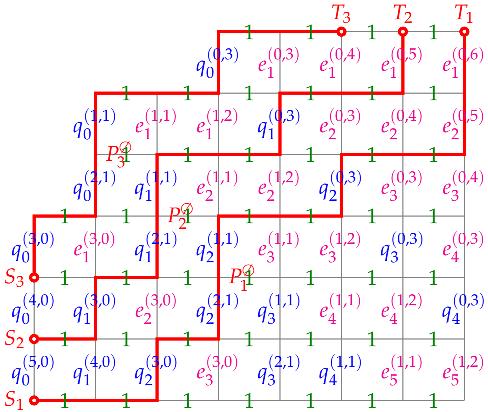

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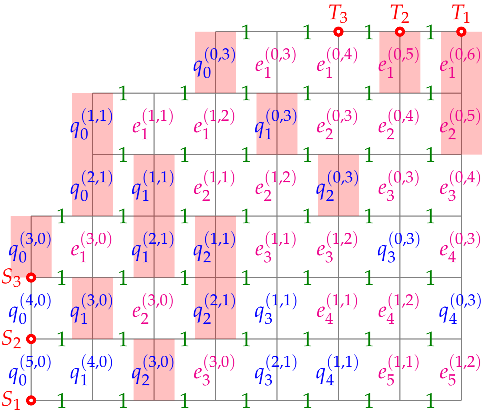

1. One-to-one correspondence with non-intersecting lattice paths:

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} w(\pi) = \sum_{\substack{(P_1,\dots,P_c) \\ \in \mathrm{NILP}(\lambda,c)}} \prod_{k=1}^{c} w(P_k) \Big/ \prod_{k=1}^{c} w(P_k^{\emptyset});$$

2. Nice formula for non-intersecting lattice paths:

$$\sum_{\substack{(P_1,\dots,P_c) \\ \in \mathrm{NILP}(\lambda,c)}} \prod_{k=1}^{c} w(P_k) = \prod_{i=1}^{a} \prod_{k=1}^{c} q_{c-k}^{(a-i,b)} \prod_{0 \leq j < k < c} e_{c-k}^{(0,b+j)};$$

3. And:

$$\prod_{k=1}^{c} w(P_k^{\emptyset}) = \prod_{i=1}^{a} \prod_{k=1}^{c} q_{c-k}^{(a-i,b-\lambda_i)} \prod_{0 \leq j < k < c} e_{c-k}^{(0,b+j)}.$$

Therefore

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} w(\pi) = \prod_{i=1}^{a} \prod_{k=1}^{c} \frac{q_{c-k}^{(a-i,b)}}{q_{c-k}^{(a-i,b-\lambda_i)}}.$$

Each solution $q_n^{(s,t)} \neq 0, e_n^{(s,t)} \neq 0$ to the discrete 2D Toda molecule gives a nice (product) formula for $\mathrm{RPP}(\lambda, c)$.

Example: MacMahon's formula

$$\sum_{\pi \in \mathrm{RPP}((b^a),c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

is derived from the solution

$$q_n^{(s,t)} = \frac{q^n(1-q^{s+t+n+1})}{1-q^{s+n+1}}, \qquad e_n^{(s,t)} = \frac{q^{s+t+n}(1-q^n)}{1-q^{s+n}}.$$

A boxed version of Gansner's formula

Theorem (Gansner)

$$\sum_{\pi\in\mathrm{RPP}(\lambda)}\prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = \prod_{(i,j)\in\lambda}\frac{1}{1-\prod_{(k,\ell)\in H_\lambda(i,j)} y_{\ell-k}}$$

where $H_\lambda(i,j)$ denotes the hook of cell $(i,j)\in\lambda$.

E. R. Gansner, *The Hillman–Grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory Ser. A **30** (1981), 71–89.

Find a boxed version of Gasner's formula.

As is the case for the hook-length formula

$$\boxed{\text{simply adding ``}\pi_{i,j}\le c\text{''}}$$

does NOT result in a nice formula.

- Notation: For $\alpha, \beta \in \mathbb{Z}$,

$$[z]_\alpha^\beta := \frac{\prod_{k \le \beta} z_k}{\prod_{\ell < \alpha} z_\ell} = \begin{cases} z_\alpha z_{\alpha+1} \cdots z_\beta & \text{if } \beta - \alpha \ge 0; \\ 1 & \text{if } \beta - \alpha = -1; \\ (z_{\beta+1} \cdots z_{\alpha-2} z_{\alpha-1})^{-1} & \text{if } \beta - \alpha < -1 \end{cases}$$

(or $[z]_\alpha^\beta [z]_{\beta+1}^\gamma = [z]_\alpha^\gamma$ for $\alpha, \beta, \gamma \in \mathbb{Z}$).

- Solution to the discrete 2D Toda lattice

$$q_n^{(s,t)} = [u]_{s+1}^{s+n}(1 - x[u]_1^s [v]_1^{t+n}), \qquad e_n^{(s,t)} = x[u]_1^{s+n-1}[v]_1^t (1 - [v]_{t+1}^{t+n})$$

including the parameters x, u_ℓ and v_ℓ for $\ell \ge 1$;

- $\lambda = (\lambda_1, \ldots, \lambda_a)$: Young diagram of a rows and b columns,

$$b = \lambda_1 \ge \cdots \ge \lambda_a > 0;$$

- $\lambda' = (\lambda'_1, \ldots, \lambda'_b)$: Young diagram conjugate with λ;
- Parameters

$$x = [y]_{b-\lambda'_b}^{-a+\lambda_a}, \qquad u_\ell = [y]_{-a+\ell+\lambda_{a-\ell+1}}^{-a+\ell+\lambda_{a-\ell}}, \qquad v_\ell = [y]_{b-\ell-\lambda'_{b-\ell}}^{b-\ell-\lambda'_{b-\ell+1}}$$

where $\lambda_i = b$ and $\lambda'_i = a$ for $i \le 0$.

Theorem (A boxed version of Gansner's formula)

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \leq c}} \omega(\pi) \prod_{(i,j)\in\lambda} y_{j-i}^{\pi_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1-[y]_{-\lambda'_{-c+j}-c+j}^{\lambda_i - i}}{1-[y]_{-\lambda'_j+j}^{\lambda_i - i}},$$

$$\omega(\pi) := \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1-[y]_{-\lambda'_{-c+j+k-1}-c+j+k-1}^{j-i-1}}{1-[y]_{-\lambda'_{-c+j+k}-c+j+k}^{j-i}}$$

where $\lambda'_i = \lambda'_1 (= a)$ for $i \leq 0$.

This formula reduces into Gansner's formula as $c \to \infty$.

Set $y_\ell = q$ for $\ell \in \mathbb{Z}$ to obtain:

Corollary (A boxed version of the hook-kength formula)

$$\sum_{\substack{\pi \in \mathrm{RPP}(\lambda) \\ \pi_{i,j} \le c}} q^{|\pi|} \omega(\pi) = \prod_{(i,j)\in\lambda} \frac{1 - q^{h_\lambda(i,j-c)}}{1 - q^{h_\lambda(i,j)}},$$

$$\omega(\pi) := \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - q^{\lambda'_{-c+j+k-1}+c-i-k+1}}{1 - q^{\lambda'_{-c+j+k}+c-i-k+1}}$$

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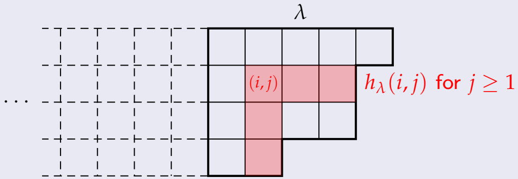

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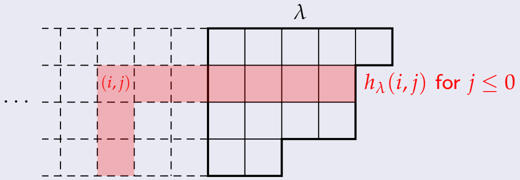

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