The Combinatorics of Alternating Sign Matrices

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Plan

Today

- Introduce & enumerate
  - Alternating sign matrices (ASMs)
  - Alternating sign triangles (ASTs)
  - Descending plane partitions (DPPs)
  - Totally symmetric self-complementary plane partitions (TSSCPPs)
  - Double-staircase semistandard Young tableaux

Thursday

- Discuss refined enumeration of ASMs with
  - Fixed values of statistics
  - Invariance under symmetry operations
- Sketch proofs for enumerations of
  - Unrestricted ASMs
  - Odd-order diagonally & antidiagonally symmetric ASMs

Alternating Sign Matrices (ASMs)

ASM: square matrix for which

- each entry is 0, 1 or -1
- each row & column contains at least one 1
- along each row & column, the nonzero entries alternate in sign, starting & ending with a 1

e.g. $$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

History:

- Arose during study of Dodgson condensation algorithm for determinant evaluation *(Mills, Robbins, Rumsey 1982; Robbins, Rumsey 1986)*
- Many subsequent appearances in combinatorics, algebra, mathematical physics, . . .

Observations:
- first/last row/column of an ASM contains single 1 & all other entries 0
- acting on an ASM with any symmetry operation of the square (reflections, rotations) gives another ASM
- any permutation matrix is an ASM

Number A_n of $n \times n$ ASMs

n=1

$$(1) \quad \Rightarrow \quad A_1 = 1$$

n=2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A_2 = 2$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A_3 = 7$$

n=4

- $4! = 24$ matrices without any -1's (permutation matrices)

- 4 matrices with one -1, at position 2,2:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

  Similarly:

  4 matrices with one -1 at 2,3
  4 matrices with one -1 at 3,2
  4 matrices with one -1 at 3,3

  So, 16 matrices with one -1

- 2 matrices with two -1's:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\Rightarrow A_4 = 24 + 16 + 2 = 42$

General Case

$$\# \text{ of } n\times n \text{ ASMs: } A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots$$

- Recursion: $\binom{2n}{n} A_{n+1} = \binom{3n+1}{n} A_n$
- Conjectured: *Mills, Robbins, Rumsey 1982*
- First proved:
  - *Zeilberger 1996* using constant term identities
  - *Kuperberg 1996* using connections with statistical mechanical model
- Book: D. Bressoud *The Story of the ASM Conjecture*, Cambridge Uni. Press (1999), 274 pages
- No combinatorial proof currently known.
- Kuperberg proof (more on Thursday):
  - Apply bijection between $n\times n$ ASMs & configurations of statistical mechanical *six-vertex model* on $n\times n$ square with *domain-wall boundary conditions*.
  - Introduce parameter-dependent weights & consider weighted sum over all configurations of model, i.e., generating function or *partition function*.
  - Use *Yang–Baxter equation* & other properties to obtain *Izergin–Korepin formula* for partition function as $n\times n$ determinant.
  - Evaluate determinant at certain values of parameters for which all weights are 1.

Alternating Sign Triangles (ASTs)

AST of order n: triangular array

$$\begin{array}{ccccccc}
a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1,2n-3} & a_{1,2n-2} & a_{1,2n-1} \\
 & a_{22} & a_{23} & \cdots\cdots & a_{2,2n-3} & a_{2,2n-2} & \\
 & & \ddots & & & \iddots & \\
 & & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} & & \\
 & & & a_{nn} & & &
\end{array}$$

such that

- each entry is 0, 1 or -1
- each row contains at least one 1
- along each row, the nonzero entries alternate in sign, starting & ending with a 1
- down each column, the nonzero entries (if there are any) alternate in sign, starting with a 1

• Introduced by *Ayyer, RB, Fischer 2016*

• e.g. AST of order 6:

$$\begin{array}{ccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 1 & -1 & 0 & 0 & 1 & & \\
 & & & 1 & -1 & 1 & 0 & 0 & & & \\
 & & & & 1 & -1 & 1 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

Observations:
- an order n AST has n^2 entries
- last row of an AST is a single 1
- first row of an AST contains a single 1 & all other entries 0
- reflecting an AST in the central vertical line gives another AST
- (# of order n ASTs without any -1's) $=$ $n!$

Number A'_n of order n ASTs

n=1

$$(1) \quad \Rightarrow \quad A'_1 = 1$$

n=2

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ & 1 & \end{pmatrix} \quad \Rightarrow \quad A'_2 = 2$$

n=3

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & \\ & & 1 & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & \\ & & 1 & & \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & -1 & 1 & \\ & & 1 & & \end{pmatrix} \quad \Rightarrow \quad A'_3 = 7$$

General Case

$$\boxed{\#\text{ of order } n \text{ ASTs: } A'_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots}$$

- Stated & proved: *Ayyer, RB, Fischer 2016*
- Proof analogous to that of Kuperberg for ASMs:
  - Apply bijection between order n ASTs & configurations of statistical mechanical six-vertex model on a *triangle* with certain boundary conditions.
  - Introduce parameter-dependent weights & consider weighted sum over all configurations of model, i.e., generating function or *partition function*.
  - Use *Yang–Baxter equation*, *reflection equation* & other properties to obtain formula for partition function as $n \times n$ determinant.
  - Evaluate determinant at certain values of parameters for which all weights are 1.
- Therefore

$$\boxed{(\#\text{ of } n \times n \text{ ASMs}) = (\#\text{ of order } n \text{ ASTs})}$$

- No explicit bijection currently known between $n \times n$ ASMs & order n ASTs for arbitrary n.

Descending Plane Partitions (DPPs)

DPP of order n: array

$$\begin{array}{ccccccc} d_{11} & d_{12} & d_{13} & \cdots\cdots\cdots\cdots\cdots\cdots & & & d_{1,\lambda_1} \\ & d_{22} & d_{23} & \cdots\cdots\cdots\cdots & & d_{2,\lambda_2+1} & \\ & & d_{33} & \cdots\cdots\cdots & d_{3,\lambda_3+2} & & \\ & & & \ddots \quad\quad \iddots & & & \\ & & & d_{tt} \;\cdots\cdots\; d_{t,\lambda_t+t-1} & & & \end{array}$$

such that

- each part d_{ij} is a positive integer
- parts decrease weakly along rows
- parts decrease strictly down columns
- $n \geq d_{11} > \lambda_1 \geq d_{22} > \lambda_2 \geq \ldots > \lambda_{t-1} \geq d_{tt} > \lambda_t$

- Arose during study of cyclically symmetric plane partitions. *(Andrews 1979)*

- e.g. DPP of order 6:

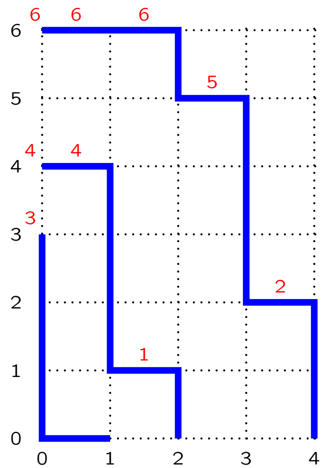
$$\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array}$$

$$n \geq d_{11} > \lambda_1 \geq d_{22} > \lambda_2 \geq d_{33} > \lambda_3$$
$$\leftrightarrow \quad 6 \geq 6 > 5 \geq 4 > 3 \geq 3 > 1$$

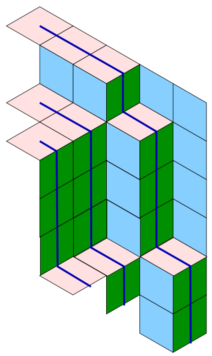
- There are simple bijections between
  - order n DPPs
  - sets of nonintersecting paths from $(0, k+2)$ to $(k, 0)$ with $0 \leq k \leq n-2$ & steps $(1,0)$ or $(0,-1)$
  - cyclically symmetric rhombus tilings of a hexagon with alternating sides of lengths $n \pm 1$ & central equilateral triangular hole of side length 2

- e.g.

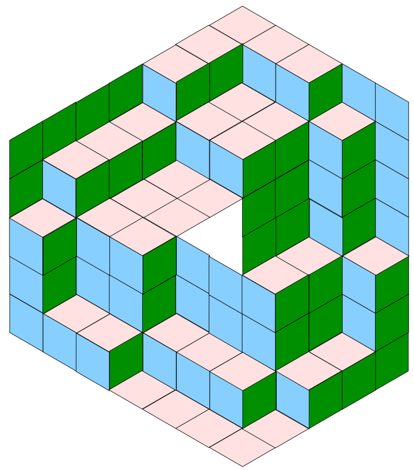
$$\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \quad \longleftrightarrow$$



$\longleftrightarrow$



$\longleftrightarrow$



Number P_n of DPPs of order n

n=1

$\emptyset \qquad \Rightarrow \qquad P_1 = 1$

n=2

$\emptyset,\ 2 \qquad \Rightarrow \qquad P_2 = 2$

n=3

$\emptyset,\ 2,\ 3,\ 3\,1,\ 3\,2,\ 3\,3,\ \begin{array}{cc}3&3\\&2\end{array} \qquad \Rightarrow \qquad P_3 = 7$

n=4

$\emptyset,\ 2,\ 3,\ 3\,1,\ 3\,2,\ 3\,3,\ \begin{array}{cc}3&3\\&2\end{array},\ 4,\ 4\,1,\ 4\,2,\ 4\,3,\ 4\,4,\ 4\,1\,1,\ 4\,2\,1,\ 4\,3\,1,\ 4\,4\,1,$

$4\,2\,2,\ 4\,3\,2,\ 4\,4\,2,\ 4\,3\,3,\ 4\,4\,3,\ 4\,4\,4,\ \begin{array}{cc}4&3\\&2\end{array},\ \begin{array}{cc}4&4\\&2\end{array},\ \begin{array}{ccc}4&3&1\\&2&\end{array},\ \begin{array}{ccc}4&4&1\\&2&\end{array},\ \begin{array}{ccc}4&3&2\\&2&\end{array},\ \begin{array}{ccc}4&4&2\\&2&\end{array},\ \begin{array}{ccc}4&3&3\\&2&\end{array},$

$\begin{array}{ccc}4&4&3\\&2&\end{array},\ \begin{array}{ccc}4&4&4\\&2&\end{array},\ \begin{array}{ccc}4&4&1\\&3&\end{array},\ \begin{array}{ccc}4&4&2\\&3&\end{array},\ \begin{array}{ccc}4&4&3\\&3&\end{array},\ \begin{array}{ccc}4&4&4\\&3&\end{array},\ \begin{array}{ccc}4&4&2\\&3&1\end{array},\ \begin{array}{ccc}4&4&3\\&3&1\end{array},\ \begin{array}{ccc}4&4&4\\&3&1\end{array},\ \begin{array}{ccc}4&4&3\\&3&2\end{array},\ \begin{array}{ccc}4&4&4\\&3&2\end{array},\ \begin{array}{ccc}4&4&4\\&3&3\end{array},\ \begin{array}{ccc}4&4&4\\&3&3\\&&2\end{array}$

$\Rightarrow \qquad P_4 = 42$

General Case

$$\boxed{\#\text{ of order } n \text{ DPPs: } P_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots}$$

- Stated & first proved: *Andrews 1979*

- Proof:
  - Apply bijection between order n DPPs & sets of nonintersecting lattice paths.
  - Use Lindström–Gessel–Viennot theorem to give $P_n = \det_{1 \le i,j \le n-1} \left(\delta_{ij} + \binom{i+j}{i-1} \right)$.
  - Show that determinant is given by product formula.

- Therefore

$$\boxed{(\#\text{ of } n\times n \text{ ASMs}) = (\#\text{ of order } n \text{ ASTs}) = (\#\text{ of order } n \text{ DPPs})}$$

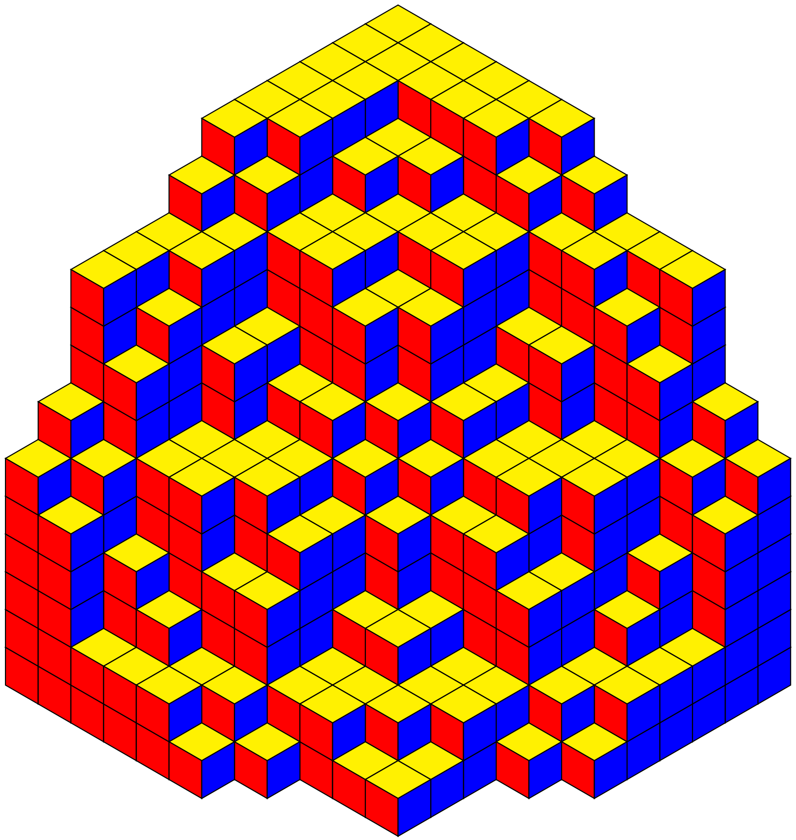
- No explicit bijection currently known between any pair of these three objects for arbitrary n.

Totally Symmetric Self-Complementary Plane Partitions (TSSCPPs)

TSSCPP: plane partition in a box, which is invariant under reflections, rotations & box-complementation

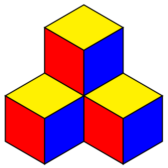
- Introduced: *Stanley 1986*

  e.g. TSSCPP in $12 \times 12 \times 12$ box:



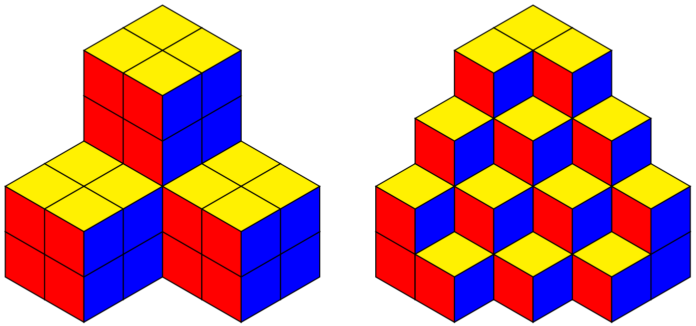
Number P'_n of TSSCPPs in $2n \times 2n \times 2n$ box

n=1



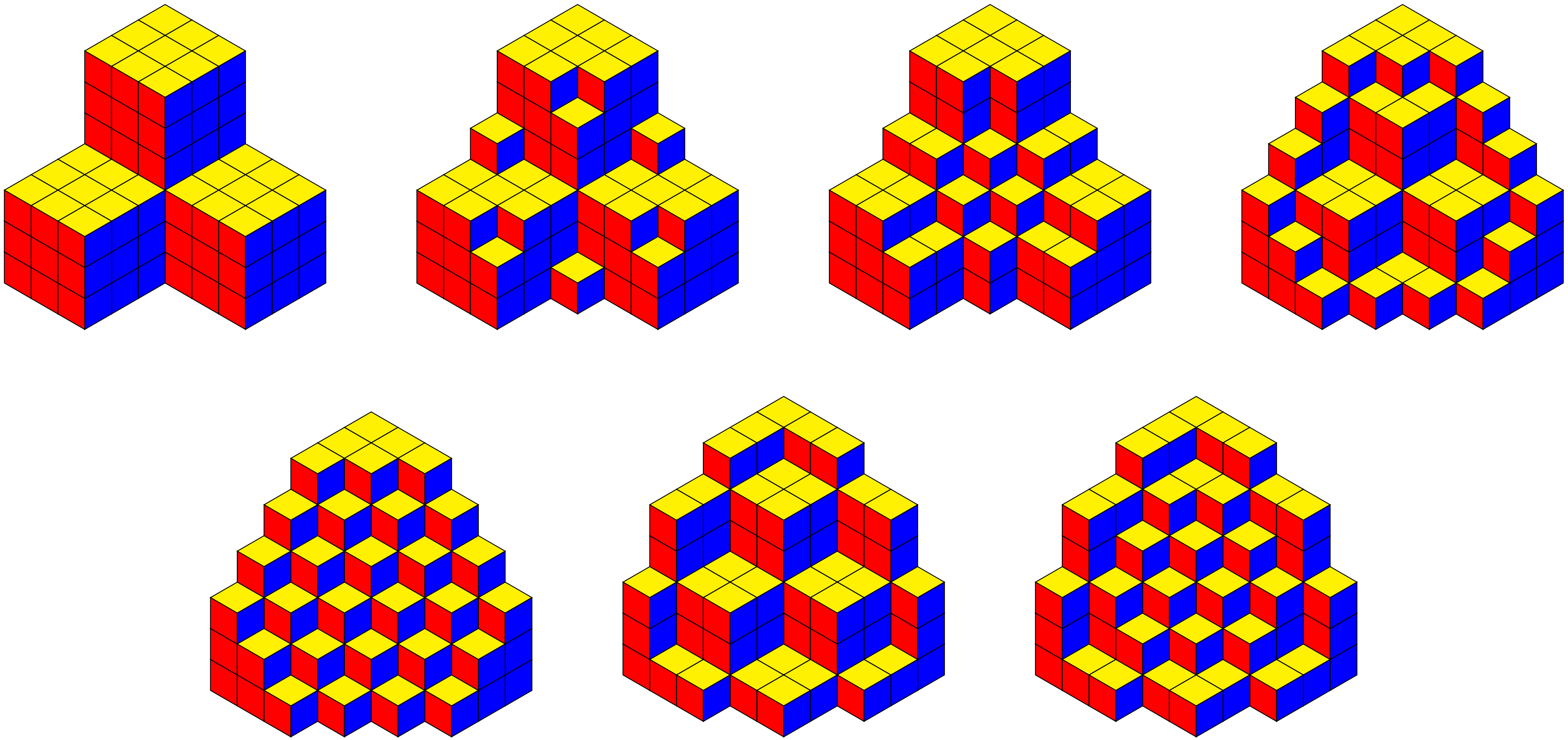
$\Rightarrow$ $P'_1 = 1$

n=2



$\Rightarrow$ $P'_2 = 2$

n=3



$$\Rightarrow \quad P'_3 = 7$$

General Case

$$\#\text{ of TSSCPPs in } 2n\times 2n\times 2n \text{ box: } P'_n = \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, 7436, \ldots$$

- Conjectured: *Mills, Robbins, Rumsey 1986*
- First proved: *Andrews 1994*
- Proof analogous to that for DPPs:
  - Apply bijection between TSSCPPs in $2n \times 2n \times 2n$ box & certain sets of nonintersecting lattice paths with fixed start points & partially-free end points. *(Doran 1993)*
  - Use Lindström–Gessel–Viennot-type theorem to express P'_n as Pfaffian,

    $$P'_n = \operatorname*{Pf}_{\sigma_n \le i < j \le n-1}\left(\sum_{k=2i-j+1}^{2j-i}\binom{i+j}{k}\right), \quad \sigma_n = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd.}\end{cases}$$

    (Okada 1989, Stembridge 1990)
  - Show that Pfaffian is given by product formula.

- Therefore

$$\begin{aligned}(\#\text{ of } n\times n \text{ ASMs}) &= (\#\text{ of order } n \text{ ASTs}) \\ &= (\#\text{ of order } n \text{ DPPs}) = (\#\text{ of TSSCPPs in } 2n\times 2n\times 2n \text{ box})\end{aligned}$$

- No explicit bijection currently known between any pair of these four objects for arbitrary n.

Summary

- The following are all equal
  - $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$
  - # of $n \times n$ ASMs
  - # of order n ASTs
  - # of order n DPPs
  - # of TSSCPPs in $2n \times 2n \times 2n$ box
- No bijective proofs currently known for the equality between any of the $\binom{5}{2} = 10$ pairs of numbers.
- *"This is one of the most intriguing open problems in the area of bijective proofs."* *(R. Stanley 2009)*
- Other comments:
  - *"These conjectures are of such compelling simplicity that it is hard to know how any mathematician can bear the pain of living without understanding <u>why</u> they are true. . . . I expect that these problems will remain with us for some time."* *(D. Robbins 1991)*
  - *"The greatest, still unsolved, mystery concerns the question of what plane partitions have to do with alternating sign matrices."* *(C. Krattenthaler 2016)*

Double-staircase semistandard Young tableaux

- Hook-content formula for semistandard Young tableaux gives

$$\frac{\mathsf{SSYT}\big((n-1,n-1,\ldots,2,2,1,1),2n\big)}{\mathsf{SSYT}\big((2n-2,2n-4,\ldots,6,4,2),n\big)} = \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!} \qquad \textit{(Okada 2006)}$$

where $\mathsf{SSYT}(\lambda,k) := \left(\begin{array}{l}\text{\# of semistandard Young tableaux of}\\ \text{shape } \lambda \text{ with entries from } \{1,2,\ldots,k\})\end{array}\right)$

- Observe that $(n-1,n-1,\ldots,2,2,1,1)$ & $(2n-2,2n-4,\ldots,6,4,2)$ are conjugate partitions of double-staircase shape.

- Also $\mathsf{SSYT}\big((2n-2,2n-4,\ldots,6,4,2),n\big) = 3^{n(n-1)/2}$

- e.g. for $n=3$:

$$\frac{\mathsf{SSYT}\big((2,2,1,1),6\big)}{\mathsf{SSYT}\big((4,2),3\big)} = \mathsf{SSYT}\left(\ydiagram{2,2,1,1},6\right)\Big/\mathsf{SSYT}\left(\ydiagram{4,2},3\right) = 189/3^3 = 189/27 = 7$$

- No bijective proofs currently known for equality between $\frac{\mathsf{SSYT}((n-1,n-1,\ldots,2,2,1,1),2n)}{\mathsf{SSYT}((2n-2,2n-4,\ldots,6,4,2),n)}$ & # of $n\times n$ ASMs, order n ASTs, order n DPPs or TSSCPPs in $2n\times 2n\times 2n$ box.

- Can show $q^{-(2n-1)n(n-1)/6}\, s_{(n-1,n-1,\ldots,2,2,1,1)}(1,q,\ldots,q^{2n-1})\Big/ s_{(2n-2,\ldots,6,4,2)}(1,q,\ldots,q^{n-1})$

$$= \prod_{i=0}^{n-1}\frac{[3i+1]_q!}{[n+i]_q!} = \sum_{\text{order } n \text{ DPPs } \pi} q^{\sum_{ij}\pi_{ij}} \qquad \textit{(Mills, Robbins, Rumsey 1982)}$$

& $\quad s_{(n-1,n-1,\ldots,2,2,1,1)}(x_1,\ldots,x_{2n}) \approx$ certain case of six-vertex model partition function

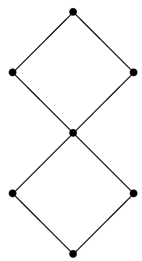
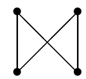
(Okada 2006)

Posets & Polytopes

- "The biggest lesson I learned from Richard Stanley is *combinatorial objects want to be partially ordered*. . . . A related lesson Stanley has taught me is *combinatorial objects want to belong to polytopes*." *(J. Propp 2016)*

- With certain natural order relations, $\{n\times n$ ASMs$\}$, $\{$order n ASTs$\}$, $\{$order n DPPs$\}$ & $\{$TSSCPPs in $2n\times 2n\times 2n$ box$\}$ are (nonisomorphic) distributive lattices.
  (Mills, Robbins, Rumsey 1982; Elkies, Kuperberg, Larsen, Propp 1992; Striker 2011; RB)

- # of join irreducibles is $\binom{n+1}{3}$ for $\{n\times n$ ASMs$\}$ & $\{$TSSCPPs in $2n\times 2n\times 2n$ box$\}$ & $(n^2+n-3)(n-1)/3$ for $\{$order n ASTs$\}$ & $\{$order n DPPs$\}$.

- e.g. the lattice of 3×3 ASMs is  & its poset of join irreducibles is .

- A polytope in $\mathbb{R}^{n^2}$ with $n\times n$ ASMs as vertices has also been studied.
  This contains the Birkhoff polytope. *(RB, Knight 2008, Striker 2009)*

More specifically, for any fixed n:

- the *Birkhoff polytope (polytope of doubly stochastic matrices)* is

$$\left\{ n\times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry is nonnegative} \\ \bullet \text{ each complete row \& column sum is 1} \end{array} \right\}$$

- the *alternating sign matrix polytope* is

$$\left\{ n\times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each partial row \& column sum extending from} \\ \text{ each end of the row or column is nonnegative} \\ \bullet \text{ each complete row \& column sum is 1} \end{array} \right\}$$

Some Reviews

- R. Stanley *A baker's dozen of conjectures concerning plane partitions* Lecture Notes in Math. **1234** (1985)
- D. Robbins *The story of 1, 2, 7, 42, 429, 7436, . . .* Math. Intelligencer **13** (1991)
- D. Bressoud & J. Propp *How the alternating sign matrix conjecture was solved* Notices Amer. Math. Soc. **46** (1999)
- D. Bressoud *Proofs and confirmations: the story of the alternating sign matrix conjecture* Cambridge University Press (1999) 274 pages
- J. Propp *The many faces of alternating sign matrices* Disc. Math. and Theor. Comp. Sci. Proc. **AA** (2001)
- J. de Gier *Loops, matchings and alternating sign matrices* Discrete Math. **298** (2005)
- RB *Multiply-refined enumeration of alternating sign matrices* Adv. Math. **245** (2013)
- C. Krattenthaler *Plane partitions in the work of Richard Stanley and his school* Amer. Math. Soc. (2016)