d-Complete Posets and Hook Formulas

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Introduction

Hook formulas for Young diagrams

Let λ be a partition of n.

- Frame–Robinson–Thrall

$$\#\text{standard tableaux of shape } \lambda = \frac{n!}{\prod_{v \in D(\lambda)} h_\lambda(v)}.$$

- Stanley (univariate q)

$$\sum_{\substack{\pi \,:\, \text{reverse plane partition} \\ \text{of shape } \lambda}} q^{|\pi|} = \frac{1}{\prod_{v \in D(\lambda)} (1 - q^{h_\lambda(v)})}.$$

- Gansner (multivariate $\boldsymbol{z} = (\cdots, z_{-1}, z_0, z_1, \cdots)$)

$$\sum_{\substack{\pi \,:\, \text{reverse plane partition} \\ \text{of shape } \lambda}} \boldsymbol{z}^\pi = \frac{1}{\prod_{v \in D(\lambda)} (1 - \boldsymbol{z}[H_{D(\lambda)}(v)])}.$$

Hook Formulas for d-Complete Posets

Theorem (Peterson) Let P be a d-complete poset. Then the number of linear extensions of P is given by

$$\frac{n!}{\prod_{v \in P} h_P(v)},$$

where $n = \#P$.

Theorem (Peterson–Proctor) Let P be a d-complete poset. Then the multivariate generating function of P-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P)} \boldsymbol{z}^{\pi} = \frac{1}{\prod_{v \in P} (1 - \boldsymbol{z}[H_P(v)])}.$$

Hook formulas for skew Young diagrams

Let $\lambda \supset \mu$ be a partitions such that $|\lambda| - |\mu| = n$.

- Naruse

  $$\#\text{standard tableaux of skew shape } \lambda/\mu = n! \sum_{D} \frac{1}{\prod_{v \in D(\lambda) \setminus D} h_\lambda(v)},$$

  where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$.

- Morales–Pak–Panova (univariate q)

  $$\sum_{\substack{\pi \,:\, \text{reverse plane partition} \\ \text{of skew shape } \lambda/\mu}} q^{|\pi|} = \sum_{D} \frac{\prod_{v \in B(D)} q^{h_\lambda(v)}}{\prod_{v \in D(\lambda) \setminus D} (1 - q^{h_\lambda(v)})},$$

  where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$.

Goal : Generalize these skew hook formulas to d-complete posets.

Plan

1. **Survey of d-complete posets**

2. **Skew hook formula for d-complete poset**

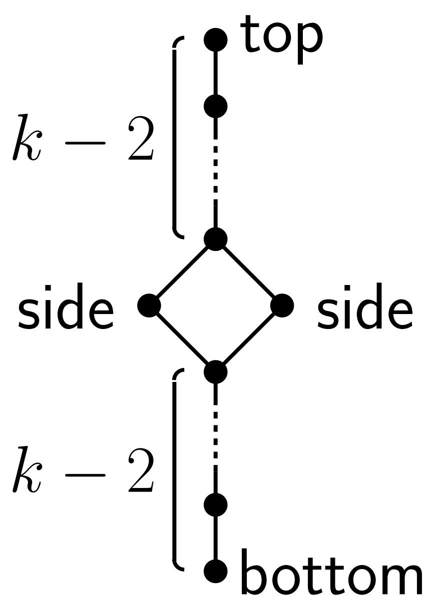
   (joint work with H. Naruse)

   Proof will be given in Naruse's talk (tomorrow morning) based on the equivariant K-theory of Kac–Moody partial flag varieties.

Survey of d-Complete Posets

Double-tailed Diamond

- The double-tailed diamond poset $d_k(1)$ ($k \geq 3$) is the poset depicted below:



- A d_k-interval is an interval isomorphic to $d_k(1)$.
- A d_k^--convex set is a convex subset isomorphic to $d_k(1) - \{\text{top}\}$.

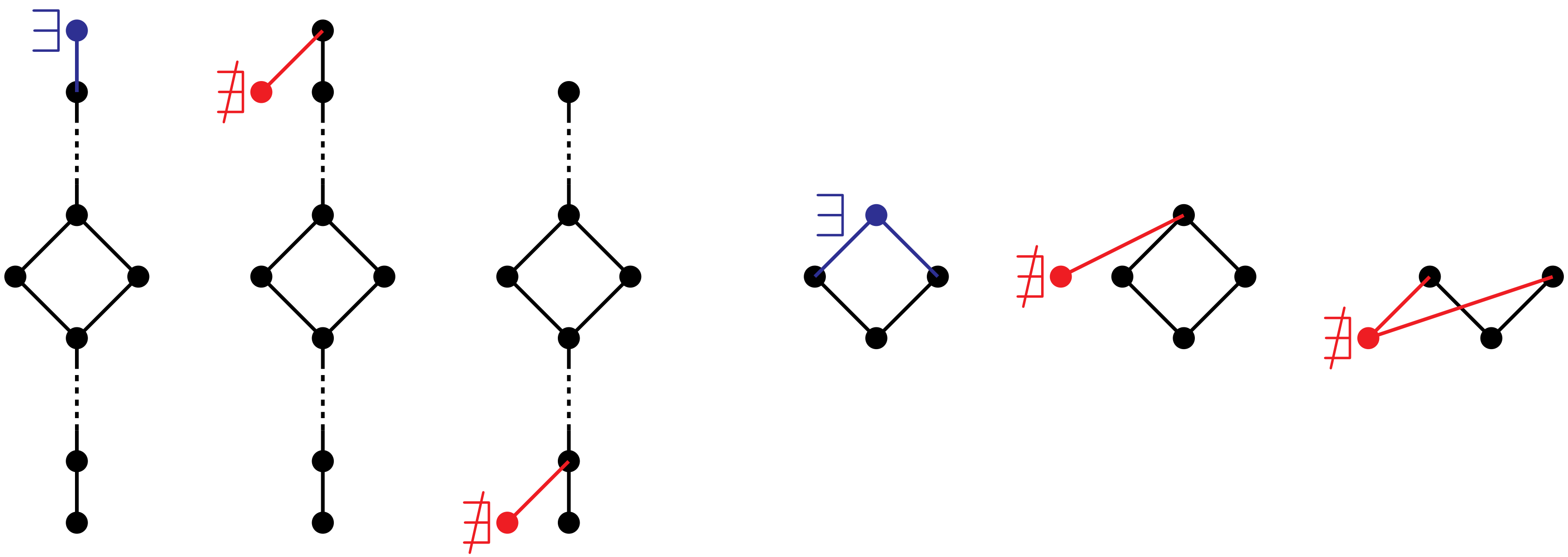
d-Complete Posets

Definition A finite poset P is d-complete if it satisfies the following three conditions for every $k \geq 3$:

(D1) If I is a d_k^--convex set, then there exists an element v such that v covers the maximal elements of I and $I \cup \{u\}$ is a d_k-interval.

(D2) If $I = [v, u]$ is a d_k-interval and u covers w in P, then $w \in I$.

(D3) There are no d_k^--convex sets which differ only in the minimal elements.



Example : Shape

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ ($\lambda_1 \geq \lambda_2 \geq \dots$), let $D(\lambda)$ be the Young diagram of λ given by

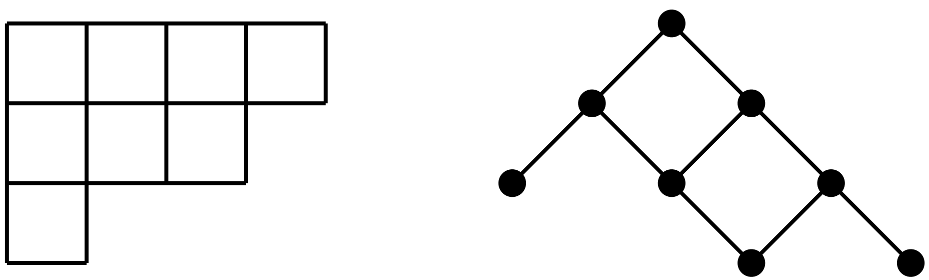
$$D(\lambda) = \{(i,j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\}.$$

The Young diagram is usually represented by replacing the lattice points with unit squares. We endow $D(\lambda)$ with a partial ordering defined by

$$(i,j) \geq (i',j') \iff i \leq i' \text{ and } j \leq j'.$$

The resulting poset is called a shape.

If $\lambda = (4,3,1)$, then the Young diagram $D(4,3,1)$ and the corresponding Hasse diagram are given as follows:



Example : Shifted Shape

For a strict partition $\mu = (\mu_1, \mu_2, \dots)$ ($\mu_1 > \mu_2 > \dots$), let $S(\mu)$ be the shifted Young diagram given by

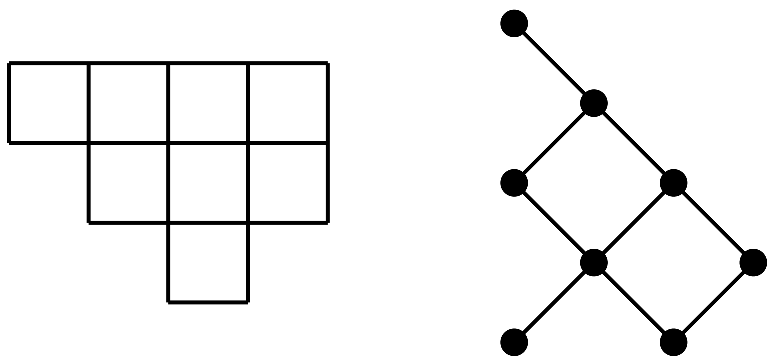
$$S(\mu) = \{(i,j) \in \mathbb{Z}^2 : i \le j \le \mu_i + i - 1\}.$$

We endow $S(\mu)$ with a partial ordering defined by

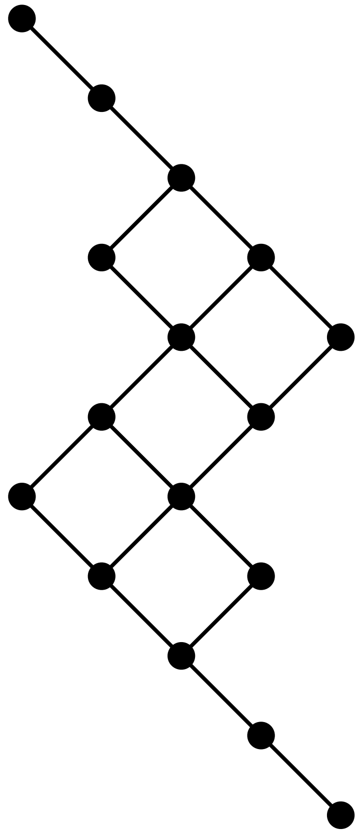
$$(i,j) \ge (i',j') \iff i \le i' \text{ and } j \le j'.$$

The resulting poset is called a shifted shape.

If $\mu = (4,3,1)$, then the shifted Young diagram $S(4,3,1)$ and the corresponding Hasse diagram are given as follows:



Example : Swivel



Classification

A poset is called connected if its Hasse daigram is connected. Each connected component of a d-complete poset is d-complete.

Fact

(a) If P is a connected d-complete poset, then P has a unique maximal element.

(b) Any connected d-complete poset is uniquely decomposed into a slant sum of one-element posets and slant-irreducible d-complete posets.

(c) Slant-irreducible d-complete posets are classified into 15 families :

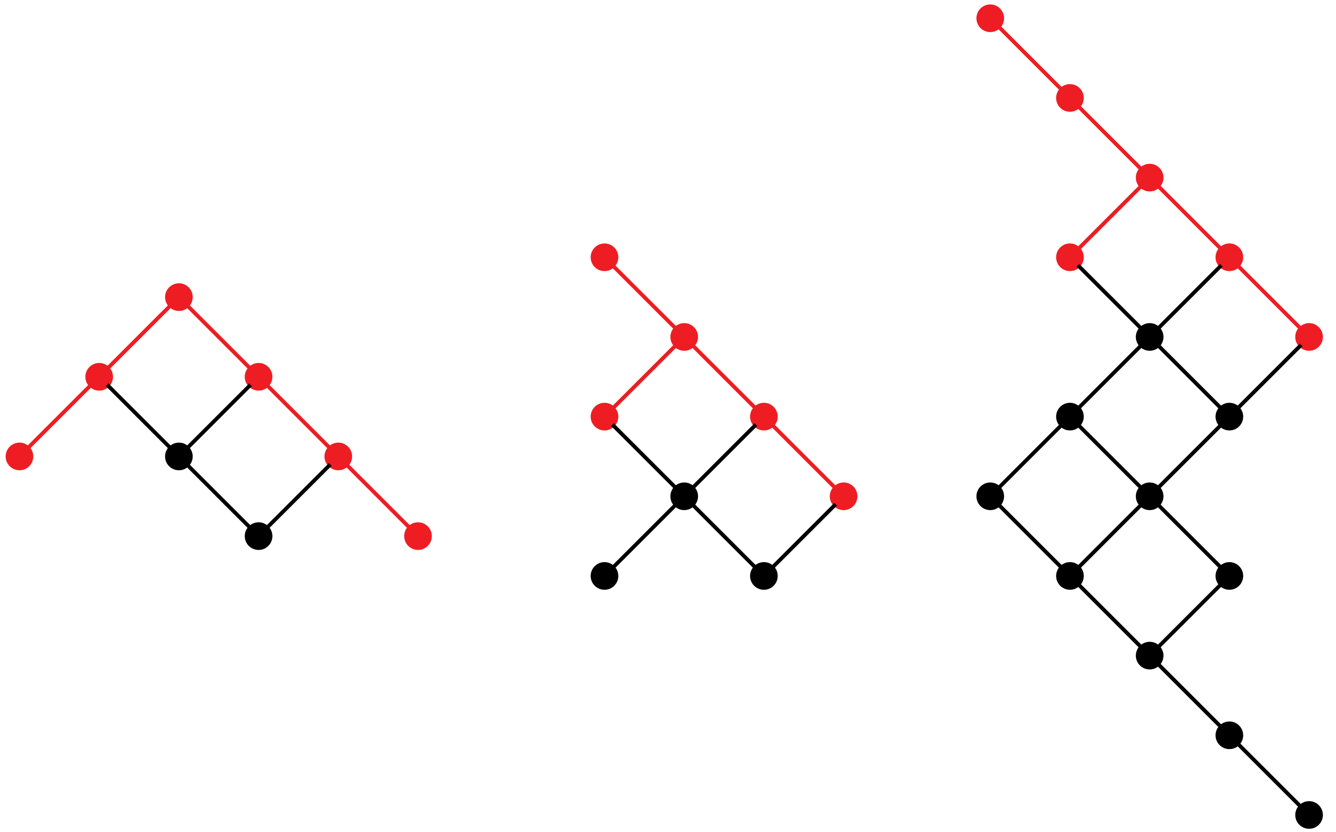
shapes, shifted shapes, birds, insets, tailed insets, banners, nooks, swivels, tailed swivels, tagged swivels, swivel shifts, pumps, tailed pumps, near bats, bat.

Top Tree

For a connected d-complete poset P, we define its top tree by putting

$$\Gamma = \{x \in P : \text{ every } y \geq x \text{ is covered by at most one other element } \}$$

Example



d-Complete Coloring

Fact Let P be a connected d-complete poset with top tree Γ. Let I be a set of colors such that $\#I = \#\Gamma$. Then a bijective labeling $c : \Gamma \to I$ can be uniquely extended to a map $c : P \to I$ satisfying the following three conditions:

- If x and y are incomparable, then $c(x) \neq c(y)$.
- If an interval $[v.u]$ is a chain, then the colors $c(x)$ ($x \in [v, u]$) are distinct.
- If $[v, u]$ is a d_k-interval then $c(v) = c(u)$.

Such a map $c : P \to I$ is called a d-complete coloring.

A d-complete coloring $c : P \to I$ satisfies

- If $c(x) = c(y)$ or the nodes labeled by $c(x)$ and $c(y)$ are adjacent in Γ, then x and y are comparable.

Example : d-Complete Coloring of a Shape

Let λ be a partition. Then the "content" function

$$\begin{array}{rcl} c : D(\lambda) & \longrightarrow & I = \{-\lambda'_1 + 1, \ldots, -1, 0, 1, \ldots, \lambda_1 - 1\} \\ (i, j) & \longmapsto & j - i \end{array}$$

is a d-complete coloring, where λ' is the conjugate partition of λ.

If $\lambda = (4, 3, 1)$, then this d-complete coloring is given by

0	1	2	3
-1	0	1	
-2			

Example : d-Complete Coloring of a Shifted Shape

Let μ be a strict partition. Then the "content" function

$$c : S(\mu) \longrightarrow \quad I = \{0, 0', 1, 2, \dots, \mu_1 - 1\}$$

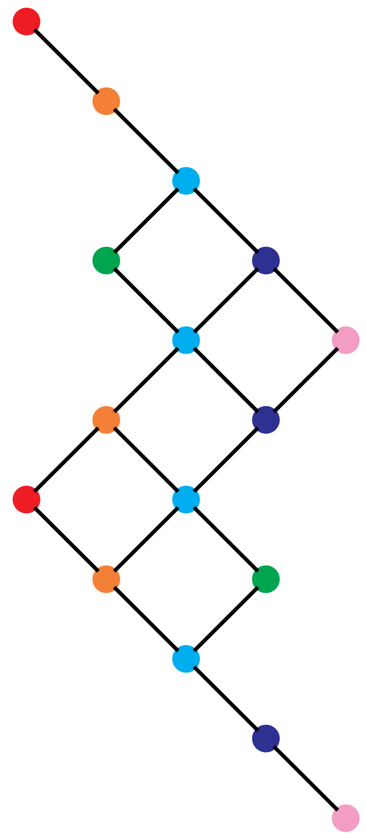
$$(i,j) \longmapsto \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd}, \\ 0' & \text{if } i = j \text{ and } i \text{ is even}. \end{cases}$$

is a d-complete coloring.

If $\mu = (4, 3, 1)$, then this d-complete coloring is given by

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 0' & 1 & 2 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

Example : d-Complete Coloring of a Swivel



Hook Lengths and Hook Monomials

Let P be a connected d-complete poset with top tree Γ and d-complete coloring $c : P \to I$. Let z_i $(i \in I)$ be indeterminate. For each $u \in P$, we define the hook length $h_P(u)$ and the hook monomial $\boldsymbol{z}[H_P(u)]$ inductively as follows:

(a) If u is not the top of any d_k-interval, then we define

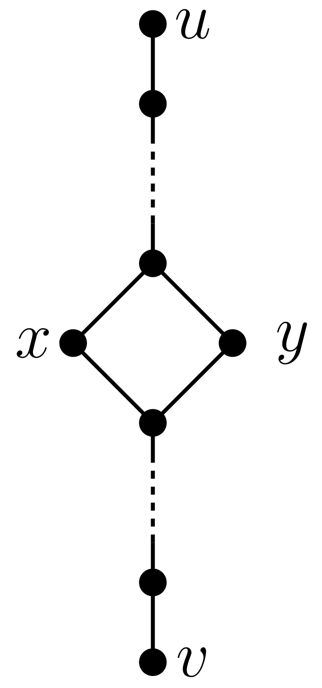
$$h_P(u) = \#\{w \in P : w \leq u\}, \quad \boldsymbol{z}[H_P(u)] = \prod_{w \leq u} z_{c(w)}.$$

(b) If u is the top of a d_k-interval $[v, u]$, then we define

$$h_P(u) = h_P(x) + h_P(y) - h_P(v),$$

$$\boldsymbol{z}[H_P(u)] = \frac{\boldsymbol{z}[H_P(x)] \cdot \boldsymbol{z}[H_P(y)]}{\boldsymbol{z}[H_P(v)]},$$

where x and y are the sides of $[v, u]$.



Example : Hooks in a Shape

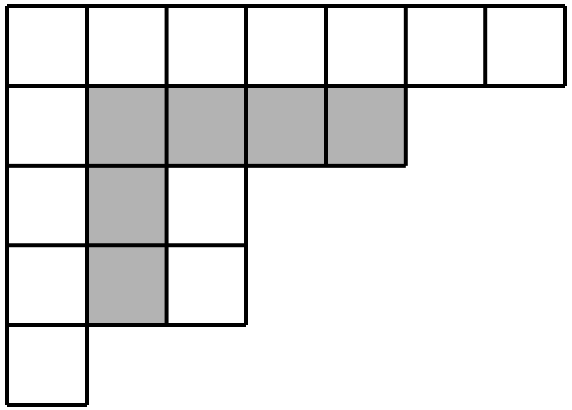
For a partition λ, the hook at (i,j) in $D(\lambda)$ is defined by

$$H_{D(\lambda)}(i,j) = \{(i,j)\} \cup \{(i,l) \in D(\lambda) : l > j\} \\ \cup \{(k,j) \in D(\lambda) : k > i\}.$$

Then we have

$$h_{D(\lambda)}(i,j) = \# H_{D(\lambda)}(i,j), \quad \boldsymbol{z}[H_{D(\lambda)}(i,j)] = \prod_{(k,l) \in H_{D(\lambda)}(i,j)} z_{c(i,j)}.$$

If $\lambda = (7,5,3,3,1)$, then the hook at $(2,2)$ in $D(7,5,3,3,1)$ is



Example : Hooks in a Shifted Shape

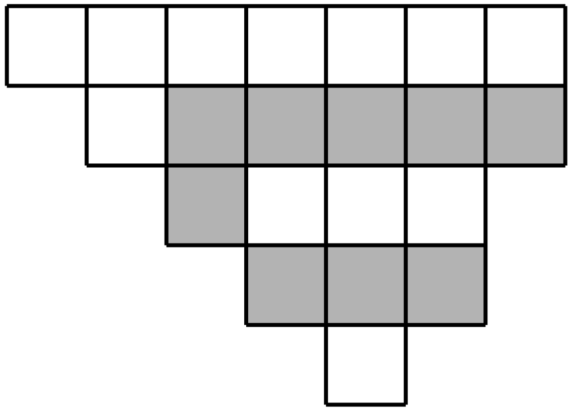
For a strict partition μ, the shifted hook at (i,j) in $S(\mu)$ is defined by

$$\begin{aligned} H_{S(\mu)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in S(\mu) : l > j\} \\ &\cup \{(k,j) \in S(\mu) : k > i\} \\ &\cup \{(j+1,l) \in S(\mu) : l > j\}. \end{aligned}$$

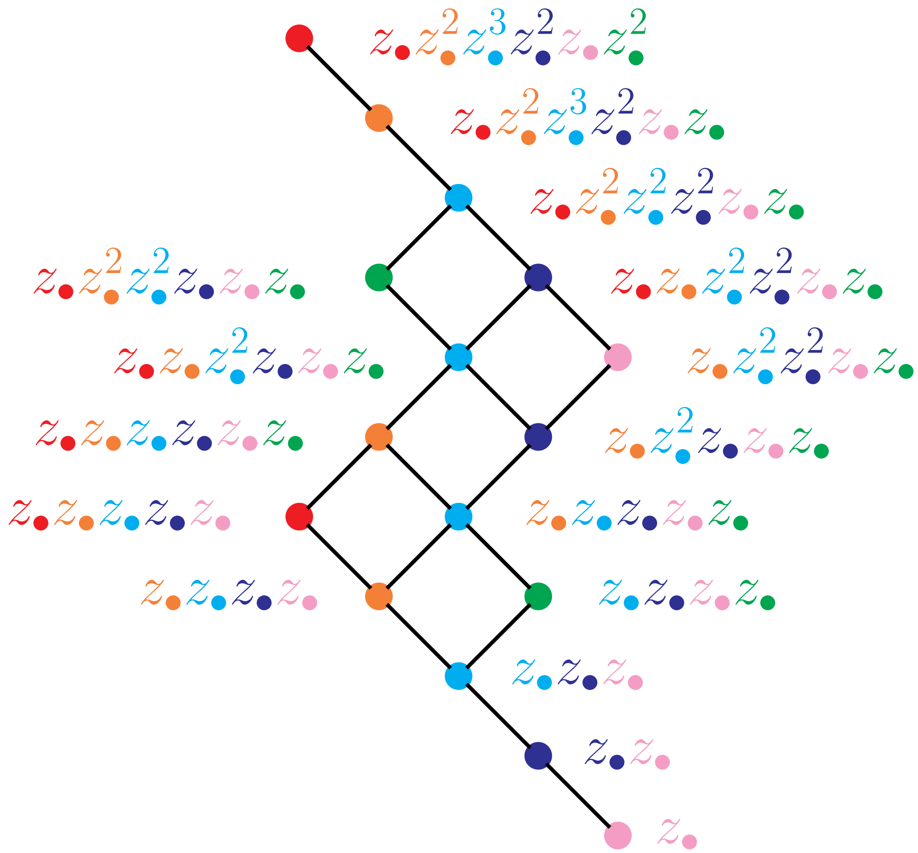
Then we have

$$h_{S(\mu)}(i,j) = \# H_{S(\mu)}(i,j), \quad \boldsymbol{z}[H_{S(\mu)}(i,j)] = \prod_{(k,l) \in H_{S(\mu)}(i,j)} z_{c(i,j)}.$$

If $\mu = (7,6,4,3,1)$, then the shifted hook at $(i,j) = (2,3)$ in $S(7,6,4,3,1)$ is



Example : Hook Monomials in Swivel



Linear Extensions and P-partitions

Let P be a poset. A linear extension of P is a bijection $\sigma : P \to \{1, 2, \dots, n\}$ ($n = \#P$) satsifying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \sigma(x) \leq \sigma(y) \text{ in } \mathbb{Z}.$$

A P-partition is a map $\pi : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \pi(x) \geq \pi(y) \text{ in } \mathbb{N}.$$

Let $\mathcal{A}(P)$ be the set of P-partitions:

$$\mathcal{A}(P) = \{\sigma : P \to \mathbb{N} : P\text{-partition}\}.$$

Let P be a connected d-complete poset with d-complete coloring $c : P \to I$ and $\boldsymbol{z} = (z_i)_{i \in I}$ indeterminates. For a P-partition π, we define

$$\boldsymbol{z}^{\pi} = \prod_{v \in P} z_{c(v)}{}^{\pi(v)}.$$

Example : Reverse Plane Partitions

If $P = D(\lambda)$ is a shape, then

$$D(\lambda)\text{-partition} = \text{reverse plane partition of shape } \lambda,$$

and

$$\boldsymbol{z}^{\pi} = \prod_{(i,j) \in D(\lambda)} {z_{j-i}}^{\pi(i,j)}.$$

For example,

$$\pi = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 3 & 3 \\ \hline 1 & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

is a reverse plane partition of shape $(4, 3, 1)$ and

$$\boldsymbol{z}^{\pi} = z_{-2}^{2} z_{-1}^{1} z_{0}^{0+1} z_{1}^{1+3} z_{2}^{3} z_{3}^{3}.$$

Example : Shifted Reverse Plane Partitions

If $P = S(\mu)$ is a shifted shape, then

$S(\mu)$-partition $=$ shifted reverse plane partition of shifted shape μ,

and

$$\boldsymbol{z}^{\pi} = \prod_{\substack{(i,i)\in S(\mu) \\ i:\text{odd}}} z_0^{\pi(i,i)} \prod_{\substack{(i,i)\in S(\mu) \\ i:\text{even}}} z_{0'}^{\pi(i,i)} \prod_{\substack{(i,j)\in S(\mu) \\ i<j}} z_{j-i}^{\pi(i,j)}.$$

For example,

$$\pi = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 1 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}$$

is a shifted reverse plane partition of shited shape $(4,3,1)$ and

$$\boldsymbol{z}^{\pi} = z_0^{0+3} z_{0'}^{1} z_1^{1+2} z_2^{1+3} z_3^{3}.$$

Hook Formulas for d-Complete Posets

Theorem (Peterson) Let P be a d-complete poset. Then the number of linear extensions of P is given by

$$\frac{n!}{\prod_{v \in P} h_P(v)},$$

where $n = \#P$.

Theorem (Peterson–Proctor) Let P be a d-complete poset. Then the multivariate generating function of P-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P)} \boldsymbol{z}^{\pi} = \frac{1}{\prod_{v \in P}(1 - \boldsymbol{z}[H_P(v)])}.$$

Different proofs are given by Peterson–Proctor, Ishikawa–Tagawa and Nakada. Our skew hook formula provides an alternate proof.

d-Complete Posets and Weyl Groups, Root Systems

Let P be a d-complete poset with top tree Γ and d-complete coloring $c : P \to I$. By regarding Γ as a simply-laced Dynkin diagram with node set labeled by I, we obtain

- Weyl group $W = \langle s_i : i \in I \rangle$,
- root system Φ and simple roots $\{\alpha_i : i \in I\}$,
- coroot system $\Phi^\vee$ and simple coroots $\{\alpha_i^\vee : i \in I\}$,
- fundamental weight λ_P corresponding to the color i_P of the maximum element of P,
- Weyl group element

$$w_P = s_{c(p_1)} s_{c(p_2)} \cdots s_{c(p_n)},$$

where we label the elements of P with $p_1, p_2, \dots, p_n$ so that $p_i < p_j$ implies $i < j$.

d-Complete Posets and Weyl Groups, Root Systems (cont.)

(1) w_P is λ_P-minuscule, i.e.,

$$\langle \alpha^\vee_{c(p_k)}, s_{c(p_{k+1})} \cdots s_{c(p_n)} \lambda_P \rangle = 1 \quad (1 \le k \le n).$$

(2) There is a poset isomorphism

$$\{\text{order filters of } P\} \ni F \overset{\sim}{\longmapsto} w_F \in [e, w_P],$$

where $[e, w_P]$ is the interval in $W^{\lambda_P} = W/W_{\lambda_P}$, the set of minimum length coset representatives w.r.t. the stabilizer W_{λ_P}.

(3) There exists a bijection $P \ni v \mapsto \beta(v) \in \Phi(w_P) = \Phi_+ \cap w_P \Phi_-$ such that

$$\boldsymbol{z}[H_P(v)]\Big|_{z_i = e^{\alpha_i}} = e^{\beta(v)} \quad (v \in P).$$

(4) P is ismorphic to the order dual of $\Phi^\vee(w_P^{-1}) = \Phi^\vee_+ \cap w_P^{-1} \Phi^\vee_-$ as posets.

Skew Hook Formula for d-Compete Posets

Hook formulas for skew Young diagrams

Let $\lambda \supset \mu$ be a partitions such that $|\lambda| - |\mu| = n$.

- Naruse

  $$\#\text{standard tableaux of skew shape } \lambda/\mu = n! \sum_{D} \frac{1}{\prod_{v \in D(\lambda) \setminus D} h_\lambda(v)},$$

  where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$.

- Morales–Pak–Panova (univariate q)

  $$\sum_{\substack{\pi \,:\, \text{reverse plane partition} \\ \text{of skew shape } \lambda/\mu}} q^{|\pi|} = \sum_{D} \frac{\prod_{v \in B(D)} q^{h_\lambda(v)}}{\prod_{v \in D(\lambda) \setminus D} (1 - q^{h_\lambda(v)})},$$

  where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$, and $B(D)$ is the set of excited peaks.

Excited Diagrams for Young Diagrams

Let D be a subset of the Young diagram $D(\lambda)$.

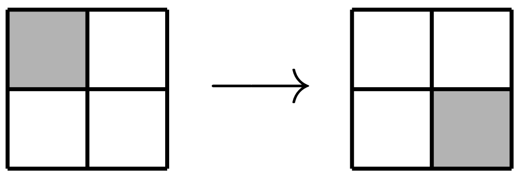
- We say that $u = (i, j) \in D$ is D-active if

$$(i, j+1),\ (i+1, j),\ (i+1, j+1) \in D(\lambda) \setminus D.$$

- If $u = (i, j)$ is D-active, then we define

$$\alpha_u(D) = D \setminus \{(i, j)\} \cup \{(i+1, j+1)\}.$$

- We say that D is an excited diagram of $D(\mu) \subset D(\lambda)$ if D is obtained from $D(\mu)$ after a sequence of elementary excitations $D \to \alpha_u(D)$.



(Cells of excited diagrams are shaded.)

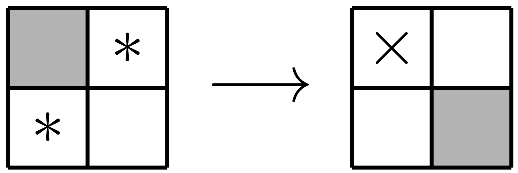
Excited Peaks for Young Diagrams

To an exicted diagram D of $D(\mu)$ in $D(\lambda)$, we associate a subset $B(D) \subset D(\lambda) \setminus D$, called the set of excited peaks of D as follows:

(a) If $D = D(\mu)$, then we define $B(D(\mu)) = \emptyset$.

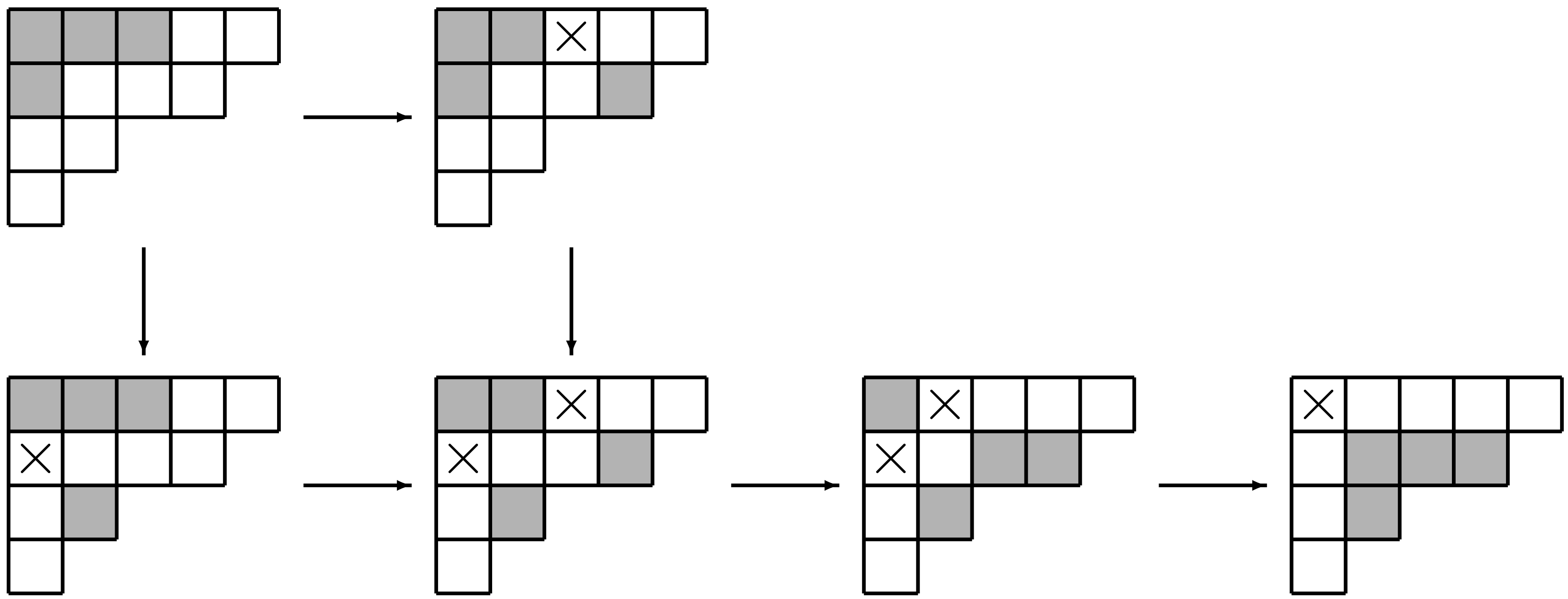
(b) If $D' = \alpha_u(D)$ is obtained from D by an elementary excitation at $u = (i, j) \in D$, then

$$B(\alpha_u(D)) = B(D) \setminus \{(i, j+1), (i+1, j)\} \cup \{(i, j)\}.$$



(Excitec peaks are marked with $\times$, and the symbol $*$ stands for $\times$ or empty.)

Example If $\lambda = (5, 4, 2, 1)$ and $\mu = (3, 1)$, then there are 6 excited diagrams of $D(3, 1)$ in $D(5, 4, 2, 1)$.



Excited Diagrams for Shifted Young Diagrams

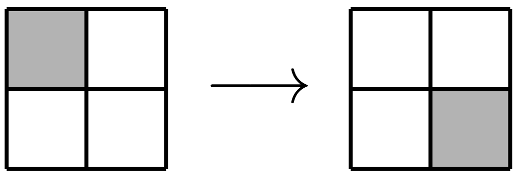
Let D be a subset of the shifted Young diagram $S(\mu)$.

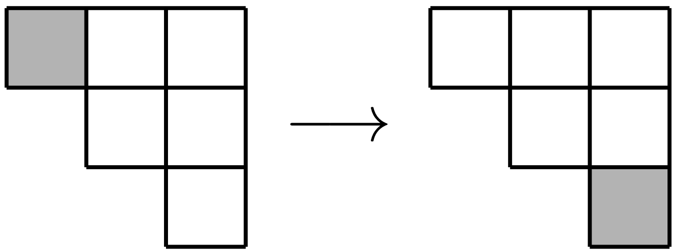
- We say that $u = (i, j) \in D$ is D-active if

$$i < j \text{ and } (i, j+1),\ (i+1, j),\ (i+1, j+1) \in S(\mu) \setminus D, \text{ or}$$
$$i = j \text{ and } (i, i+1),\ (i+1, i+2),\ (i+2, i+2) \in S(\mu) \setminus D.$$

- If $u = (i, j)$ is D-active, then we define

$$\alpha_u(D) = \begin{cases} D \setminus \{(i,j)\} \cup \{(i+1, j+1)\} & \text{if } i < j, \\ D \setminus \{(i,i)\} \cup \{(i+2, i+2)\} & \text{if } i = j. \end{cases}$$





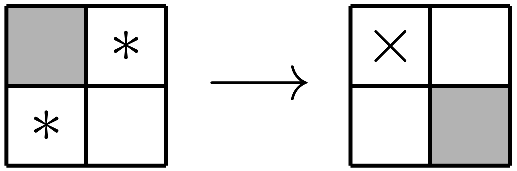
Excited Peaks for Shifted Young Diagrams

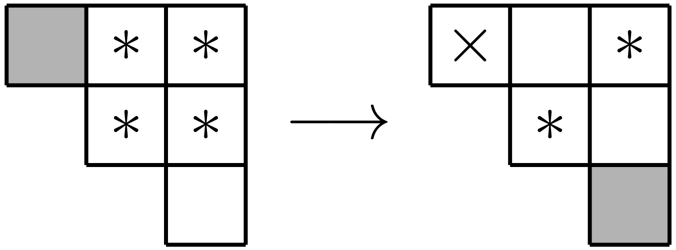
To an excited diagram D of $S(\nu)$ in $S(\mu)$, we associate a subset $B(D) \subset S(\mu) \setminus D$, called the subset of excited peaks of D as follows:

(a) If $D = S(\nu)$, then we define $B(S(\nu)) = \emptyset$.

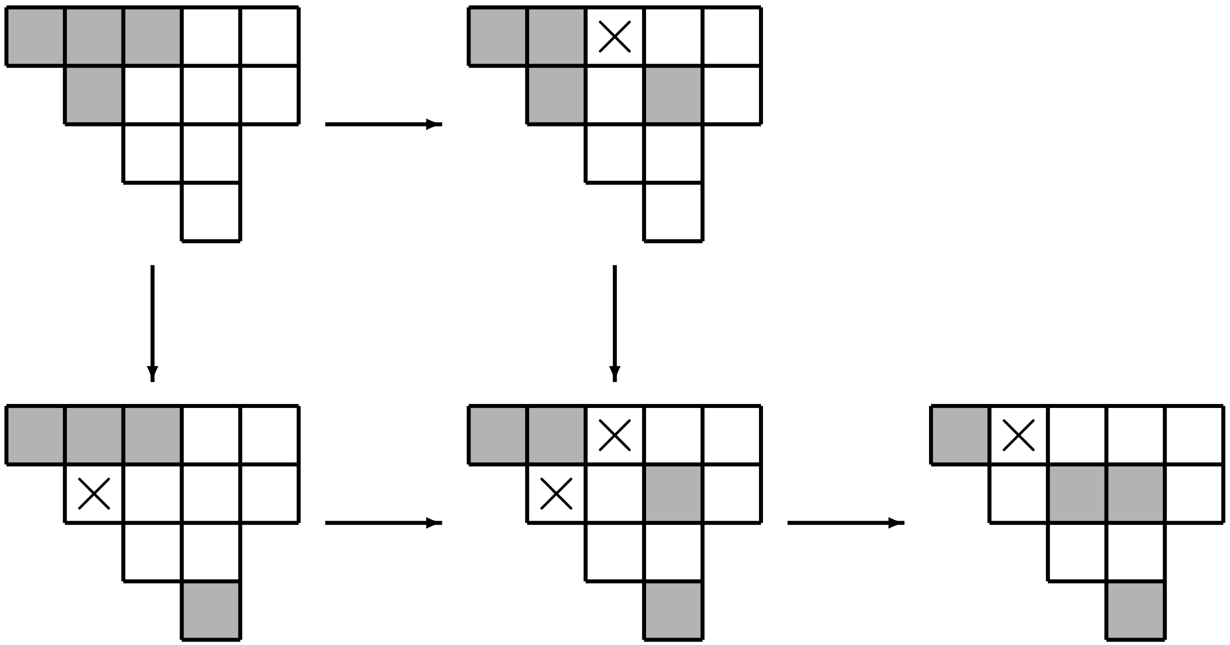
(b) If $D' = \alpha_u(D)$ is obtained from D by an elementary excitation at $u = (i, j) \in D$, then

$$B(\alpha_u(D)) = \begin{cases} B(D) \setminus \{(i, j+1), (i+1, j)\} \cup \{(i, j)\} & \text{if } i < j, \\ B(D) \setminus \{(i, i+1), (i+1, i+2)\} \cup \{(i, i)\} & \text{if } i = j. \end{cases}$$





Example If $\mu = (5,4,2,1)$ and $\mu = (3,1)$, then there are 5 excited diagrams of $S(3,1)$ in $S(5,4,2,1)$.

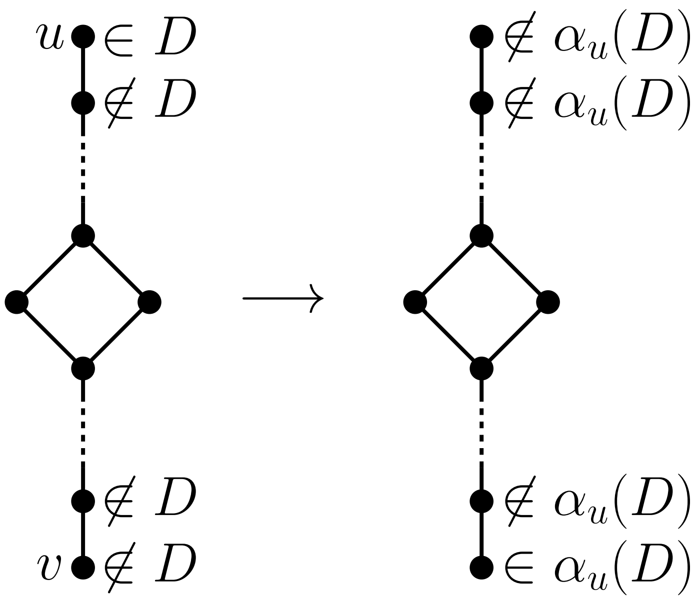


Excited Diagrams for d-Complete Posets

Let P be a connected d-complete poset.

- We say that $u \in D$ is D-active if there is a d_k-interval $[v, u]$ with $v \notin D$ such that

$$z \in [v, u] \text{ and } \begin{cases} z \text{ is covered by } u \\ \text{or} \\ z \text{ covers } v \end{cases} \implies z \notin D.$$



- If $u \in D$ is D-active, then we define

$$\alpha_u(D) = D \setminus \{u\} \cup \{v\}.$$

Let F be an order filter of P.

- We say that D is an excited diagram of F in P if D is obtained from F after a sequence of elementary excitations $D \to \alpha_u(D)$.

Excited Peaks for d-Complete Posets

Let P be a d-complete poset and F an order filter of P. To an exicted diagram D of F in P, we associate a subset $B(D) \subset P \setminus D$, called the subset of excited peaks of D as follows:

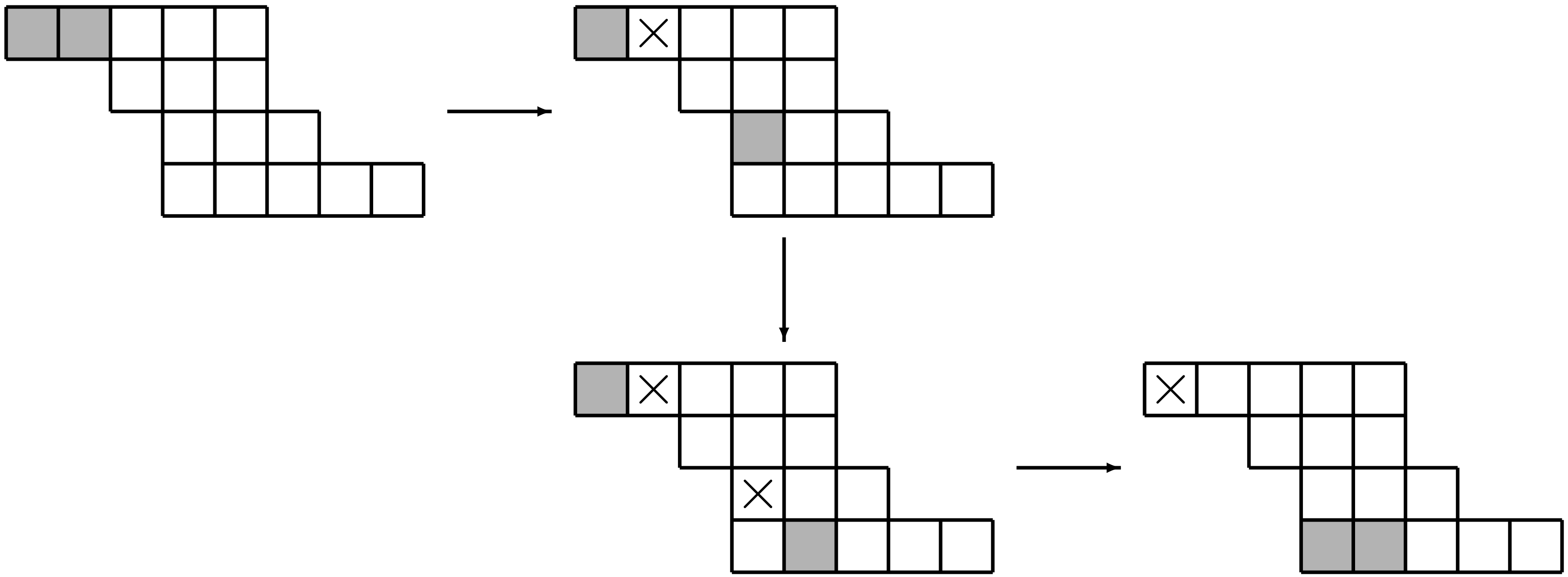
(a) If $D = F$, then we define $B(F) = \emptyset$.

(b) If $D' = \alpha_u(D)$ is obtained from D by an elementary excitation at $u \in D$, then

$$B(\alpha_u(D)) = D \setminus \left\{ z \in [v,u] : \begin{array}{l} z \text{ is covered by } u \\ \text{or } z \text{ covers } v \end{array} \right\} \cup \{v\},$$

where $[v,u]$ is the d_k-interval with top u.

Example If P is the Swivel and an order filter F has two elements, then there are 4 exited diagrams of F in P.



Main Theorem

Theorem (Naruse–Okada) Let P be a connected d-complete poset and F an order filter of P. Then the multivariate generating function of $(P \setminus F)$-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} \boldsymbol{z}^{\pi} = \sum_{D} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D} (1 - \boldsymbol{z}[H_P(v)])},$$

where D runs over all excited diagrams of F in P.

Corollary Let P be a connected d-complete poset and F an order filter of P. Then the number of linear extensions of $P \setminus F$ is given by

$$n! \sum_{D} \frac{1}{\prod_{v \in P \setminus D} h_P(v)},$$

where $n = \#(P \setminus F)$ and D runs over all excited diagrams of F in P.

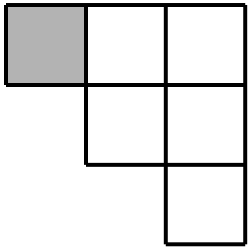
Example If $P = S(3,2,1)$ and $F = S(1)$, then we have

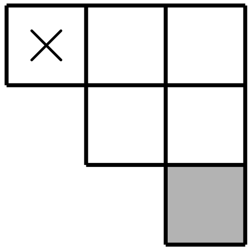
$$\sum_{\pi \in \mathcal{A}(S(3,2,1)\setminus S(1))} \boldsymbol{z}^{\pi}$$

$$= \frac{1}{(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)(1-z_0)}$$

$$+ \frac{z_0z_{0'}z_1^2z_2}{(1-z_0z_{0'}z_1^2z_2)(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)}$$

$$= \frac{1-z_0^2z_{0'}z_1^2z_2}{(1-z_0z_{0'}z_1^2z_2)(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)(1-z_0)}$$





Idea of Proof

Given a connected d-complete poset P with top tree Γ, we can associate the Weyl group W, the fundamental weight λ_P, ..., and the Kac–Moody partial flag variety $\mathcal{X}$. By using the equivariant K-theory $K_{\mathcal{T}}(\mathcal{X})$ of $\mathcal{X}$, we obtain

$$\xi^v|_w \in \mathbb{Z}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Z}e^{\lambda} \quad (v, w \in W^{\lambda_P}),$$

where Λ is the weight lattice. Main Theorem follows from

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} \boldsymbol{z}^{\pi} = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}} = \sum_{D} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D}(1 - \boldsymbol{z}[H_P(v)])},$$

where $z_i = e^{\alpha_i}$ $(i \in I)$ and D runs over all excited diagrams of F in P.

Excited Diagrams, Excited Peaks and Weyl Groups

Fix a labeling of elements of P with $p_1, \dots, p_n$ such that $p_i < p_j$ implies $i < j$. For a subset $D = \{i_1, \dots, i_r\}$ ($i_1 < \cdots < i_r$), we define

$$w_D = s_{c(p_{i_1})} s_{c(p_{i_2})} \cdots s_{c(p_{i_r})}, \quad w^*_D = s_{c(p_{i_1})} * s_{c(p_{i_2})} * \cdots * s_{c(p_{i_r})},$$

where $*$ is the Demazure product given by

$$s_i * w = \begin{cases} s_i w & \text{if } l(s_i w) = l(w) + 1, \\ w & \text{if } l(s_i w) = l(w) - 1. \end{cases}$$

Proposition Let F be an order filter of P. For a subset E of P, we have

$$w_E = w_F \text{ and } \#E = \#F \iff E \text{ is an excited diagram of } F \text{ in } P,$$

$$w^*_E = w_F \iff \begin{cases} E = D \sqcup S \\ \text{for some excited diagrams } D \text{ of } F \text{ in } P \\ \text{and a subset } S \subset B(D) \end{cases}$$