LABELED CONFIGURATION SPACES AND GROUP COMPLETIONS

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Abstract. Given a pair of an abelian partial monoid $M$ and a pointed space $X$, let $C^M(R^\infty, X)$ denote the configuration space of finite distinct points in $R^\infty$ parametrized by the partial monoid $X \wedge M$. In this note we will show that if $M$ is embedded in a topological abelian group then the natural map $C^M(R^\infty, X) \to C^\pm M(R^\infty, X)$, induced by the inclusion $M \subset \pm M = \{a - b | a, b \in M\}$, is a group completion. This generalizes the result of Caruso [1] that the space of “positive and negative particles” in $R^\infty$ parametrized by $X$ is weakly equivalent to $\Omega^\infty \Sigma^\infty X$.

1. Introduction

By an abelian partial monoid we shall mean a pointed space $M$ with subspaces $M_J \subset M^J$ for every finite ordered set $J$ and maps $M_J \to M$, written $(a_j) \mapsto \sum_{j \in J} a_j$, satisfying the conditions below. For $J = \{1, 2, \ldots, n\}$ we write $M_J = M_n$.

1. $M_0$ consists of a unique element 0, called the unit of $M$.
2. $M_1$ coincides with $M$ and the map $M_1 \to M$ is the identity of $M$.
3. Suppose that $J$ is the union $J_1 \cup \cdots \cup J_r$, where $J_m \cap J_n = \emptyset$ if $m \neq n$. Let $(a_j)$ be an element of $M^J$ such that $(a_j)_{j \in J_k} \in M_{J_k}$ for $1 \leq k \leq r$. Then $(a_j) \in M_J$ if and only if $(\sum_{j \in J_1} a_j, \ldots, \sum_{j \in J_r} a_j) \in M_r$, and

$$\sum_{j \in J} a_j = \sum_{j \in J_1} a_j + \cdots + \sum_{j \in J_r} a_j$$

holds whenever either side of the equation exists.

For example, arbitrary subset $M$ of a topological abelian monoid such that $0 \in M$ can be regarded as an abelian partial monoid, where $M_J = \{(a_j) \in M^J | \sum_{j \in J} a_j \in M\}$.

In [5] we assigned to any space $Y$ and any abelian partial monoid $M$ the configuration space $C^M(Y)$ of finite subsets of $Y$ with labels in $M$. As a set $C^M(Y)$ consists of those pairs $(S, \sigma)$, where $S$ is a finite subset of $Y$ and $\sigma$ is a map $S \to M$. But $(S, \sigma)$ is identified with $(S', \sigma')$ when $S \subset S'$, $\sigma'|S = \sigma$, and $\sigma'(x) = 0$ if $x \not\in S$. It should be noted that the topology of $C^M(Y)$ depends not only on the topologies of $Y$ and $M$, but also on the partial monoid structure of $M$.

For any pointed space $X$ let

$$C^M(R^\infty, X) = C^{X \wedge M}(R^\infty).$$

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Here $X \wedge M$ is regarded as an abelian partial monoid such that $x_1 \wedge a_1, \cdots, x_k \wedge a_k$ are summable if and only if $x_1 = \cdots = x_k$ and $a_1, \cdots, a_k$ are summable in $M$. In such a case we have $\sum_{j=1}^{k}(x \wedge a_j) = x \wedge (\sum_{j=1}^{k}a_j)$.

Let $E^M(X) = \Omega C^M(\mathbb{R}^\infty, \Sigma X)$, where $\Sigma X$ is the reduced suspension of $X$. As $C^M(\mathbb{R}^\infty, X)$ is a continuous functor of $X$, there exists a natural map

$$C^M(\mathbb{R}^\infty, X) \to \Omega C^M(\mathbb{R}^\infty, \Sigma X) = E^M(X).$$

The results of [5] imply the following.

1. The map $C^M(\mathbb{R}^\infty, X) \to E^M(X)$ is a group completion, that is, induces an isomorphism of Pontrjagin ring $H_\bullet(C^M(\mathbb{R}^\infty, X))[\pi^{-1}] \cong H_\bullet(E^M(X))$ for any commutative coefficient ring, where $\pi = \pi_0 C^M(\mathbb{R}^\infty, X)$.
2. The correspondence $X \mapsto \pi_\bullet E^M(X)$ defines a generalized homology theory on the category of pointed spaces.

Various homology theories arise in this way. For example, the stable homotopy and the ordinary homology theories correspond, respectively, to the subsets ${0}$ and $\mathbb{N} = \{0,1,2,\ldots\}$ of the additive group of integers $\mathbb{Z}$. (The former is a consequence of the Barratt-Priddy-Quillen theorem and the latter is the Dold-Thom theorem.) On the other hand, the connective $K$-homology theory arises from $\text{Gr}(\mathbb{R}^\infty)$, the Grassmannian of finite-dimensional subspaces of $\mathbb{R}^\infty$. Here $\text{Gr}(\mathbb{R}^\infty)_J$ consists of those tuples $(V_j)$ such that $V_i$ and $V_j$ are perpendicular if $i \neq j$, and $\sum_{j \in J} V_j$ is defined to be the direct sum $\bigoplus_{j \in J} V_j$. (Compare [4].)

The objective of this note is to show that when $M$ is nicely embedded in a topological abelian group then we can take $C^{\pm M}(\mathbb{R}^\infty, X)$ as a group completion of $C^M(\mathbb{R}^\infty, X)$, where $\pm M = \{a - b \mid a, b \in M\}$.

More precisely, we say that a subset $M$ of a topological abelian group is acceptable if $M$ contains $0$ and the map $M^2 \to \pm M$, $(a, b) \mapsto a - b$, is a fibration in the sense of Serre. Then we have

**Theorem 1.** Let $M$ be an acceptable subset a topological abelian group. Then the natural map $C^M(\mathbb{R}^\infty, X) \to C^{\pm M}(\mathbb{R}^\infty, X)$, induced by the inclusion $M \subset \pm M$, is a group completion for any pointed space $X$.

This implies that there is a natural isomorphism of homology theories $\pi_\bullet E^M(X) \cong \pi_\bullet C^{\pm M}(\mathbb{R}^\infty, X)$.

In particular, let $M = \{0,1\} \subset \mathbb{Z}$. Then $C^M(\mathbb{R}^\infty, X) = C(\mathbb{R}^\infty, X)$ is the standard configuration space of finite subsets of $\mathbb{R}^\infty$ parametrized by $X$. On the other hand, $C^{\pm M}(\mathbb{R}^\infty, X) = C^{\pm}(\mathbb{R}^\infty, X)$ is the space of positive and negative particles introduced by Mcduff [2]. Caruso has shown in [1] that $C^{\pm}(\mathbb{R}^\infty, X)$ is weakly homotopy equivalent to $\Omega^{\infty} \Sigma^{\infty} X$ if $X$ is locally equi-connected. By using Theorem 1 this can be generalized, both in $M$ and in $X$, as follows.

**Theorem 2.** Let $M$ be a finite set of integers such that $0 \in M$. Suppose that $M$ contains at least one non-zero element and is stable under the involution $n \mapsto -n$. 
Let $\permutative category with respect to the operation $\Phi: D \to X$ be the realization of the diagonal simplicial set $X$, then there are positive integers $k$ and $l$ such that

$$\pm\{0,d\} \subset (\pm)^k M \subset (\pm)^l \{0,d\} \subset (\pm)^{k+l-1} M$$

holds, for we have $(\pm)^l \{0,d\} = \{0, \pm d, \ldots, \pm ld\}$. By Theorem 1 we have

$$\pi_* C^{±,0,d}(R^\infty, X) \cong \pi_* C^{±,k,M}(R^\infty, X).$$

We also have

$$\pi_* C^M(R^\infty, X) \cong \pi_* C^{±,k,M}(R^\infty, X),$$

because $C^M(R^\infty, X)$ is a grouplike $H$-space. Thus $C^M(R^\infty, X)$ is weakly equivalent to $C^{±,0,d}(R^\infty, X)$. But $C^{0,d}(R^\infty, X)$ is homeomorphic to the standard configuration space $C(R^\infty, X)$, hence its group completion $C^{±,0,d}(R^\infty, X)$ is weakly equivalent to $P^\infty \Sigma^\infty X$ by the Barratt-Priddy-Quillen theorem.

**Corollary 3.** If $M$ is a finite set of integers then $\pi_* E^M(X)$ is the stable homotopy of a pointed space $X$.

**2. Proof of Theorem 1**

In [5] we showed that there exists a $CW$ monoid $D^M(X)$ and a weak equivalence $\Phi: D^M(X) \to C^M(R^\infty, X)$ natural in $X$. Let us briefly recall the definitions.

Let $Q^M$ be the total topological category whose space of objects is the disjoint union $\coprod_{p \geq 0} M^p$, and whose morphisms from $(a_1) \in M^p$ to $(b_1) \in M^q$ are maps of finite sets $\theta: \{1, \ldots, p\} \to \{1, \ldots, q\}$ such that $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ holds for $1 \leq j \leq q$. Let $Q^M$ denote the classifying space of $Q^M$, that is, the realization of the nerve $[k] \mapsto N_k Q^M$. Then $Q^M$ is a homotopy commutative monoid, because $Q^M$ is a permutative category with respect to the operation

$$(a_1, \ldots, a_p) \cdot (b_1, \ldots, b_q) = (a_1, \ldots, a_p, b_1, \ldots, b_q).$$

For a pointed space $X$ let $D^M(X) = |S_* Q^{X\wedge M}|$ be the realization of the total singular complex of $Q^{X\wedge M}$. Then $D^M(X)$ inherits from $Q^{X\wedge M}$ a monoid structure with respect to which the weak equivalence

$$D^M(X) = |S_* Q^{X\wedge M}| \to Q^{X\wedge M}$$

is a map of topological monoids. Note that $D^M(X)$ is homeomorphic to the realization of the diagonal simplicial set

$$[k] \mapsto S_k N_k Q^{X\wedge M} = N_k Q^{S_k(X\wedge M)}.$$

Let us define $\Phi: D^M(X) \to C^M(R^\infty, X)$ to be the composite

$$D^M(X) = |N_* Q^{S_*(X\wedge M)}| \xrightarrow{\Phi'} |S_* C^{X\wedge M}(R^\infty)| \to C^{X\wedge M}(R^\infty)$$

where $\Phi'$ is a weak equivalence constructed in [5, §4]. Since $\Phi$ is a weak equivalence between $H$-spaces, Theorem 1 follows from
Proposition 4. If $M$ is an acceptable subset of a topological abelian group then the natural map $D^M(X) \to D^{\pm M}(X)$ is a group completion.

The rest of the note is devoted to the proof of this proposition.

Given a map of topological monoids $f: D \to D'$ let $B(D, D')$ denote the realization of the category $B(D, D')$ whose space of objects is $D'$ and whose space of morphisms is the product $D \times D'$, where $(d, d') \in D \times D'$ is regarded as a morphism from $d'$ to $f(d) \cdot d'$. Then there is a sequence of maps

$$D' = B(0, D') \to B(D, D') \to B(D, 0) = BD$$

induced by the maps $0 \to D$ and $D' \to 0$ respectively. Observe that $BD$ is the standard classifying space of the monoid $D$ and $B(D, D)$ is contractible when $f$ is the identity.

Let us consider the commutative diagram

$$
\begin{array}{ccc}
D^M(X) & \longrightarrow & B(D^M(X), D^M(X)) \\
\downarrow & & \downarrow_{B(1, i)} \\
D^{\pm M}(X) & \longrightarrow & B(D^M(X), D^{\pm M}(X))
\end{array}
$$

(2.1)

in which the upper and the lower sequences are associated with the identity and the inclusion $i: D^M(M) \to D^{\pm M}(X)$, respectively.

Lemma 5. The natural map $D^M(X) \to \Omega BD^M(X)$ is a group completion.

This follows from the fact that $D^M(X)$ is a homotopy commutative, hence admissible, monoid.

Lemma 6. The lower sequence in the diagram (2.1) is a homotopy fibration sequence with contractible total space $B(D^M(X), D^{\pm M}(X))$.

Proposition 4 can be deduced from this, because $D^M(X) \to D^{\pm M}(X)$ is equivalent to the group completion map $D^M(X) \to \Omega BD^M(X)$ under the equivalence $D^{\pm M}(X) \simeq \Omega BD^M(X)$.

It remains to prove Lemma 6. Since $D^M(X)$ acts on $D^{\pm M}(X)$ through homotopy equivalences, the diagram

$$
\begin{array}{ccc}
D^{\pm M}(X) & \longrightarrow & B(D^{\pm M}(X)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B(D^{\pm M}(X), 0)
\end{array}
$$

is homotopy-cartesian by Proposition 1.6 of [3]. This implies that the lower sequence in the diagram (2.1) is a fibration sequence.

To prove that $B(D^M(X), D^{\pm M}(X))$ is contractible, we may suppose both $M$ and $X$ are discrete sets, for $B(D^M(X), D^{\pm M}(X))$ is homeomorphic to the realization of the simplicial space

$$[k] \to B(D^{S_k M}(S_k X), D^{S_k (\pm M)}(S_k X)),$$
and $S_k(\pm M) = \pm S_k M$ holds by the assumption that $M$ is acceptable. We may also suppose $X = S^0$, for the argument below can be applied to the general case if we replace $M$, $-M$ and $\pm M$ by $X \wedge M$, $X \wedge (-M)$ and $X \wedge (\pm M)$, respectively.

Observe that $B(Q^M, Q^{\pm M})$ is homeomorphic to the realization of the diagonal simplicial set

$$[k] \mapsto E_k = N_k B(N_k Q^M, N_k Q^{\pm M}).$$

Let $\|E_*\|$ denote the thick realization of $E_*$. We shall show that the natural map $p: \|E_*\| \to |E_*| = B(Q^M, Q^{\pm M})$ is homotopic to the constant map. This implies that $B(Q^M, Q^{\pm M})$ is contractible, for $p$ is a homotopy equivalence. (See [3, Appendix A].)

For every element $a$ of $\pm M$ choose $a^+ \in M$ and $a^- \in -M$ such that $a = a^+ + a^-$. If $S = (a_j)$ is an element of $N_0 Q^{\pm M}$ then we write

$$S_+ = (a^+_j), \quad S_- = (a^-_j), \quad \overline{S} = (-a_j).$$

To any $S \in (\pm M)^p$ assign a chain of elements in $E_1$,

$$S \xleftarrow{(S_+ \cdot \nabla)} S_\pm \xrightarrow{\nabla \cdot (\nabla \cdot \nabla)} 0^p \xrightarrow{\nabla \cdot (\nabla \cdot \nabla)} 0$$

where $\nabla$ is the map $\{1, \ldots, 2p\} \to \{1, \ldots, p\}$ such that $\nabla(j) = \nabla(p + j) = j$ for $1 \leq j \leq p$ and $\nu$ is the unique map $0 \to \{1, \ldots, p\}$.

Let $\alpha^S: I \to B(Q^M, Q^{\pm M})$ be the corresponding path joining $S$ to $\emptyset$. Then the correspondence $(S, t) \mapsto \alpha^S(t)$ defines a homotopy

$$h: \|E_*\|_0 \times I \to B(Q^M, Q^{\pm M})$$

between the restriction of $p$ and the constant map. Here $\|E_*\|_0$ denotes the image of $E_0 \times \Delta^0$ in $\|E_*\|$.

We shall extend $h$ to a homotopy over $\|E_*\|$. Let $\theta: S \to T$ be an element of $N_1 Q^{\pm M}$, where $S \in (\pm M)^p$ and $T \in (\pm M)^q$. Let $[\theta]$ be the 1-cell of $B(Q^M, Q^{\pm M})$ corresponding to $\theta$. Then the commutative diagram below induces a homotopy $[\theta] \simeq *$ which extends the one already defined on $\partial [\theta] = S \cup T$:

$$\begin{array}{ccccccccc}
T & \xrightarrow{\theta} & T & \leftarrow & T_- & \xrightarrow{\theta} & 0^q & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{\alpha^S} & S \cdot 0^q & \xleftarrow{(S_+ \cdot T_+ \cdot \nabla)} & S_\pm \cdot T_+ \xrightarrow{(S_+ \cdot T_+ \cdot \nabla)} & 0^p \cdot 0^q & \leftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
S & \xrightarrow{\alpha^S} & S \cdot 0^q & \xleftarrow{(S_+ \cdot \nabla)} & S_- \cdot 0^q & \xleftarrow{(S_+ \cdot \nabla)} & 0^p \cdot 0^q & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{\alpha^S} & S & \leftarrow & S_- & \xrightarrow{\alpha^S} & 0^p & \leftarrow & 0 \\
\end{array}$$

(2.2)

Here $\tau$ is the switching map $T_+ \cdot S_- \cdot T_+ \to S_- \cdot T_+ \cdot T_+$. Thus $h$ can be extended to a homotopy over the image of $N_1 Q^{\pm M} \times \Delta^1$. 

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We need to extend the construction above to $N_n\mathbb{Q}^{\pm M}$ for all $n \geq 0$. Suppose that for every $D \in N_k\mathbb{Q}^{\pm M}$ with $k < n$, there exists a null homotopy of $D$ given by a chain of commutative diagrams
\begin{equation}
D \to D_0 \leftarrow D_- \to 0^D \leftarrow \emptyset
\end{equation}
which is compatible with face operators. Let $D' = (\theta_n, \ldots, \theta_1)$ be an element of $N_n\mathbb{Q}^{\pm M}$. Then the diagram similar to (2.2), but $S$ and $\theta$ are replaced by $D = (\theta_{n-1}, \ldots, \theta_1)$ and $\theta_n$ respectively, defines a null homotopy of $D'$ which extends the ones already defined on the faces $\partial_i D'$. One easily observes that the diagram (2.3) is compatible, up to natural isomorphism, with the multiplication of $\mathbb{Q}^{\pm M}$, and hence with the action by $\mathbb{Q}^M$. Thus we can show by induction on $n$ that $h$ extends to a homotopy $p \simeq \ast$ over $\|E_\bullet\|_n$, the image of $\prod_{k \leq n} E_k \times \Delta^k$ in $\|E_\bullet\|$. This completes the proof of the lemma.

**References**


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