# Short Time Asymptotics of a Certain Infinite Dimensional Diffusion Process

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ABSTRACT The main objective of this contribution is to prove the Varadhan type short-time asymptotics of an infinite dimensional diffusion process associated with a certain Dirichlet form. This paper gives a generalization of Fang's results of the Ornstein-Uhlenbeck process on an abstract Wiener space.

## 1 Introduction

Let $(E, H, \mu)$ be an abstract Wiener space, i.e., $E$ is a separable Banach space, $H$ is the Cameron-Martin space and $\mu$ is the Wiener measure on $E$. We will consider the following Dirichlet form on $E$:

$$\mathcal{E}(u, v) = \int_E (A(z)Du(z), Dv(z))_H \, d\mu(z), \tag{1.1}$$

where $A(z)$ is a positive symmetric bounded linear operator on $H$ and $Du$ denotes the $H$-derivative. It is well known that there exist diffusion processes $X = (X_t, P_z)$ associated with (1.1). (See Kusuoka [19] for details.)

In this paper, we will study the short time behavior of the transition probability $P_\mu(t, A, B)$ itself, where $P_\mu(t, A, B)$ denotes the probability of the diffusion processes $X$ starting from a set $A$ and reaching a set $B$ at time $t$. In the strict sense it is defined as follows:

$$P_\mu(t, A, B) := \int_A P_z(t, B)\mu(dz) \ .$$

The main object is to prove in Corollary 2.17 that there exists an appropriate distance $d(A, B)$ between two subsets $A$ and $B$ of $E$ (cf. Definition 2.7) such that

$$\lim_{t \to 0} 4t \log P_\mu(t, A, B) = -d(A, B)^2. \tag{1.2}$$

This small time asymptotic formula is called the Varadhan type asymptotics.

Similar results have been obtained for symmetric diffusion processes on some infinite dimensional spaces. S. Fang [9] proved the asymptotics of this type for the standard Ornstein-Uhlenbeck process on $(E, H, \mu)$. Note that this case is obtained by setting $A(z) = I_H$. T.S. Zhang [35], [36] and M. Hino [15] studied some general cases. Actually diffusion processes in [35] are given by the solution to a stochastic differential equation on $E$ and the square root of the diffusion coefficient satisfies Lipschitz continuity. In such a case, standard large deviation theory is applicable to the present problem. In our approach to this problem, we can include the case where $A(z)$ is not continuous in the topology of $E$ but it has a certain regularity in the $H$-direction (cf. Definition 2.1).

Let us explain our approach briefly. We prove the upper bound of the left-hand side in (1.2) similarly to [9], [35] by using Lyons-Zheng's decomposition theorem (cf. Proposition 5.1). Our proof of the lower bound is totally different from the previous works. We use Wang's parabolic Harnack inequality [33], which is "dimension free" and is valid under the condition that "Ricci curvature" of the diffusion process has a global lower bound.

Since Wang's original inequality was proved on Riemannian manifolds, we have to prove an extension to our settings. The proof of the lower bound of the transition probability by using a parabolic Harnack inequality is in some sense an infinite dimensional version of Li and Yau's argument [8] on Riemannian manifolds. A similar kind of argument was found in [24], [1].

The organization of this paper is as follows. In Section 2, we state our problems and main results. In Section 3, we study fundamental properties of our distance function. In Section 4, we formulate a dimensional free Harnack inequality for our setting. In Section 5, our main results are proved. In Section 6, we show the calculation of Ricci curvature of the Dirichlet form which we study in this paper. In Section 7, we show that the method using the Harnack inequality works for proving (1.2) without the assumption on the Ricci curvature in the case where $A(z)$ is smooth in the Fréchet sense. In Section 8, we present two examples. In Example 1, we discuss the diffusion process whose diffusion coefficient is discontinuous. In particular we shall consider a diffusion coefficient which is defined by multiple Wiener integrals. We think that standard large deviation method is not applicable to this example. In Example 2, we note that our lower estimates may hold in the case of the diffusion process arising from statistical mechanics. We will discuss the details in a forthcoming paper.

## 2 Formulation of Main Results

Let $(E, H, \mu)$ be an abstract Wiener space. We will consider a diffusion process $X := (X_t, P_z)$ on $E$ corresponding to the following Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$:

$$\begin{aligned} \mathcal{E}(u,v) &:= \int_E (A(z)Du(z), Dv(z))_H \mu(dz) , \qquad (2.3) \\ \mathcal{D}(\mathcal{E}) &:= \text{the completion of } \mathcal{FC}_b^\infty(E,\mathbb{R}) \text{ with respect to } \mathcal{E}_1^{1/2}\text{-norm,} \end{aligned}$$

where $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(E)}$. In this paper, $L_{(2)}(H,H)$ denotes the set of Hilbert-Schmidt operators on $H$ and $L(H,H)$ denotes the set of bounded linear operators on $H$.

We will assume the following regularity conditions for the coefficient operator $A(\cdot)$.

(A1) $A(\cdot) : E \to L(H,H)$ are measurable maps such that

$$\mathrm{esssup}_{z\in E} \|A(z)\|_{L(H,H)} \leq M_1 .$$

(A2) There exists $M_2 > 0$ such that $A(z) - M_2 I_H$ is a positive definite symmetric operator for any $z \in E$.

(A3) $A(\cdot)$ is $H$-continuous, i.e., for any $z \in E$, $A(z + \cdot) : H \to L(H,H)$ is continuous.

(A4)

$$A^{-1}(\cdot) \in H\text{-}UC(E, L(H,H))$$

The definition of $H$-$UC(E, L(H,H))$ is the following:

**Definition 2.1.** We will say a map $F(\cdot) : E \to L(H,H)$ belongs to $H$-$UC(E, L(H,H))$ if and only if the following holds.

(1) There exists a sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset E$ such that

$$\lim_{n\to\infty} \mu(K_n^C) = 0,$$

(2) For any $K_n \subset E, y \in K_n$ and $r > 0$,

$$\lim_{x\to y,\ x\in K_n} \left( \sup_{\|v\|_H \leq r} \|F(x+v) - F(y+v)\|_{L(H,H)} \right) = 0. \qquad (2.4)$$

Moreover we call $\{K_n\}_{n=1}^\infty$ the $H$-$UC$ nest and we say that $H$-$UC$ property holds for $K$ if (2.4) holds replacing $K_n$ by $K$.

**Remark 2.2.** In finite dimensional case, Norris [27] proved the Varadhan type asymptotics under ( $\mathrm{A}_1$ ) and ( $\mathrm{A}_2$ ) only. Hence it is natural to hope that our estimates are still valid without assuming ( $\mathrm{A}_3$ ) and ( $\mathrm{A}_4$ ). This will be studied in separate papers.

By definition, we note the following fundamental property.

**Proposition 2.3.** *Let* $F(\cdot) \in C(E, L(H,H))$ *. Then* $F(\cdot)$ *belongs to* $H-UC(E, L(H,H))$ *.*

*Proof.* By virtue of the tightness of the Wiener measure, there exists a sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset E$ such that $\lim_{n\to\infty} \mu(K_n^C) = 0$. Here we denote $U_H(r) := \{v \in H| \ \|v\|_H \le r\}$. We note that $F(\cdot)$ is uniformly continuous on compact set in $E$ and $K_n + U_H(r)$ is compact in $E$. Then the following holds for any $K_n \subset E$ and $y \in K_n$:

$$\begin{aligned}
&\lim_{x\to y,\ x\in K_n} \left( \sup_{\|v\|_H \le r} \|F(x+v) - F(y+v)\|_{L(H,H)} \right) \\
&\quad \le \lim_{x\to y} \left( \sup \left\{ \|F(w) - F(\eta)\|_{L(H,H)} \Big| \ w, \eta \in K_n + U_H(r), \right. \right. \\
&\qquad \left. \left. \|w - \eta\|_E \le \|x - y\|_E \right\} \right) \\
&\quad = 0.
\end{aligned}$$

This completes the proof. □

**Remark 2.4.** The solution of stochastic differential equation $X(t,x,w)$ and multiple Wiener integral $I_p(f)(w)$ are not continuous in the Fréchet sense. But we note that these are typical examples in $H$-$UC(E, L(H,H))$. Especially, we shall prove the $H$-$UC$ property of the multiple Wiener integral in Section 8.

We shall define $H_g$-*distance* as a generalization of $H$-*distance* in Fang [9].

**Definition 2.5.** For $x \in E$, we define $(H_x, g)$ as a Hilbert manifold with a Riemannian metric $g$ by $H_x := H + x, g^{-1}(x) := A(x)$. Let us define $d_g(x,y)$. If $y \notin H_x$, define $d_g(x,y) = \infty$ and if $y \in H_x$,

$$\begin{aligned}
d_g(x,y) \quad = \quad \inf \Bigg\{ & \left( \int_0^1 (g(x+h(s))\dot{h}(s), \dot{h}(s))_H ds \right)^{1/2} \\
& \Big| \ h \in C^1([0,1], H) \text{ and } h(0) = 0, h(1) = y - x \Bigg\},
\end{aligned}$$

with the convention $d_g(x,y) = \infty$ if the above set is empty.

**Remark 2.6.** By virtue of $(\mathrm{A}_1)$ and $(\mathrm{A}_2)$, the following identity holds:

$$M_1^{-1}\|x-y\|_H \leq d_g(x,y) \leq M_2^{-1}\|x-y\|_H \text{ for any } x \in E \text{ and } y \in H_x.$$

Next, we shall give the distance between two Borel measurable sets $A, B \subset E$.

**Definition 2.7.** Let $A, B \subset E$ be Borel measurable sets with $\mu(A), \mu(B) > 0$. We denote $d_g(x,A) := \inf_{y\in A} d_g(x,y)$ and define

$$\begin{aligned} S_A \quad := \quad & \Big\{ M \subset E \ \Big| \ M = \bigcup_{n=1}^{\infty} L_n \text{with } \mu(A \triangle M) = 0, \\ & \quad \text{where } L_n \text{ is a compact set} \\ & \quad \text{and} L_n \subset K_m \text{ holds for a certain } m \in \mathbb{N} \Big\}. \end{aligned} \tag{2.5}$$

Then we define $d_g(A,B)$ as follows:

$$d_g(A,B) := \sup_{M\in S_A, N\in S_B} \Big( \operatorname{essinf}_{x\in A} d_g(x,N), \operatorname{essinf}_{y\in B} d_g(y,M) \Big).$$

**Remark 2.8.** In the above definition, we have used the Borel measurability of the distance function $d_g(\cdot, M)$ and $d_g(\cdot, N)$. See Lemma 3.1 for details.

We will state the fundamental properties of distance $d_g$ as follows:

**Proposition 2.9.** (1) *Let $\mu(A), \mu(B) > 0$. Then $d_g(A,B) < \infty$ holds.*

(2) *Let $A', B' \subset E$ with $\mu(A \triangle A') = 0$ and $\mu(B \triangle B') = 0$. Then $d_g(A,B) = d_g(A',B')$ holds.*

(3)

$$\begin{aligned} d_g(A,B) \quad = \quad & \sup \Big\{ \operatorname{essinf}_{x\in A} d_g(x,N), \operatorname{essinf}_{y\in B} d_g(y,M) \Big| \\ & \qquad M \in S_A, N \in S_B, M \subset A, N \subset B \Big\}. \end{aligned}$$

*Proof.* We shall recall the ergodicity of Wiener measure $\mu$. That is, for $\mu(A), \mu(B) > 0$, there exists $v \in H$ such that

$$\mu\left(A \cap (B+v)\right) > 0.$$

Using this property, we can easily see that (1) holds. By the definition, we can get (2) and (3). □

Using this distance, we will state our main results.

**Theorem 2.10 (Upper Estimate).** *Let $A, B \subset E$ be Borel measurable sets such that $\mu(A), \mu(B) > 0$. Then the following estimate holds.*

$$\overline{\lim}_{t\to 0} 4t \cdot \log P_\mu(t, A, B) \le -d_g(A, B)^2.$$

**Theorem 2.11 (Diagonal Lower Estimate).** *Let $A \subset E$ be a Borel measurable set with $\mu(A) > 0$. We denote $A_r := \{x \in E | d_g(\cdot, A) \le r\}$. Then the following estimate holds for any $1 < p < \infty$ and $t > 0$:*

$$P_\mu(t, A, A) \ge \frac{\mu(A)^2}{\mu(A_{K_*\sqrt{t}})} \cdot (1 - \frac{1}{p})^2,$$

*where* $K_* := 2\sqrt{\log\left(\frac{2p}{\mu(A)}\right)}$.

Next, we shall consider the lower estimate. Let us consider the following special case only.

(A5) $A(z)$ is given by

$$A(z) = I_H + a(z) := I_H + \sigma^*(z)\sigma(z),$$

where $\sigma(\cdot) \in \mathbb{D}^\infty_{\infty-}(E, L_{(2)}(H, H))$.

To state the lower estimate, we will recall the definition of the Ricci curvature of a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. For details the reader is referred to Bakry-Emery [4], Bakry [5] and Getzler [13].

**Definition 2.12 (Ricci Curvature of the Dirichlet Form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$).** The Ricci curvature of Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the operator valued function $\mathrm{Ric}(\cdot)$ with values in $I_H + \mathbb{D}^\infty_{\infty-}(E, L_{(2)}(H, H))$ which satisfies that for any $f \in \mathbb{D}^\infty_{\infty-}(E, \mathbb{R})$,

$$\big(\mathrm{Ric}(z)Df(z), Df(z)\big)_{T_zH^*} := \Gamma_2(f, f)(z) - \|\nabla Df(z)\|^2_{\otimes^2 T_zH^*},$$

where $\nabla$ denotes the covariant derivative associated with the Levi-Civita connection which is defined by the Riemannian metric $g(z) = (I_H + a(z))^{-1}$.

**Remark 2.13.** By the definition of the Ricci curvature of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, we note that

$$\mathrm{Ric}(z) \ge -K$$

$\mu$-a.e. $z$ is equivalent to

$$\Gamma_2(f, f)(z) \ge -K\Gamma(f, f)(z)$$

$\mu$-a.s. $z \in E$, for any $f \in \mathbb{D}^\infty_{\infty-}(E, \mathbb{R})$. For details the reader is referred to [5].

In our problem, the Ricci curvature of Dirichlet form (2.3) is given as follows. We shall show the calculation in Section 6.

**Lemma 2.14.** *Assume* (A5)*. Then the Ricci curvature of Dirichlet form* (2.3) *is*

$$\begin{aligned}\mathrm{Ric}(z) &= (I_H - La)(z) + \frac{1}{2}(I_H + a)^{-1}\mathcal{L}a(z) \\ &\quad -(I_H + a)^{ij}(I_H + a)^{-1}\Gamma^{\cdot *}_{\cdot j}(I_H + a)\Gamma^{\cdot}_{\cdot i}(z)\,.\end{aligned}$$

*Here* $\mathcal{L}$ *is the generator of* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $L = -D^*D$, $\Gamma^{\cdot}_{\cdot k}$ *is a Hilbert-Schmidt operator on* $H$ *defined by*

$$\Gamma^{\cdot}_{\cdot k} f = \sum_{i,j=1}^{\infty} \Gamma^i_{jk}(f, h_i)_H h_j\,,$$

*where* $f \in H$, $\{h_i\}_{i=1}^{\infty} \subset E^*$ *is a complete orthonormal basis in* $H$ *and* $\Gamma^i_{jk}$ *are the coefficients of the Levi-Civita connection on* $(H_z, g)$.

Fang and Zhang [9, 35] proved the lower estimate under that $A$ or $B$ is open in $E$. Here we will introduce the notion of $H$-open set which is weaker property than open set.

**Definition 2.15.** A Borel measurable set $A \subset E$ is $H$-open if and only if the following holds. For any $z \in A$, there exists $\varepsilon > 0$ such that

$$\{z + h \mid h \in H, \|h\|_H < \varepsilon\} \subset A.$$

We are in a position to state our lower estimate.

**Theorem 2.16 (Lower Estimate).** *Assume* (A5) *and that there exists a positive number* $K$ *such that* $\mathrm{Ric}(z) \geq -K$ $\mu$*-a.e.* $z \in E$. *Let* $A, B \subset E$ *be Borel measurable sets with* $\mu(A), \mu(B) > 0$ *and assume that* $A$ *or* $B$ *is* $H$*-open. Then we have*

$$\underline{\lim}_{\,t\to 0} 4t \cdot \log P_\mu(t, A, B) \geq -d_g(A, B)^2.$$

As a corollary of Theorem 2.10 and Theorem 2.16, we see that

**Corollary 2.17 (Varadhan Type Asymptotic Formula).** *Under the same assumptions as in Theorem 2.16,*

$$\lim_{t\to 0} 4t \cdot \log P_\mu(t, A, B) = -d_g(A, B)^2.$$

**Remark 2.18.** S. Kusuoka kindly informed us of his previous works [21, 22], in which he had independently defined a notion similar to "$H - UC$-map" of Section 2. We explain his notion of "compact $H - C^0$-map" below.

**Definition 2.19 (Compact $H$-$C^0$ Map [21] [22]).** We will say that $F(\cdot) : E \to L(H, H)$ is a compact $H$-$C^0$ map if the following holds. For any $z \in E$ and $\{h_n\}_{n=1}^{\infty} \subset H$ with $h_n \to 0$ weakly in $H$ as $n \to \infty$, it holds that

$$\lim_{n\to\infty} \|F(z + h_n) - F(z)\|_{L(H,H)} = 0.$$

By virtue of Proposition 1.3 of [21], we can easily state the following relation between the $H$-$UC$ *map* and the *compact* $H$-$C^0$ *map*. (Proof is omitted. See Kawabi [17] for details.)

**Proposition 2.20.** (1) *If $F$ is a compact $H$-$C^0$ map, then $F$ is an $H$-$UC$ map.*

(2) *Under the following condition, any $H$-$UC$ map becomes a compact $H$-$C^0$ map:*
*for any $H$-$UC$ nest $K_n$, there exist $\varepsilon > 0$ and a positive integer $m > n$ such that $K_n + U_H(\varepsilon) \subset K_m$ holds.*

Here we mention that the multiple Wiener integrals are compact $H$-$C^0$ maps by virtue of Proposition 8.1. Therefore, all statements in this paper which include (1) in Section 8 are still valid by assuming that the coefficient operators $A(\cdot)$ are compact $H$-$C^0$ maps instead of assumptions $(\mathrm{A}_3)$ and $(\mathrm{A}_4)$.

# 3 Basic Properties of the $H_g$-Distance

In this section, we shall prepare some basic properties of $H_g$-distance defined in Section 2. First, we shall show the fundamental property of distance function.

**Lemma 3.1.** *Let $K \subset E$ be a compact set with $H$-$UC$ property. Then $d_g(\cdot, K) : E \to \mathbb{R}$ is a Borel measurable function. Moreover $K_r := \{x \in E \mid d_g(\cdot, K) \le r\}$ is a compact set in $E$.*

*Proof.* First, we fix a positive integer $n$. For $n$ and $r > 0$, we will construct an approximate set $K(n, r)$ of $K_r$. By using the $H$-$UC$ property of the compact set $K$, there exist $p_1, p_2 \ldots, p_{m(n)} \in K$ and $a(n) > 0$ such that

$$\begin{aligned} K &\subset \bigcup_{i=1}^{m(n)} U_E(p_i, a(n)), \\ \big(g(x+h)\xi, \xi\big)_H &\le (1 + \frac{1}{n})\big(g(y+h)\xi, \xi\big)_H \end{aligned} \tag{3.6}$$

for any $x, y \in U_E(p_i, a(n)) \cap K,\ \xi,\, h \in H$ with $\|h\|_H \le 4(r+2)M_1^{1/2}$, here we denote $U_E(p, a) := \{x \in E \mid \|x - p\|_E \le a\}$. We now prove the following claim:

**Claim 1.** Let $n \ge 6$ and $u \in H$. We suppose that there exists $z \in U_E(p_i, a(n)) \cap K$ such that $d_g(z, z + u) \le 3(r + 2)$. Then for any $x, y \in$

$U_E(p_i, a(n)) \cap K$ it holds that

$$(1) \qquad d_g(x, x+u) \quad \leq \quad \frac{7}{2}(r+2). \tag{3.7}$$

$$(2) \qquad d_g(x, x+u) \quad \leq \quad (1+\frac{1}{n})\ d_g(y, y+u). \tag{3.8}$$

*Proof of Claim 1.* For $h \in C^1([0,1] \to H;\ h(0) = 0,\ h(1) = u)$, we assume that the following holds.

$$\max_{0 \leq t \leq 1} \|h(t)\|_H \geq 4(r+2)M_1^{1/2}.$$

By recalling Remark 2.6, the following estimate holds for any $x \in E$:

$$\begin{aligned}
\left\{ \int_0^1 \Big( g(x+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2} \quad &\geq \quad \left\{ \int_0^1 M_1^{-1} \|\dot{h}(t)\|_H^2 dt \right\}^{1/2} \\
&\geq \quad M_1^{-1/2} \int_0^1 \|\dot{h}(t)\|_H dt \\
&\geq \quad M_1^{-1/2} \max_{0 \leq t \leq 1} \|h(t)\|_H \\
&\geq \quad 4(r+2).
\end{aligned} \tag{3.9}$$

Hence, for any $w \in E$ and $u \in H$ with $d_g(w, w+u) \leq \dfrac{7}{2}\ (r+2)$, the following identity holds:

$$d_g(w, w+u) = \inf_{h \in C^1_{\#}([0,1] \to H)} \left\{ \int_0^1 \Big( g(w+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2},$$

where

$$\begin{aligned}
&C^1_{\#}([0,1] \to H) \\
&\quad := \Big\{ h \in C^1([0,1] \to H) \ \Big|\ h(0) = 0, h(1) = u, \\
&\qquad\qquad \max_{0 \leq t \leq 1} \|h(t)\|_H \leq 4(r+2)M_1^{1/2} \Big\}.
\end{aligned}$$

For any $x \in U_E \cap K$, applying (3.10) to the case where $w = z$, we can get

$$\begin{aligned}
d_g(x, x+u) &= \inf_{h \in C^1([0,1] \to H)} \left\{ \int_0^1 \Big( g(x+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2} \\
&\leq (1+\frac{1}{6}) \inf_{h \in C^1_{\#}([0,1] \to H)} \left\{ \int_0^1 \Big( g(z+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2} \\
&= (1+\frac{1}{6})\ d_g(z, z+u) \\
&\leq \frac{7}{2}\ (r+2).
\end{aligned} \tag{3.10}$$

Moreover, using (3.6), (3.10) and (3.10), we can conclude that for any $x, y \in U_E(p_i, a(n)) \cap K$,

$$\begin{aligned} d_g(x, x+u) &= \inf_{h \in C^1_{\#}([0,1] \to H)} \left\{ \int_0^1 \Big( g(x+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2} \\ &\le (1+\frac{1}{n}) \inf_{h \in C^1_{\#}([0,1] \to H)} \left\{ \int_0^1 \Big( g(y+h(t))\dot{h}(t), \dot{h}(t) \Big)_H dt \right\}^{1/2} \\ &\le (1+\frac{1}{n})\ d_g(y, y+u). \end{aligned}$$

This completes the proof of Claim 1.

We return to the proof of Lemma 3.1. Set $B_x(r) := \{u \in H \mid d_g(x, x+u) \le r\}$. Then we define an approximation set $K(n,r)$ by

$$K(n,r) := \bigcup_{i=1}^{m(n)} \Big[ \Big\{ U_E(p_i, a(n)) \cap K \Big\} + B_{p_i}(r) \Big].$$

Since $B_{p_i}(r)$ is a bounded closed set in $H$, $K(n,r)$ is a compact set in $E$.

To prove the measurability of $K_r$, we need the following claim.

**Claim 2.** For any $n \ge 6$, the following inclusion holds:

$$K_{(1+\frac{3}{n})^{-1}r} \subset K(n,r) \subset K_{(1+\frac{1}{n})r}. \tag{3.11}$$

The proof is as follows: for any $w \in K(n,r)$, there exist $\eta \in U_E(p_i, a(n)) \cap K$ and $u \in B_{p_i}(r)$ such that $w = \eta + u$. By using Claim 1, we get

$$\begin{aligned} d_g(\eta, \eta+u) &\le (1+\frac{1}{n})\ d_g(p_i, p_i+u) \\ &\le (1+\frac{1}{n})\ r. \end{aligned}$$

This implies that $K(n,r) \subset K_{(1+\frac{1}{n})r}$ holds. On the other hand, for any $w \in K_{(1+\frac{3}{n})^{-1}r}$, there exist $p_i$, $z \in U_E(p_i, a(n)) \cap K$ and $u \in H$ such that $w = z + u$ and $d_g(z, z+u) \le (1+\frac{2}{n})^{-1} r$ hold. Again by using Claim 1, we see

$$\begin{aligned} d_g(p_i, p_i+u) &\le (1+\frac{1}{n})\ d_g(z, z+u) \\ &\le (1+\frac{1}{n})(1+\frac{2}{n})^{-1} r \\ &\le \frac{n+1}{n+2}\ r < r. \end{aligned}$$

So we can deduce $u \in B_{p_i}(r)$. This means that $K_{(1+\frac{3}{n})^{-1}r} \subset K(n,r)$ holds. This completes the proof of Claim 2.

By virtue of (3.11), we have for any integer $n \geq 6$,

$$K_r \subset K\big(n, (1+\frac{3}{n})r\big) \subset K_{(1+\frac{1}{n})(1+\frac{3}{n})r}.$$

Noting the identity

$$K_r = \bigcap_{r' > r, r' \in \mathbb{Q}} K_{r'} = \bigcap_{n \geq 6} K\big(n, (1+\frac{3}{n})r\big),$$

we conclude that $K_r$ is a compact set, since $K\big(n, (1+\frac{3}{n})r\big)$ is compact. $\square$

Next, we state the following lemma which will be used to prove the upper estimate.

**Lemma 3.2.** *Let $K \subset E$ be a $H$-UC compact set with $\mu(K) > 0$. For any $n > 0$, we define the function $u(x) := d_g(x, K) \wedge n$. Then $u \in \mathcal{D}(\mathcal{E})$ and $\Gamma(u,u)(x) \leq 1$ holds for $\mu$-a.e. $x \in E$. Here $\Gamma(u,u)$ denotes the carré du champ of the Dirichlet form $\mathcal{E}$.*

*Proof.* Let $x \in E$ such that $d_g(x,K) < \infty$ holds. For any $h \in H$, the following inequality holds.

$$\begin{aligned} d_g(x+h, K) &= \inf_{y \in K} d_g(x+h, y) \\ &\leq \inf_{y \in K} (d_g(x+h, x) + d_g(x,y)) \\ &= d_g(x+h, x) + d_g(x, K). \end{aligned}$$

Therefore, we have obtained the following.

$$\big| d_g(x+h, K) - d_g(x, K) \big| \leq d_g(x, x+h). \tag{3.12}$$

By using Remark 2.6 and (3.12), we can get the following inequality:

$$\|u(x+h) - u(x)\| \leq d_g(x, x+h) \leq M_2^{-1} \|h\|_H,$$

for any $x \in E$ and $h \in H$.

By using Lemma 1.3 of Kusuoka [19], we can conclude $u \in \mathcal{D}(\mathcal{E})$ and $\|Du(x)\|_H \leq M_2^{-1}$ for $\mu-$a.e. $x \in E$.

Next, we will show that $\Gamma(u,u)(x) \leq 1$ holds. By the definition of $d_g(x, x+h)$, we have

$$\|u(x+h) - u(x)\| \leq d_g(x, x+h) \leq \int_0^1 (A(x+sh)^{-1}h, h)_H^{1/2} ds.$$

By using the assumption ( $\mathrm{A}_3$ ), for $\mu-$a.e. $x \in E$, we get

$$\begin{aligned}
\|(Du(x), h)_H\| &= \left|\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(u(x+\varepsilon h)-u(x)\Big)\right| \\
&\le \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}(A(x+sh)^{-1}h,h)_H^{1/2}ds \\
&= (A(x)^{-1}h,h)_H^{1/2}
\end{aligned}$$

which implies that $\Gamma(u,u)(x) = \|A(x)^{1/2}Du(x)\|_H^2 \le 1$ *for* $\mu-$a.e. $x \in E$. This completes the proof. □

The following lemma is important to give another definition of the distance between two subsets in $E$. This lemma will play an important role in the proof of Theorem 2.16.

**Lemma 3.3.** *Let $A \subset E$ be an $H$-open set and $B \subset E$ a Borel measurable set with $\mu(A), \mu(B) > 0$. Let $\lambda := d_g(A,B)$. Then for any $\varepsilon > 0$, there exist a Borel measurable set $C \subset B$ with $\mu(C) > 0$ and $v \in H$ such that $C + v \subset A$ and*

$$d_g(z, z+v) \le \lambda + \varepsilon,$$

*for any $z \in C$.*

*Proof.* By Proposition 2.9, we may assume $M \subset A$ for $M \in S_A$. By the definition of $d_g(A,B)$, for any $\varepsilon > 0$, there exists a Borel measurable set $B' \subset B$ with $\mu(B') > 0$ and $M \in S_A$ such that for any $z \in B'$ there exists $v(z) \in H$ with

$$z + v(z) \in M \subset A, \quad d_g(z, z+v(z)) \le \lambda + \frac{\varepsilon}{2}. \tag{3.13}$$

Let us take a dense subset $V := \{v_k\}_{k=1}^{\infty}$ in $H$. Since $A$ is $H$-open, by (3.13), for any $\varepsilon > 0$ and $z \in B'$, there exists $v_k \in V$ such that

$$z + v_k \in A, \quad d_g(z, z+v) \le \lambda + \varepsilon. \tag{3.14}$$

We set $C_k := \big\{z \in B' \mid z + v_k \in A,\ d_g(z, z+v_k) \le \lambda + \varepsilon\big\}$ $(k \ge 1)$. By the definition of $C_k$, we have

$$\bigcup_{k=1}^{\infty} C_k = B'.$$

Let us prove that $f_v := d_g(\cdot, \cdot + v) : E \to \mathbb{R}$ is a Borel measurable function for all $v \in H$. By the assumption ( A3 ), $\big(g(z+h(s))\dot h(s), \dot h(s)\big)_H$ is continuous in $s$ for fixed $z$, and Borel measurable in $z$ for fixed $s$. So, $\big(g(z+h(s))\dot h(s), \dot h(s)\big)_H$ is $\mathcal{B}(E)\otimes\mathcal{B}([0,1])$ measurable. Therefore, we can conclude the measurability of $f_v$ by using Fubini's theorem. Hence

$C_k$ is Borel measurable. Since $\mu(B') > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(C_{k_0}) > 0$. Therefore $C := C_{k_0}$, $v := v_{k_0}$ is a desired pair. $\square$

Now, we define another distance $d_g^*$.

**Definition 3.4.** Let $A, B \subset E$ be Borel measurable sets with $\mu(A)$, $\mu(B) > 0$. We define

$$d_g^*(A,B) := \inf K$$

where $K$ is the set of positive numbers $k$ such that there exists a posive measurable set $C \subset B$ and $v \in H$ which satisfy

$$\begin{aligned} C + v &\subset A, \\ d_g(z, z+v) &\leq k \quad \text{for any } z \in C. \end{aligned}$$

Note that the set $K$ above is nonempty.

Before closing this section, we summarize some relations between two distances, $d_g$ and $d_g^*$:

**Lemma 3.5.** (1) $d_g^*(A,B) \geq d_g(A,B)$ .

(2) *If* $A \subset E$ *is* $H$*-open, then*

$$d_g^*(A,B) = d_g(A,B)$$

*holds.*

*Proof.* We note that Lemma 3.3 says $d_g^*(A,B) \leq d_g(A,B)$ if $A$ is $H$-open. Then we need only to prove that $d_g^*(A,B) \geq d_g(A,B)$ for any Borel measurable set $A, B \subset E$. By the definition of $d_g^*(A,B)$, we have for any $\varepsilon > 0$, there exists a measurable set $C \subset B$ with $\mu(C) > 0$ and $v \in H$ which satisfy

$$C + v \subset A, \quad d_g(z, z+v) \leq \ d_g^*(A,B) + \varepsilon \quad \text{for any } z \in \ C. \tag{3.15}$$

Let $\{L_n\}_{n=1}^\infty$ be a sequence which appeared in (2.5). There exists $N_0$ such that for $N \geq N_0$, $\mu\big((C+v) \cap (\bigcup_{n=1}^N L_n)\big) > 0$. By using (3.15), we have

$$\begin{aligned} d_g(z, \bigcup_{n=1}^N L_n) &\leq d_g\Big(z, (C+v) \cap (\bigcup_{n=1}^N L_n)\Big) \\ &\leq d_g(z, z+v) \\ &\leq d_g^*(A,B) + \varepsilon \ . \end{aligned}$$

Then by replacing the role of $A$ and $B$ and recalling the definition of $d_g(A,B)$, we get

$$d_g(A,B) \leq d_g^*(A,B) + \varepsilon.$$

This completes the proof. $\square$

# 4 Dimension Free Harnack Inequality

In this section, we shall state a dimension free Harnack inequality. This is a key lemma to prove Theorem 2.16. Here, we will assume $(\mathrm{A}_5)$ for diffusion coefficient $A(\cdot)$. First, we will state the fundamental differentiability property of functions in $\mathbb{D}_2^1(E,\mathbb{R})$.

**Lemma 4.1.** *Let $f \in \mathbb{D}_2^1(E,\mathbb{R})$ and $h(\cdot) \in C^1([0,1] \to H; h(0)=0)$. Then there exists a measurable function $F(z,t) : E \times [0,1] \to \mathbb{R}$ such that*

(1) *$f(z+h(t)) = F(z,t)$ for any $t \in [0,1]$ and $\mu$-a.e. $z \in E$.*

(2) *The function $t(\in [0,1]) \mapsto F(z,t) \in \mathbb{R}$ is an absolutely continuous function.*

(3) *For any $t \in [0,1]$ and $\mu$-a.e. $z \in E$,*

$$F(z,t) = F(z,0) + \int_0^t (Df(z+h(s)), \dot{h}(s))_H ds$$

*Proof.* For $f \in \mathbb{D}_2^1(E,\mathbb{R})$, we can choose $\{f_n\}_{n=1}^\infty \subset \mathcal{FC}_b^\infty(E,\mathbb{R})$ such that $f_n \to f$ in $\mathbb{D}_2^1(E,\mathbb{R})$. Note that for any $t \in [0,1]$ and $z \in E$,

$$f_n(z+h(t)) = f_n(z+h(0)) + \int_0^t (Df_n(z+h(s)), \dot{h}(s))_H ds. \tag{4.16}$$

By the quasi-invariance of $\mu$, we have for any $t \in [0,1]$ and $1 < p < 2$,

$$f_n(z+h(t)) \to f(z+h(t))$$

in $\mathbb{D}_p^1(E,\mathbb{R})$ and

$$\int_0^t (Df_n(z+h(s)), \dot{h}(s))_H ds \to \int_0^t (Df(z+h(s)), \dot{h}(s))_H ds$$

in $L^p(E,\mathbb{R})$. Set

$$F(z,t) := f(z+h(0)) + \int_0^t (Df(z+h(s)), \dot{h}(s))_H ds.$$

Then $F(z,t)$ satisfies the assertion in the lemma above. □

The following is the main result in this section. See Wang [33].

**Lemma 4.2 (Dimension Free Harnack Inequality).** *Let $u : E \to \mathbb{R}$ be a bounded measurable function and $T_t u(z) = \mathbb{E}_z(u(X_t))$. We assume that $\mathrm{Ric}(z) \geq -K$ holds for $\mu$-a.e. $z \in E$. Then for any $v \in H$ and $\alpha > 0$, the following inequality holds for $\mu$-a.e. $z \in E$:*

$$|T_t u(z)|^\alpha \leq T_t |u|^\alpha (z+v) \cdot \exp\left( \frac{\alpha d_g(z,z+v)^2}{4(\alpha-1)} \cdot \frac{2K}{1-e^{-2Kt}} \right).$$

To prove the inequality above, we recall that for two fundamental results about the Markov semigroups $\{T_t\}_{t\geq 0}$ on $L^2(E,\mathbb{R})$ .

**Lemma 4.3 (Kusuoka [24]).** *Let* $T_t$ *be a Markov semigroup associated with the generator* $\mathcal{L}$ *defined by :*

$$\mathcal{L}u(z) = -D^*\left((A_0 + K(z))Du(z)\right) ,$$

*where*

(1) $A_0 : H \to H$ *is a bounded symmetric operator such that*

$$(A_0 h, h)_H \geq \|h\|_H^2 ,$$

*for any* $h \in H$ *.*

(2) $K(\cdot) \in \mathbb{D}^\infty_{\infty-}(E, L_{(2)}(H,H))$ *satisfies the following properties:*

$$K(z) = K^*(z) \quad \mu\text{-a.e. } z \in E$$

$$\operatorname{esssup}_{z\in E} \|K(z)\|_{L(H,H)} < \infty.$$

*Then there exists* $\lambda \in (0,1/2)$ *such that for any* $\tau \in \mathbb{R}$ *,* $\sigma \geq 0$ *,* $p \in [(1-\lambda)^{-1}, \lambda^{-1}]$ *,* $q > p$ *,* $t \in (0,1]$ *and* $u \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ *, there exists* $C > 0$ *such that*

$$\|T_t u\|_{\mathbb{D}_q^{\tau+2\sigma}} \leq C t^{-\sigma} \|u\|_{\mathbb{D}_p^\tau}.$$

**Lemma 4.4 (Bakry [5]).** *We assume that* $\operatorname{Ric}(z) \geq -K$ *for* $\mu$ *-a.e.* $z \in E$ *and we denote* $\Gamma(f)(z) := \Gamma(f,f)(z)$ *and* $\Gamma_2(f)(z) := \Gamma_2(f,f)(z)$ *. Then for any* $f \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ *and* $t > 0$ *, the following inequality holds for* $\mu - a.e.$ $z \in E$ *.*

$$\Gamma(T_t f)^{1/2}(z) \leq e^{Kt} T_t(\Gamma^{1/2}(f))(z).$$

*Proof.* We fix $t > 0$ . For $f \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ , we consider

$$\Phi(s) := e^{Ks} T_s\Big(\Gamma(T_{t-s}f)^{1/2}(z)\Big). \tag{4.17}$$

By Lemma 4.3, we see that $T_t f \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ . Therefore $\Phi(s) \in \mathcal{D}(\mathcal{E})$ . By virtue of Lemma 4.1, for $\mu$-a.e. $z \in E$ , $\Phi(s)$ is differentiable with respect to a.e. $s \in [0,t]$ . Hence the following identity holds for a.e. $s \in [0,t]$ :

$$\Phi'(s) = T_s\Big\{\frac{1}{4}\Gamma(g)^{-3/2}\Big\{4\Gamma(g)\big(\Gamma_2(g) + K\Gamma(g)\big) - \Gamma(\Gamma(g))\Big\}\Big\}(z) , \tag{4.18}$$

where $g(z) := T_{t-s}f(z)$ . Next, we recall the following condition which is equivalent to $\operatorname{Ric} \geq -K$ : for any $f \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ , it holds that

$$4\Gamma(f)\big\{\Gamma_2(f) + K\Gamma(f)\big\}(z) \geq \Gamma(\Gamma(f))(z). \tag{4.19}$$

Hence integrating (4.18) with respect to $s$ from 0 to $t$, we get $\Phi(t) \geq \Phi(0)$ and this completes the proof. $\square$

*Proof of Lemma 4.2:* We may assume that $f \in \mathbb{D}^{\infty}_{\infty-}(E,\mathbb{R}), f(z) > \delta > 0$ since $|T_t f(z)| \leq T_t|f|(z)$ holds generally. We choose $h \in C^1([0,1] \to H; h(0)=0, h(1)=v)$. For fixed $t>0$, we set $\tilde{h}(\cdot) \in C^1([0,t] \to H)$ such that

$$\tilde{h}(s) = h(\tau), \quad \tau := \left(\int_0^s e^{-2Kr}dr\right) \Big/ \left(\int_0^t e^{-2Kr}dr\right).$$

For $f \in \mathbb{D}^{\infty}_{\infty-}(E,\mathbb{R}), \alpha > 1$, we shall consider

$$\Phi(s) := \log T_s (T_{t-s}f)^{\alpha}(z+\tilde{h}(s)).$$

We shall recall $T_t f \in \mathcal{D}(\mathcal{E})$ for any $f \in \mathcal{D}(\mathcal{E})$. So we conclude $\Phi(s) \in \mathcal{D}(\mathcal{E})$. By Lemma 4.1, $\Phi(s)$ is differentiable with respect to a.e. $s \in [0,t]$. So the following identity holds:

$$\Phi'(s) = \frac{d}{ds}\Big\{T_s(T_{t-s}f)^{\alpha}(z+\tilde{h}(s))\Big\}\Big/ T_s(T_{t-s}f)^{\alpha}(z+\tilde{h}(s)) \quad \text{a.e. } s \in [0,t].$$

Therefore

$$\begin{aligned}\frac{d}{ds}\Big\{T_s(T_{t-s}f)^{\alpha}(z+\tilde{h}(s))\Big\} &= T_s\mathcal{L}(T_{t-s}f)^{\alpha}(z+\tilde{h}(s)) \\ &\quad + T_s\Big\{\alpha(T_{t-s}f)^{\alpha-1}(-\mathcal{L}T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\ &\quad + \Big(D\{T_s(T_{t-s}f)^{\alpha}\}(z+\tilde{h}(s)), \dot{\tilde{h}}(s)\Big)_H, \end{aligned} \tag{4.20}$$

where $\mathcal{L}$ is the generator of $T_t$. Noting the identity (c.f.[7]),

$$\mathcal{L}(F^{\alpha}(z)) = \alpha F^{\alpha-1}(z)\mathcal{L}(F(z)) + \alpha(\alpha-1)F^{\alpha-2}(z)\Gamma(F,F)(z)$$

for $F \in \mathcal{D}(\mathcal{L})$, we have

$$\begin{aligned}&\frac{d}{ds}\Big\{T_s(T_{t-s}f)^{\alpha}(z+\tilde{h}(s))\Big\} \\ &\qquad = T_s\Big\{\alpha(T_{t-s}f)^{\alpha-1}\mathcal{L}T_{t-s}f\Big\}(z+\tilde{h}(s)) \\ &\qquad\quad + T_s\Big\{\alpha(\alpha-1)(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\ &\qquad\quad - T_s\Big\{\alpha(T_{t-s}f)^{\alpha-1}(\mathcal{L}T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\ &\qquad\quad + \Big(D\{T_s(T_{t-s}f)^{\alpha}\}(z+\tilde{h}(s)), \dot{\tilde{h}}(s)\Big)_H \\ &\qquad = \alpha(\alpha-1)T_s\Big\{(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\ &\qquad\quad + \Big(D\{T_s(T_{t-s}f)^{\alpha}\}(z+\tilde{h}(s)), \dot{\tilde{h}}(s)\Big)_H\end{aligned}$$

$$
\begin{aligned}
&\geq \alpha(\alpha-1)T_s\Big\{(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\
&\qquad -\Big\|g(z+\tilde{h}(s))^{-1/2}D\{T_s(T_{t-s}f)^{\alpha}\}(z+\tilde{h}(s))\Big\|_H \\
&\qquad \cdot\Big\|g(z+\tilde{h}(s))^{1/2}\dot{\tilde{h}}(s)\Big\|_H \\
&= \alpha(\alpha-1)T_s\Big\{(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\
&\qquad -\Gamma\Big\{T_s(T_{t-s}f)^{\alpha}\Big\}^{1/2}(z+\tilde{h}(s))\cdot\|\dot{\tilde{h}}(s)\|_{T_{z+\tilde{h}(s)}H}.
\end{aligned}
$$

Furthermore using Lemma 4.4,

$$
\begin{aligned}
&\frac{d}{ds}\Big\{T_s(T_{t-s}f)^{\alpha}(z+\tilde{h}(s))\Big\} \\
&\quad\geq \alpha(\alpha-1)T_s\Big\{(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f)\Big\}(z+\tilde{h}(s)) \\
&\qquad\quad -e^{Ks}T_s\Gamma\big\{(T_{t-s}f)^{\alpha}\big\}^{1/2}(z+\tilde{h}(s))\cdot\big\|\dot{\tilde{h}}(s)\big\|_{T_{z+\tilde{h}(s)}H} \\
&\quad= \alpha T_s\Big\{(\alpha-1)(T_{t-s}f)^{\alpha-2}\Gamma(T_{t-s}f) \\
&\qquad\quad -e^{Ks}\left|(T_{t-s})f^{\alpha-1}\right|\cdot\Gamma(T_{t-s}f)^{1/2}\|\dot{\tilde{h}}(s)\|_{T_{z+\tilde{h}(s)}H}\Big\}(z+\tilde{h}(s)) \\
&\quad\geq -\alpha T_s\left\{(T_{t-s}f)^{\alpha}\cdot\frac{e^{2Ks}\|\dot{\tilde{h}}(s)\|^2_{T_{z+\tilde{h}(s)}H}}{4(\alpha-1)}\right\}(z+\tilde{h}(s)). \\
&\quad= -\alpha T_s(T_{t-s}f)^{\alpha}(z+h(\tau))\cdot\|\dot{h}(\tau)\|^2_{T_{z+h(\tau)}H} \\
&\qquad\qquad\qquad\qquad \times\frac{e^{2Ks}}{4(\alpha-1)}\cdot\frac{4K^2e^{-4Ks}}{(1-e^{-2Kt})^2}.
\end{aligned}
$$

So we get

$$
\dot{\Phi}(s)\geq-\alpha\|\dot{h}(\tau)\|^2_{T_{z+h(\tau)}H}\cdot\frac{e^{-2Ks}}{4(\alpha-1)}\cdot\frac{4K^2}{(1-e^{-2Kt})^2}.
$$

By integrating over $s$ from 0 to $t$, we get

$$
\begin{aligned}
&|T_tu(z)|^{\alpha}\leq T_t|u|^{\alpha}(z+v) \\
&\qquad\qquad \exp\left(\frac{\alpha\int_0^1\|\dot{h}(\tau)\|^2_{T_{z+h(\tau)}H}d\tau}{4(\alpha-1)}\cdot\frac{2K}{1-e^{-2Kt}}\right).
\end{aligned}
$$

By taking infimum over $h\in C^1([0,1]\to H;h(0)=0,h(1)=v)$, we complete the proof. $\square$

# 5 Proof of Main Theorems

First, we shall recall Lyons-Zheng's decomposition theorem to prove Theorem 2.10.

**Proposition 5.1 ([11, 25, 26]).** *Assume that* $X = (X_t, P_z)$ *is a conservative Markov process. We set* $\mathcal{F}_t = \sigma(X_s, 0 \le s \le t)$ *and* $\bar{\mathcal{F}}_t = \sigma(X_s, T-t \le s \le T)$ *for fixed* $T > 0$. *For* $f \in \mathcal{D}(\mathcal{E})$, *we denote by* $\tilde{f}$ *a quasi-continuous version of* $f$.

*Then for any* $f \in \mathcal{D}(\mathcal{E})$, *there exists a continuous* $\mathcal{F}_t$*-martingale* $M_t^f$ *and a continuous* $\bar{\mathcal{F}}_t$*-martingale* $\bar{M_t}^f$ *which satisfy the following identities.*

(1) $N_t := M_t^f + (\bar{M}_T^f - \bar{M}_{T-t}^f)$ *is a continuous additive functional of zero energy.*

(2) $M_t^f$ *and* $\bar{M}_{T-t}^f$ *are* $P_\mu$*-square integrable, with* $M_0^f = \bar{M}_0^f = 0$.

(3) *It holds that*

$$\tilde{f}(X_t) - \tilde{f}(X_0) = \frac{1}{2}\{M_t^f - (\bar{M}_T^f - \bar{M}_{T-t}^f)\} \tag{5.21}$$

$P_\mu$*-almost surely.*

*Moreover the quadratic variation of* $M_t^f$ *and of* $\bar{M}_t^f$ *are representated as follows:*

$$\begin{aligned}\langle M^f\rangle_t &= 2\int_0^t \Gamma(f,f)(X_s)ds, \\ \langle \bar{M}^f\rangle_t &= 2\int_0^t \Gamma(f,f)(X_{T-s})ds,\end{aligned} \tag{5.22}$$

*for* $f \in \mathcal{D}(\mathcal{E})$.

We shall prove now our upper bound estimate.

*Proof of Theorem 2.10.* We proceed as in [9, 10]. We notice that $\mu$ is the invariant measure of the diffusion process $X = (X_t, P_z)$ and $\mu(A \triangle M) = 0$ for certain $M \in S_A$. Hence we can suppose that $A \in S_A, B \in S_B$ and $d_g(A,B) > 0$ hold. Take $0 < \lambda < d_g(A,B)$. We may suppose that $\operatorname{essinf}_{x\in B} d_g(x,A) > \lambda$ holds. Then, there exists a Borel measurable set $K \subset B$ with $\mu(K) = \mu(B)$ such that For any $x \in K$,

$$d_g(x,A) > \lambda. \tag{5.23}$$

We set $A_\lambda := \{x \in E \mid d_g(x,A) \le \lambda\}$. Then we have $K \subset (A_\lambda)^c$ by virtue of (5.23). Now fix an integer $n > \lambda$. We set $u(x) := d_g(x,A) \wedge n$.

By Lemma 3.2, we can get $u \in \mathcal{D}(\mathcal{E})$. Then by Proposition 5.1, under $P_\mu$ it holds that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = \frac{1}{2}\{M_t^u - M_T^u - \bar{M}_{T-t}^u)\} \qquad \text{for } 0 \le t \le T.$$

Here $M^u$ is an $\mathcal{F}_t$-square integrable martingale satisfying

$$\langle M^u \rangle_t = 2\int_0^t \Gamma(u,u)(X_s)ds \ .$$

By taking $t = T$, we have

$$\tilde{u}(X_t) - \tilde{u}(X_0) = \frac{1}{2}(M_t^u - \bar{M}_t^u) \ .$$

By noting that $\mu$ is the invariant measure of the diffusion process $X = (X_t, P_z)$, we can get the following estimate.

$$\begin{aligned}
P_\mu\Big(\big\{X_0 \in A, X_t \in B\big\}\Big) &= P_\mu\Big(\big\{X_0 \in A, X_t \in K\big\}\Big) \\
&\le P_\mu\Big(\big\{X_0 \in A, X_t \in A_\lambda^c\big\}\Big) \\
&\le P_\mu\Big(\big\{u(X_t) - u(X_0) > \lambda\big\}\Big) \\
&= P_\mu\Big(\big\{(M_t^u - \bar{M}_t^u) > 2\lambda\big\}\Big) \\
&\le P_\mu\Big(\big\{M_t^u > \lambda\big\}\Big) + P_\mu\Big(\big\{-\bar{M}_t^u > \lambda\big\}\Big) \\
&\le P_\mu\Big(\big\{\sup_{0\le s\le t}(M_s^u) > \lambda\big\}\Big) + P_\mu\Big(\big\{\sup_{0\le s\le t}(-\bar{M}_s^u) > \lambda\big\}\Big).
\end{aligned} \tag{5.24}$$

By the time change, we have

$$M_s^u = B_1\Big(2\int_0^s \Gamma(u,u)(X_\tau)d\tau\Big), \qquad \bar{M}_s^u = B_2\Big(2\int_0^s \Gamma(u,u)(X_{T-\tau})d\tau\Big),$$

where $B_1$ and $B_2$ are 1-dimensional Brownian motions. Then (5.24) can be estimated as follows:

$$\begin{aligned}
&P_\mu\Big(\big\{\sup_{0\le s\le t}(M_s^u) > \lambda\big\}\Big) + P_\mu\Big(\big\{\sup_{0\le s\le t}(-\bar{M}_s^u) > \lambda\big\}\Big) \\
\le\ & P\Big(\big\{\sup_{0\le s\le t} B_1(2\int_0^s \Gamma(u,u)(X_\tau)d\tau) > \lambda\big\}\Big) \\
&+P\Big(\big\{\sup_{0\le s\le t} B_2(2\int_0^s \Gamma(u,u)(X_{T-\tau})d\tau) > \lambda\big\}\Big) \\
=\ & 2P\Big(\big\{B_1(2\int_0^t \Gamma(u,u)(X_\tau)d\tau) > \lambda\big\}\Big) \\
&+2P\Big(\big\{B_2(2\int_0^t \Gamma(u,u)(X_{t-\tau})d\tau) > \lambda\big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{2\pi}}\int_{\lambda}^{\infty}\exp\Big(-\frac{s^2}{2\int_0^t\Gamma(u,u)(X_\tau)d\tau}\Big)ds \\
&\quad+\frac{2}{\sqrt{2\pi}}\int_{\lambda}^{\infty}\exp\Big(-\frac{s^2}{2\int_0^t\Gamma(u,u)(X_{t-\tau})d\tau}\Big)ds \\
&\leq \frac{4}{\sqrt{2\pi}}\int_{\lambda/\sqrt{t}}^{\infty}\exp(-\frac{s^2}{2})ds. \qquad (5.25)
\end{aligned}
$$

We have used in the last step $\Gamma(u,u)(z)\leq 1$ which follows from Lemma 3.2. Thus we can get from (5.25) that

$$
\overline{\lim}_{t\to 0}4t\log P_\mu(t,A,B)\leq -\lambda^2.
$$

Letting $\lambda\to d_g(A,B)$, we complete the proof. □

*Proof of Theorem 2.11.* We will proceed as in Proposition VII-6.6 of [6]. By using the proof of Theorem 2.10, we get the following esimate:

$$
\begin{aligned}
P_\mu\Big(\big\{X_0\in A,\sup_{0\leq s\leq t/2}d_g(X_s,A)>K_*\sqrt{t}\big\}\Big) &\leq \frac{4}{\sqrt{2\pi}}\int_{K_*}^{\infty}\exp(-\frac{s^2}{2})ds \\
&\leq 2\exp(-\frac{K_*^2}{4}).
\end{aligned}
$$

By taking $K_*:=2\sqrt{\log(\frac{2p}{\mu(A)})}$, we obtain

$$
P_\mu\Big(\big\{X_0\in A,\sup_{0\leq s\leq t/2}d_g(X_s,A)>K_*\sqrt{t}\big\}\Big)\leq\frac{1}{p}\mu(A).
$$

Therefore we have

$$
P_\mu\Big(\big\{X_0\in A,X_{t/2}\in A_{K_*\sqrt{t}}\big\}\Big)\geq\mu(A)(1-\frac{1}{p}). \qquad (5.26)
$$

Then, noting that $\mu$ is the reversible measure of the diffusion process $X=(X_t,P_z)$ and (5.26),

$$
\begin{aligned}
\mu(A)^2(1-\frac{1}{p})^2 &\leq P_\mu\Big(\big\{X_0\in A,X_{t/2}\in A_{K_*\sqrt{t}}\big\}\Big)^2 \\
&= \Big(\int_E P_{t/2}(z,A)\cdot\mathbf{1}_{A_{K_*\sqrt{t}}}(z)\mu(dz)\Big)^2 \\
&\leq \Big(\int_E P_{t/2}(z,A)\cdot P_{t/2}(A,z)\mu(dz)\Big)\cdot\mu(A_{K_*\sqrt{t}}) \\
&= P_\mu(t,A,A)\cdot\mu(A_{K_*\sqrt{t}})\ .
\end{aligned}
$$

So, we get the result. □

Next, we prepare the following lemma to prove Theorem 2.16.

**Lemma 5.2.** *Let* $A \subset E$ *a measurable set with* $\mu(A) > 0$ *. We consider bounded measurable function* $\Psi : E \to \mathbb{R}$ *with* $\Psi(z) = 1$ *on* $A$ *, and* $0 \leq \Psi(z) \leq 1$ *. Then , for any sequence* $\{t_n\}_{n=1}^{\infty} \downarrow 0$ *, there exist measurable subset* $B \subset A$ *with* $\mu(B) > 0$ *and a subsequence* $\{t_{n(k)}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *such that the following property holds: there exists* $N_0 \in \mathbb{N}$ *such that, for any* $z \in B$ *and* $k \geq N_0$ *,*

$$T_{t(k)}\Psi(z) \geq \frac{1}{2}. \tag{5.27}$$

*Proof.* Since $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on $L^2(E,\mathbb{R})$ , the following property holds:

$$\lim_{t\to 0}\int_A |T_t\Psi(z) - 1|^2 d\mu(z) = 0. \tag{5.28}$$

We denote $d\mu_A := d\mu|_A/\mu(A)$ . By virtue of (5.28), for any $\{t_n\}_{n=1}^{\infty} \downarrow 0$ , there exists a subsequence $\{t_{n(k)}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ such that

$$\mu_A\left(\left\{z \in A \;\middle|\; T_{t_{n(k)}}\Psi(z) \geq 1 - \frac{1}{k}\right\}\right) \geq 1 - \frac{1}{k^2} \ .$$

Set $A_k = \{x \in A \mid T_{t_{n(k)}}\Psi(z) \geq 1 - \dfrac{1}{k}\}$ , $B_k = \bigcap\limits_{l=k}^{\infty} A_l$ . Then, we see that

$$\mu_A(B_k) = 1 - \mu_A(\bigcup_{l=k}^{\infty} A_l^c) \geq 1 - \sum_{l=k}^{\infty}\frac{1}{l^2},$$

which implies (5.27). □

We are now in a position to prove Theorem 2.16.

*Proof of Theorem 2.16.* Let $\lambda = d_g(A,B)$ . By recalling Lemma 3.3, for any $\varepsilon > 0$ , there exist a measurable set $C \subset B$ with $\mu(C) > 0$ and $v \in H$ , which satisfy the following properties:

$$\begin{array}{rcll} C + v & \subset & A & \\ d_g(z, z+v) & \leq & \lambda + \varepsilon & \text{for any } z \in C. \end{array} \tag{5.29}$$

Let $\Psi(\cdot)$ be the indicater function of $C + v$ . Then by Lemma 5.2, there exist a Borel measurable set $C' \subset C$ , a sequence $\{t_{n(k)}\}_{k=1}^{\infty} \downarrow 0$ , and $N_0 \in \mathbb{N}$ such that

$$T_{t_{n(k)}}\Psi(z) \geq \frac{1}{2} \quad \text{for any } z \in C' + v \text{ and } k \geq N_0. \tag{5.30}$$

By using Lemma 4.2, for any $\alpha > 1$, we can estimate $P_\mu(t, A, B)$ as follows:

$$\begin{aligned} P_\mu(t, A, B) &= \int_B T_t \mathbf{1}_A(z)\mu(dz) \\ &\geq \int_{C'} T_t|\Psi|^\alpha(z)\mu(dz) \\ &\geq \int_{C'} |T_t\Psi|^\alpha(z+v) \cdot \exp\left(-\frac{\alpha d_g(z, z+v)^2}{4(\alpha-1)} \cdot \frac{2K}{1-e^{-2Kt}}\right). \end{aligned} \tag{5.31}$$

By using (5.29), (5.30) and (5.31), we can get

$$\begin{aligned} &4t_{n(k)} \log P_\mu(t_{n(k)}, A, B) \\ &\quad \geq 4t_{n(k)} \log\left\{\left(\frac{1}{2}\right)^\alpha \cdot \mu(C') \cdot \exp\left(-\frac{\alpha(\lambda+\varepsilon)^2}{4(\alpha-1)} \cdot \frac{2K}{1-e^{-2Kt_{n(k)}}}\right)\right\}. \end{aligned}$$

Finally, we complete the proof by letting $k \to \infty$, $\alpha \to \infty$ and $\varepsilon \downarrow 0$. □

## 6 The Ricci Curvature of Dirichlet Form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$

Throughout this section, we always assume $(\mathrm{A}_5)$ and we will get into the detail of the calculation of the Ricci curvature of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ given in (2.3) and give a condition under which the Ricci curvature is bounded from below.

By Definition 2.12, The Ricci curvature of the Dirichlet form is given as follows:

$$\big(\mathrm{Ric}(z)Df(z), Df(z)\big)_{T_zH^*} = \Gamma_2(f,f)(z) - \|\nabla Df(z)\|^2_{\otimes^2 T_zH^*}$$

for $f \in \mathbb{D}^\infty_{\infty-}(E, \mathbb{R})$. First, we will give the proof of Lemma 2.14.

*Proof of Lemma 2.14.* We will calculate the Ricci curvature in several steps. We use the summation convention in the calculation below.

**Step1 (Calculation of $\Gamma_2(f,f)(z)$):** First, we fix a complete orthonormal basis of $H$, $\mathcal{H} := \{h_i\}_{i=1}^\infty \subset E^*$. We denote $a(z)^i = a(z)h_i$, $a(z)^{ij} = (a(z)h_i, h_j)_H$ and $D_if(z) = (Df(z), h_i)_H$. Then the generator of $\mathcal{E}$ is given as follows:

$$\begin{aligned} \mathcal{L}f(z) &= \Big\{D_iD_jf(z) - h_i(z)D_if(z)\Big\} \\ &\quad + \Big\{a(z)^{ij}D_iD_jf(z) - (D^*a(z), Df(z))_H\Big\} \\ &:= Lf(z) + L_af(z). \end{aligned} \tag{6.32}$$

By the definition of $\Gamma_2$,

$$
\begin{aligned}
\Gamma_2(f,f)(z) &= \frac{1}{2}\Big\{\mathcal{L}\Gamma(f,f)(z) - 2\Gamma(\mathcal{L}f,f)(z)\Big\} \\
&= \frac{1}{2}\Big\{L\big(Df(z),Df(z)\big)_H - 2\big(DLf(z),Df(z)\big)_H\Big\} \\
&\quad +\frac{1}{2}\Big\{L\big(\sigma(z)Df(z),\sigma(z)Df(z)\big)_H \\
&\qquad\quad -2\big(\sigma(z)DLf(z),\sigma(z)Df(z)\big)_H\Big\} \\
&\quad +\frac{1}{2}\Big\{L_a\big(Df(z),Df(z)\big)_H - 2\big(DL_af(z),Df(z)\big)_H\Big\} \\
&\quad +\frac{1}{2}\Big\{L_a\big(\sigma(z)Df(z),\sigma(z)Df(z)\big)_H \\
&\qquad\quad -2\big(\sigma(z)DL_af(z),\sigma(z)Df(z)\big)_H\Big\} \\
&:= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}\ .
\end{aligned}
$$

We can calculate $\mathrm{I}$, $\mathrm{II}$, $\mathrm{III}$ and $\mathrm{IV}$ as follows:

$$
\begin{aligned}
\mathrm{I} &= \big(D_iD_jf(z)\big)^2 + \big(D_if(z)\big)^2 \\
&= \|D^2f(z)\|^2_{H^{\otimes 2}} + \|Df(z)\|^2_H\ . \\
\mathrm{II} &= 2D_ka(z)^{ij}D_iD_kf(z)D_jf(z) + a(z)^{ij}D_iD_kf(z)D_jD_kf(z) \\
&\quad +a(z)^{ij}D_if(z)D_jf(z) + \frac{1}{2}La(z)^{ij}D_if(z)D_jf(z)\ . \\
\mathrm{III} &= a(z)^{ij}D_iD_kf(z)D_jD_kf(z) \\
&\quad -D_ka(z)^{ij}D_iD_jf(z)D_kf(z) \\
&\quad +D_i(D^*a(z))^jD_if(z)D_jf(z)\ . \\
\mathrm{IV} &= a(z)^{ij}a(z)^{kl}D_iD_kf(z)D_jD_lf(z) \\
&\quad +2a(z)^{kl}D_ka(z)^{ij}D_jD_lf(z)D_if(z) \\
&\quad -a(z)^{ij}D_ia(z)^{kl}D_kD_lf(z)D_jf(z) \\
&\quad +\frac{1}{2}a(z)^{kl}D_kD_la(z)^{ij}D_if(z)D_jf(z) \\
&\quad -\frac{1}{2}(D^*a(z))^kD_ka(z)^{ij}D_if(z)D_jf(z) \\
&\quad -a(z)^{ij}D_i(D^*a(z))^kD_jf(z)D_kf(z).
\end{aligned}
$$

By combining $\mathrm{I}$, $\mathrm{II}$, $\mathrm{III}$, $\mathrm{IV}$ and (6.32), we get

$$\begin{aligned}
&\Gamma_2(f,f)(z) \\
&\quad= \Big(\|D^2 f(z)\|^2_{H^{\otimes 2}} + 2a(z)^{ij} D_i D_k f(z) D_j D_k f(z) \\
&\qquad\qquad + a(z)^{ij} a(z)^{kl} D_i D_k f(z) D_j D_l f(z)\Big) \\
&\qquad + \Big(2a(z)^{ij} D_j a(z)^{kl} D_l f(z) D_i D_k f(z) \\
&\qquad\qquad + 2D_k a(z)^{ij} D_j f(z) D_i D_k f(z) \\
&\qquad\qquad - D_k a(z)^{ij} D_k f(z) D_i D_j f(z) \\
&\qquad\qquad - a(z)^{kl} D_l a(z)^{ij} D_k f(z) D_i D_j f(z)\Big) \\
&\qquad + \Big(\|Df(z)\|^2_H + a(z)^{ij} D_i f(z) D_j f(z) \\
&\qquad\qquad + \frac{1}{2} L_a a(z)^{ij} D_i f(z) D_j f(z) \\
&\qquad\qquad - \frac{1}{2} L a(z)^{ij} D_i f(z) D_j f(z) \\
&\qquad\qquad - a(z)^{ij} L a(z)^{ik} D_j f(z) D_k f(z)\Big).
\end{aligned} \tag{6.33}$$

**Step 2. (Calculation of Christoffel Symbol $\Gamma^k_{ij}(z)$):** Next we calculate Christoffel's symbol $\Gamma^k_{ij}(z)$ with respect to Levi-Civita connection on Hilbert manifold $(H_z, g)$. First, we recall the following identities:

$$(\nabla_{h_i} h_j)(z) = \Gamma^k_{ij}(z) h_k\,,$$

and

$$\begin{aligned}
&\Big(g(z)\nabla_{h_i} h_j, h_l\Big)_H \\
&\quad= \frac{1}{2}\Big\{D_i\big(g(z)h_j, h_l\big)_H + D_j\big(g(z)h_l, h_i\big)_H - D_l\big(g(z)h_i, h_j\big)_H\Big\}.
\end{aligned} \tag{6.34}$$

Noting the chain rule for vector valued functions $A, B$,

$$D(AB)(\cdot,\cdot) = DA(\cdot, B(\cdot)) + A(DB(\cdot,\cdot)).$$

and $g(z)\cdot(I_H + a(z)) = I_H$, we can get

$$D_i g(z) = -g(z) D_i a(z) g(z).$$

Putting this into (6.34),

$$\begin{aligned}\Big(\Gamma^p_{ij}(z)h_p, g(z)h_l\Big)_H &= -\frac{1}{2}\Big(g(z)D_i a(z)g(z)h_j, h_l\Big)_H \\ &\quad -\frac{1}{2}\Big(g(z)D_j a(z)g(z)h_l, h_i\Big)_H \\ &\quad +\frac{1}{2}\Big(g(z)D_l a(z)g(z)h_i, h_j\Big)_H. \qquad (6.35)\end{aligned}$$

Multiplying with $(h_l, g(z)^{-1}h_k)_H$ and summing up over $p$, we arrive at

$$\begin{aligned}\Gamma^k_{ij}(z) &= -\frac{1}{2}\Big(h_k, D_i a(z)g(z)h_j\Big)_H - \frac{1}{2}\Big(h_k, D_j a(z)g(z)h_i\Big)_H \\ &\quad +\frac{1}{2}\Big(g(z)D_{g^{-1}h_k} a(z)g(z)h_i, h_j\Big)_H. \qquad (6.36)\end{aligned}$$

**Step 3. (Calculation of Hessian Term $\|\nabla Df(z)\|^2_{\otimes^2 T_z H^*}$):** To complete the proof, we calculate the Hessian term $\|\nabla Df(z)\|^2_{\otimes^2 T_z H^*}$. $\delta^{ij}$ denotes Kronecker's delta below. By the definition of the covariant derivative, we have the following expansion.

$$\begin{aligned}\|\nabla Df(z)\|^2_{\otimes^2 T_z H^*} &= \big(g^{-1}(z)\big)^{ip}\big(g^{-1}(z)\big)^{jq}\big(D^2 f(z)\big)(h_i, h_j)\cdot\big(D^2 f(z)\big)(h_p, h_q) \\ &= \big(\delta^{ip} + a(z)^{ip}\big)\big(\delta^{jq} + a(z)^{jq}\big) \\ &\quad \times\big(D_i D_j f(z) - \Gamma^k_{ij}(z)D_k f(z)\big)\big(D_p D_q f(z) - \Gamma^r_{pq}(z)D_r f(z)\big) \\ &= \Bigg(\|D^2 f(z)\|^2_{H^{\otimes 2}} + 2a(z)^{ip}D_i D_j f(z)D_p D_j f(z) \\ &\qquad + a(z)^{ip}a(z)^{jq}D_i D_j f(z)D_p D_q f(z)\Bigg) \\ &\quad +\Bigg(-2\Gamma^k_{ij}(z)D_k f(z)D_i D_j f(z) - 2a(z)^{jq}\Gamma^k_{iq}(z)D_k f(z)D_i D_j f(z)\Bigg) \\ &\quad +\Bigg(-2a(z)^{ip}\Gamma^k_{pj}(z)D_k f(z)D_i D_j f(z) \\ &\qquad -2a(z)^{ip}a(z)^{jq}\Gamma^k_{pq}(z)D_k f(z)D_i D_j f(z)\Bigg) \\ &\quad +\Bigg(\Gamma^k_{ij}(z)D_k f(z)\Gamma^r_{ij}(z)D_r f(z) + a(z)^{jq}\Gamma^k_{ij}(z)D_k f(z)\Gamma^r_{iq}(z)D_r f(z) \\ &\qquad + a(z)^{ip}\Gamma^k_{ij}(z)D_k f(z)\Gamma^r_{pj}(z)D_r f(z) \\ &\qquad + a(z)^{ip}a(z)^{jq}\Gamma^k_{ij}(z)D_k f(z)\Gamma^r_{pq}(z)D_r f(z)\Bigg) \\ &:= \mathrm{V} + \mathrm{VI}_1 + \mathrm{VI}_2 + \mathrm{VII}\,.\end{aligned}$$

Now, we shall calculate the cross terms $\mathrm{VI}_1$, $\mathrm{VI}_2$. By using (6.36), we calculate them as follows:

$$
\begin{aligned}
\mathrm{VI}_1 &= -2\bigl(g^{-1}(z)\bigr)^{jq}\Gamma^k_{iq}(z)D_kf(z)D_iD_jf(z)\\
&= D_ia(z)^{jk}D_kf(z)D_iD_jf(z)\\
&\quad+\Bigl(D_{g^{-1}h_j}a(z)g(z)h_i,h_k\Bigr)_H D_kf(z)D_iD_jf(z)\\
&\quad-\Bigl(D_{g^{-1}h_k}a(z)g(z)h_i,h_j\Bigr)_H D_kf(z)D_iD_jf(z).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VI}_2 &= -2a(z)^{ip}\bigl(g^{-1}(z)\bigr)^{jq}\Gamma^k_{pq}(z)D_kf(z)D_iD_jf(z)\\
&= a(z)^{ip}D_pa(z)^{jk}D_kf(z)D_iD_jf(z)\\
&\quad+\Bigl(D_{g^{-1}h_j}a(z)h_k,h_j\Bigr)_H D_kf(z)D_iD_jf(z)\\
&\quad-\Bigl(g(z)D_{g^{-1}h_j}a(z)h_k,h_i\Bigr)_H D_kf(z)D_iD_jf(z)\\
&\quad-\Bigl(D_{g^{-1}h_k}a(z)h_j,h_i\Bigr)_H D_kf(z)D_iD_jf(z)\\
&\quad+\Bigl(g(x)D_{g^{-1}h_k}a(x)h_j,h_i\Bigr)_H D_kf(z)D_iD_jf(z).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{VI}_1+\mathrm{VI}_2 &= 2a(z)^{ip}D_pa(z)^{jk}D_kf(z)D_iD_jf(z)\\
&\quad+2D_ia(z)^{jk}D_kf(z)D_iD_jf(z)\\
&\quad-D_ka(z)^{ij}D_kf(z)D_iD_jf(z)\\
&\quad-a(z)^{kl}D_la(z)^{ij}D_kf(z)D_iD_jf(z).
\end{aligned}
$$

By using operator $\Gamma^{\cdot}_{\cdot k}(z):H\to H$ defined by $\Gamma^{\cdot}_{\cdot k}(z)f=\Gamma^i_{jk}(z)(f,h_i)_Hh_j$, we can write down VII as follows:

$$
\begin{aligned}
\mathrm{VII} &= \bigl(\delta^{ip}+a(z)^{ip}\bigr)\bigl(\delta^{jq}+a(z)^{jq}\bigr)\Gamma^k_{ij}(z)D_kf(z)\Gamma^r_{pq}(z)D_rf(z)\\
&= \bigl(\delta^{jq}+a(z)^{jq}\bigr)\\
&\qquad\times\Bigl((I_H+a(z))\Gamma^k_{ij}(z)(Df(z),h_k)_Hh_i,\Gamma^r_{pq}(z)(Df(z),h_r)_Hh_p\Bigr)_H\\
&= \bigl(\delta^{jq}+a(z)^{jq}\bigr)\Bigl((I_H+a(z))\Gamma^{\cdot}_{\cdot j}(z)Df(z),\Gamma^{\cdot}_{\cdot q}(z)Df(z)\Bigr)_H\\
&= \Bigl((I_H+a(z))^{jq}\Gamma^{\cdot *}_{\cdot q}(z)(I_H+a(z))\Gamma^{\cdot}_{\cdot j}(z)Df(z),Df(z)\Bigr)_H. \qquad (6.37)
\end{aligned}
$$

Combining the representation of the Hessian term above and (6.33), we arrive at

$$
\begin{aligned}
&\big(\mathrm{Ric}(z)Df(z), Df(z)\big)_{T_zH^*} \\
&\quad = \|Df(z)\|_H^2 + a(z)^{ij}D_if(z)D_jf(z) \\
&\qquad + \frac{1}{2}L_a a(z)^{ij}D_if(z)D_jf(z) \\
&\qquad - \frac{1}{2}La(z)^{ij}D_if(z) - a(z)^{ij}La(z)^{ik}D_jf(z)D_kf(z) \\
&\qquad - \Big((I_H + a(z))^{jq}\Gamma^{\cdot *}_{\cdot q}(z)(I_H + a(z))\Gamma^{\cdot}_{\cdot j}(x)Df(z), Df(z)\Big)_H \\
&\quad = \Big(\Big\{(I_H + a)\cdot(I_H - La)(z) + \frac{1}{2}\mathcal{L}a(z) \\
&\qquad - (I_H + a)^{ij}\Gamma^{\cdot *}_{\cdot j}(I_H + a)\Gamma^{\cdot}_{\cdot i}(z)\Big\}Df(z), Df(z)\Big)_H .
\end{aligned}
\tag{6.38}
$$

which completes the proof. □

In the rest of this section, we shall give the sufficient condition which assures the boundedness of the Ricci curvature of Dirichlet form $\mathcal{E}$. Here we restrict ourselves to the case where the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by

$$
\mathcal{E}(u, v) = \int_E \Big((I_H + \Phi(\Theta(z))a(z))Du(z), Dv(z)\Big)_H \mu(dz) ,
$$

where $\mathcal{D}(\mathcal{E}) = \mathbb{D}^1_2(E, \mathbb{R})$, $\Theta(\cdot) \in \mathbb{D}^\infty_{\infty-}(E, \mathbb{R})$, $\Phi(\cdot) \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $a(z) = \sigma(z)^*\sigma(z)$ and $\sigma(\cdot) \in \mathbb{D}^\infty_{\infty-}(E, L_{(2)}(H, H))$.

**Lemma 6.1.** *We assume that there exist $\varphi_1(\cdot)$ , $\varphi_2(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that the following conditions hold for $\mu$-a.e. $z \in E$ :*

(1)

$$
\begin{aligned}
&\|a(z)\|_{H^{\otimes 2}} + \|Da(z)\|_{H^{\otimes 3}} + \|D^*a(z)\|_H \\
&\qquad + \|La(z)\|_{H^{\otimes 2}} + \|D^2a(z)\|_{H^{\otimes 4}} \le \varphi_1(\Theta(z)),
\end{aligned}
\tag{6.39}
$$

(2)

$$
\|D\Theta(z)\|_H + \|D^2\Theta(z)\|_{H^{\otimes 2}} + |L\Theta(z)| \le \varphi_2(\Theta(z)). \tag{6.40}
$$

*Then there exists $K > 0$ such that*

$$
\|\mathrm{Ric}(z)\|_{L(H,H)} \le K \qquad \textit{for } \mu\textit{-a.e. } z \in E .
$$

*Proof.* Let us denote $a_\Phi(z) := \Phi(\Theta(z))a(z)$ and $g_\Phi^{-1}(z) := (I_H + a_\Phi(z))$. By Lemma 2.14, the Ricci curvature is given by

$$\begin{aligned}\mathrm{Ric}(z) &= \big\{I_H - La_\Phi(z) + \frac{1}{2}(I_H + a_\Phi)^{-1}\mathcal{L}a_\Phi(z)\big\}\\ &\quad -\big\{(I_H + a_\Phi)^{ij}(I_H + a_\Phi)^{-1}\Gamma^{\cdot *}_{\cdot j}(I_H + a_\Phi)\Gamma^{\cdot}_{\cdot i}(z)\big\}\\ &:= \mathrm{I} - \mathrm{II}\ ,\end{aligned}$$

where $\Gamma^{\cdot}_{\cdot k}(z)$ is a Hilbert-Schmidt operator on $H$ defined in Lemma 2.14. Here, $\Gamma^k_{ij}$ are the coefficients of Levi-Civita connection on $(H_z, g_\Phi)$, i.e.,

$$\begin{aligned}\Gamma^k_{ij}(z) &= -\frac{1}{2}\Big(h_k, D_i a_\Phi(z) g_\Phi(z) h_j\Big)_H\\ &\quad -\frac{1}{2}\Big(h_k, D_j a_\Phi(z) g_\Phi(z) h_i\Big)_H\\ &\quad +\frac{1}{2}\Big(g_\Phi(z) D_{g_\Phi^{-1}h_k} a_\Phi(z) g_\Phi(z) h_i, h_j\Big)_H\\ &:= \mathrm{III}_{ijk} + \mathrm{IV}_{ijk} - \mathrm{V}_{ijk}\ .\end{aligned} \tag{6.41}$$

First we calculate $\mathrm{I}$.

$$\begin{aligned}\mathrm{I} &= I_H - La_\Phi(z) + \frac{1}{2}(I_H + a_\Phi)^{-1}(z) La_\Phi(z)\\ &\quad +\frac{1}{2}(I_H + a_\Phi)^{-1}(z) L_{a_\Phi} a_\Phi(z)\\ &:= I_H - \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned}$$

Then we will calculate $\mathrm{I}_1$, $\mathrm{I}_2$ and $\mathrm{I}_3$ and estimate their $L(H,H)$ norm.

$$\begin{aligned}\mathrm{I}_3 &= \frac{1}{2}(I_H + a_\Phi(z))^{-1}\Big((a_\Phi(z))^{ij} D_i D_j a_\Phi(z) - D_{D^* a_\Phi(z)} a_\Phi(z)\Big)\\ &= \frac{1}{2}(I_H + a_\Phi(z))^{-1}\\ &\quad \times\Big\{\Big(\Phi(\Theta(z))\Phi''(\Theta(z)) D_i\Theta(z) D_j\Theta(z) a(z)^{ij} a(z)\\ &\qquad +\Phi(\Theta(z))\Phi'(\Theta(z)) D_i D_j\Theta(z) a(z)^{ij} a(z)\\ &\qquad +\Phi(\Theta(z))\Phi'(\Theta(z)) D_i\Theta(z) a(z)^{ij} D_j a(z)\\ &\qquad +\Phi(\Theta(z))^2 a(z)^{ij} D_i D_j a(z)\Big)\\ &\quad -\Big(\Phi(\Theta(z))^2 D_{D^* a(z)} a(z) - \Phi(\Theta(z))\Phi'(\Theta(z)) D_{a(z)D\Theta(z)} a(z)\\ &\quad -(\Phi'(\Theta(z)))^2 D_{a(z)D\Theta(z)} a(z)\\ &\quad +\Phi(\Theta(z))\Phi'(\Theta(z)) D_{D^* a(z)}\Theta(z) a(z)\Big)\Big\}.\end{aligned}$$

Noting

$$\|(I_H + a_\Phi(z))^{-1}\|_{L(H,H)} \le 1. \tag{6.42}$$

and by using the assumption (1), (2) and (6.42), we get

$$\begin{aligned}\|\mathrm{I}_3\|_{\mathrm{L(H,H)}} &\le C\big|\Phi(\Theta(z))\Phi'(z)\big|\varphi_1(\Theta(z))\varphi_2(\Theta(z)) \\ &\qquad \times\Big(\varphi_1(\Theta(z))+\varphi_2(\Theta(z))+1\Big) \\ &\quad +C\big|\Phi(\Theta(z))\Phi''(\Theta(z))\big|\varphi_1(\Theta(z))\varphi_2(\Theta(z))^2 \\ &\quad +C\Phi(\Theta(z))^2\varphi_1(\Theta(z))^2 \\ &\quad +C\Phi'(\Theta(z))^2\varphi_1(\Theta(z))^2\varphi_2(\Theta(z))\ .\end{aligned}$$

Since $\Phi(\cdot)\in C_0^\infty(\mathbb{R},\mathbb{R})$, $\varphi_1(\Theta(z))$ and $\varphi_1(\Theta(z))$ have upper bounds which are idependent of $z\in E$. Consequently $\|\mathrm{I}_3\|_{L(H,H)}$ is uniformly bounded with respect to $z\in E$.

Concerning $\mathrm{I}_1$, by using the same calculation, we can get

$$\begin{aligned}\|\mathrm{I}_1\|_{L(H,H)} &\le C\Big(|\Phi'(\Theta(z))|+|\Phi''(\Theta(z))|\Big)\varphi_1(\Theta(z)) \\ &\quad +C|\Phi(\Theta(z))|\cdot\varphi_2(\Theta(z))\end{aligned}$$

which implies the boundedness of $\mathrm{I}_1$. Thus the remainder term $\mathrm{I}_2$ is also bounded by (6.42). We proceed to the estimate for $\mathrm{II}$. Noting (6.42),

$$\begin{aligned}\big\|\mathrm{II}\big\|_{\mathrm{L(H,H)}} &\le \big\|\Gamma^{\cdot *}_{\cdot i}(z)\Gamma^{\cdot}_{\cdot i}(z)\big\|_{L(H,H)}+\big\|\Gamma^{\cdot *}_{\cdot i}(z)a_\Phi(z)\Gamma^{\cdot}_{\cdot i}(z)\big\|_{L(H,H)} \\ &\quad +\big\|\big(a_\Phi(z)\big)^{ij}\Gamma^{\cdot *}_{\cdot j}(z)\Gamma^{\cdot}_{\cdot i}(z)\big\|_{L(H,H)} \\ &\quad +\big\|\big(a_\Phi(z)\big)^{ij}\Gamma^{\cdot *}_{\cdot j}(z)a_\Phi(z)\Gamma^{\cdot}_{\cdot i}(z)\big\|_{L(H,H)} \\ &:= \|\mathrm{II}_1\|_{L(H,H)}+\|\mathrm{II}_2\|_{L(H,H)} \\ &\quad +\|\mathrm{II}_3\|_{L(H,H)}+\|\mathrm{II}_4\|_{L(H,H)}\ .\end{aligned}$$

Now we estimate $\|\mathrm{II}_i\|_{L(H,H)}$ $(1\le i\le 4)$:

$$\|\mathrm{II}_1\|_{L(H,H)}\le\sum_{i=1}^\infty\|\Gamma^{\cdot}_{\cdot i}(z)\|^2_{H^{\otimes 2}}, \tag{6.43}$$

$$\|\mathrm{II}_2\|_{L(H,H)}+\|\mathrm{II}_3\|_{L(H,H)}\le 2\|a_\Phi(z)\|_{H^{\otimes 2}}\sum_{i=1}^\infty\|\Gamma^{\cdot}_{\cdot i}(z)\|^2_{H^{\otimes 2}}. \tag{6.44}$$

$$\|\mathrm{II}_4\|_{L(H,H)}\le\|a_\Phi(z)\|^2_{H^{\otimes 2}}\sum_{i=1}^\infty\|\Gamma^{\cdot}_{\cdot i}(z)\|^2_{H^{\otimes 2}}\ . \tag{6.45}$$

Thus we have

$$\|\mathrm{II}\|_{L(H,H)}\le\big(1+2\varphi_1(\Theta(z))+\varphi_1(\Theta(z))^2\big)\sum_{i=1}^\infty\|\Gamma^{\cdot}_{\cdot i}(z)\|^2_{H^{\otimes 2}}\ . \tag{6.46}$$

By (6.41),

$$\begin{aligned}\sum_{i=1}^{\infty}\|\Gamma_{\cdot i}^{\cdot}(z)\|_{H^{\otimes 2}}^2 &= \sum_{i,p,q=1}^{\infty}\left|\Gamma_{pi}^q(z)\right|^2 \\ &\le 3\sum_{i,j,k=1}^{\infty}\left(\|\mathrm{III}_{ijk}\|^2+\|\mathrm{IV}_{ijk}\|^2+\|\mathrm{V}_{ijk}\|^2\right) \\ &:= 3(\mathrm{III}+\mathrm{IV}+\mathrm{V})\ . \qquad (6.47)\end{aligned}$$

Noting (6.42), we estimate $\mathrm{III},\mathrm{IV},\mathrm{V}$ as follows:

$$\begin{aligned}\mathrm{III}+\mathrm{IV} &\le \frac{1}{2}\left\|Da_{\Phi}(z)\right\|_{H^{\otimes 3}}^2, \qquad (6.48)\\ \mathrm{V} &\le \frac{1}{2}\|Da_{\Phi}(z)\|_{H^{\otimes 3}}^2\left(1+\|a_{\Phi}(z)\|_{H^{\otimes 2}}^2\right)\ . \qquad (6.49)\end{aligned}$$

By virtue of (6.46), (6.47), (6.48) and (6.49), we get

$$\begin{aligned}\|\mathrm{II}\|_{\mathrm{L(H,H)}} &\le 3\left(1+\varphi_1(\Theta(\mathrm{z}))+\varphi_1(\Theta(\mathrm{z}))^2\right)\cdot\left(1+|\Phi(\Theta(\mathrm{z}))|\cdot\varphi_1(\Theta(\mathrm{z}))^2\right) \\ &\quad\times\Big(|\Phi(\Theta(z))|\cdot\varphi_1(\Theta(z))+|\Phi'(\Theta(z))|\cdot\|\Theta(z)\|_H\cdot\varphi_1(\Theta(z))\Big)^2\ .\end{aligned}$$

which completes the proof. □

# 7 Application to Stochastic Differential Equations

In this section, we will consider the case that the diffusion coefficient is smooth in the Fréchet sense. If the diffusion coefficient is smooth, it is natural to apply large deviation theory to our problem. In fact Zhang [35] got the Varadhan type asymptotics in this way. Here we show the Varadhan type asymptotics by another way. We can apply Theorem 2.16 to get the lower estimate by using Lemma 6.1.

Let $\sigma(\cdot)\in C_b^2(E,L(E,H))$. In this case, we can construct the diffusion process $X=(X_t,P_z)$ associated with Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ as a solution to the following stochastic differential equation on $E$ :

$$\begin{aligned}dX_t &= \sqrt{2}\big(I_H+a(X_t)\big)^{1/2}\cdot dw_t-\big(D^*a(X_t)+X_t\big)\ dt\ ,\\ X_0 &= z\ ,\end{aligned}$$

where $w=(w_t)_{t\ge 0}$ is a standard Brownian motion on $E$. Throughout this section, we consider the classical Wiener space only. Namely let

$$\begin{aligned}E \ := \ \Big\{&x(\cdot)\in L^{2m}([0,1]\to\mathbb{R}^d)\ \Big|\ x(0)=0\ ,\\ &\big\|x\big\|_E:=\Big(\int_0^1\int_0^1\frac{|x(s)-x(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\Big)^{1/2m}<\infty\Big\}\ .\end{aligned}$$

where $0 < \alpha < 1/2$ and $m \in \mathbb{N}$ with $2m\alpha > 1$.

$$\begin{aligned} H \quad := \quad & \Big\{ h \in E \,\Big|\, h(t) = \int_0^t \dot{h}(s) ds, \\ & \|h\|_H := \Big( \int_0^1 |\dot{h}(t)|^2 dt \Big)^{1/2} < \infty \Big\} . \end{aligned}$$

We note that the inner product on $\mathbb{R}^d$ is denoted simply by $( \ , \ )$ and the norm by $|\cdot|$.

**Remark 7.1.** Let $(W_0^d, H, \mu_0)$ be the $d$-dimensional classical Wiener space. Namely $W_0^d$ is the space of continuous paths on $\mathbb{R}^d$ starting at $0$. Then note that the space $E$ above is one of the choice of the support of $\mu_0$. Moreover the embeddings

$$H \underset{\rightarrow}{\subsetneq} E \underset{\rightarrow}{\subsetneq} W_0^d$$

are compact. For details the reader is referred to Sugita [29].

The following are the main results in this section. Note that the following theorem holds even if the Ricci curvature of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is not bounded from below for $\mu$-a.e. $z \in E$.

**Theorem 7.2.** *Let $A, B \subset E$ be Borel measurable sets with $\mu(A), \mu(B) > 0$. We assume that $A$ or $B$ is $H$-open and $\sigma(\cdot) \in C_b^2(E, L(E, H))$. Then the following asymptotics holds*

$$\lim_{t \to 0} 4t \cdot \log P_\mu(t, A, B) = -d_g(A, B)^2.$$

To prove this theorem, we prepare the following lemma to control the Ricci curvature of Dirichlet form $\mathcal{E}$.

**Lemma 7.3.** *For $x \in E$, $0 < \alpha < 1/2$ and $m \in \mathbb{N}$ with $2m\alpha > 1$, we define $\theta(\cdot) : E \to \mathbb{R}$ as follows:*

$$\theta(x) := \|x\|_E^{2m} = \int_0^1 \int_0^1 \frac{|x(s) - x(t)|^{2m}}{|s - t|^{1+2m\alpha}} dsdt \ .$$

*Then the following estimates hold.*

(1) *For $h \in H$,*

$$|D_h \theta(x)| \le 2m \cdot \theta(x)^{(2m-1)/2m} \theta(h)^{1/2m} \ . \tag{7.50}$$

(2)

$$\begin{aligned} \|D^2\theta(x)\|_{H^{\otimes 2}} \le\ & 2\sqrt{2}dm \\ & \times (4m^2 - 8m + 5)^{1/2} (m - 2m\alpha)^{-1/m} \theta(x)^{(m-1)/m} \ . \end{aligned} \tag{7.51}$$

(3)

$$\begin{aligned} |L\theta(x)| \le\ & 2m(d + 2m - 2)(m - 2m\alpha)^{-1/m} \cdot \theta(x)^{(m-1)/m} \\ & + 2m\theta(x) \ . \end{aligned} \tag{7.52}$$

*Proof.* First we prove (1). We remark that the following identity holds for any $x \in E$ and $h \in H$:

$$D_h\theta(x) = 2m\int_0^1\int_0^1 \frac{|x(s)-x(t)|^{2(m-1)}}{|s-t|^{1+2m\alpha}}\Big(x(s)-x(t), h(s)-h(t)\Big)dsdt. \tag{7.53}$$

By using Hölder's inequality, we get

$$\begin{aligned} |D_h\theta(x)| &\leq 2m\cdot\left(\int_0^1\int_0^1 \frac{|x(s)-x(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\right)^{(2m-1)/2m} \\ &\quad\times\left(\int_0^1\int_0^1 \frac{|h(s)-h(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\right)^{1/2m} \\ &= 2m\cdot\theta(x)^{(2m-1)/2m}\theta(h)^{1/2m}. \end{aligned}$$

This completes the proof of (1). Next, we prove (2): Taking the derivative with respect to $h_i, h_j \in \mathcal{H}$, we have

$$\begin{aligned} D_iD_j\theta(x) &= 4m(m-1)\int_{[0,1]^2}\frac{|x(s)-x(t)|^{2(m-2)}}{|s-t|^{1+2m\alpha}} \\ &\quad\times(x(s)-x(t), h_i(s)-h_i(t))\cdot(x(s)-x(t), h_j(s)-h_j(t))dsdt \\ &\quad+2m\int_{[0,1]^2}\frac{|x(s)-x(t)|^{2(m-1)}}{|s-t|^{1+2m\alpha}}\left(h_i(s)-h_i(t), h_j(s)-h_j(t)\right)^2dsdt \\ :=&\ 4m(m-1)\int_0^1\int_0^1 f_{ij}(s,t)dsdt \\ &\quad+2m\int_0^1\int_0^1 g_{ij}(s,t)\ dsdt. \end{aligned} \tag{7.54}$$

Here we note that the following identity holds for $\mathcal{H} = \{h_i\}_{i=1}^\infty$.

$$\sum_{i=1}^\infty |h_i(s)-h_i(t)|^2 = d\ |s-t|\ . \tag{7.55}$$

Hence the following estimate holds by using Hölder's inequality.

$$
\begin{aligned}
&\|D^2\theta(x)\|_{H^{\otimes 2}}^2\\
&= \sum_{i,j=1}^{\infty}\Big\{4m(m-1)\int_0^1\int_0^1 f_{ij}(s,t)\ dsdt + 2m\int_0^1\int_0^1 g_{ij}(s,t)\ dsdt\ \Big\}^2\\
&\leq 32m^2(m-1)^2\sum_{i,j=1}^{\infty}\Big(\int_0^1\int_0^1 f_{ij}(s,t)\ dsdt\Big)^2\\
&\quad +8m^2\sum_{i,j=1}^{\infty}\Big(\int_0^1\int_0^1 g_{ij}(s,t)\ dsdt\ \Big)^2\\
&\leq 8d^2m^2(4m^2-8m+5)\Big(\int_0^1\int_0^1\frac{|x(s)-x(t)|^{2(m-1)}}{|s-t|^{1+2m\alpha}}|s-t|dsdt\Big)^2\\
&\leq 8d^2m^2(4m^2-8m+5)\Big(\int_0^1\int_0^1\frac{|x(s)-x(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\Big)^{2(m-1)/m}\\
&\qquad\times\Big(\int_0^1\int_0^1\frac{dsdt}{|s-t|^{1+2m\alpha-m}}\Big)^{2/m}. \qquad (7.56)
\end{aligned}
$$

By virtue of the assumptions, there exists $\varepsilon > 0$ such that $1+2m\alpha-m = 1-\varepsilon$. Hence we calculate as follows:

$$
\begin{aligned}
\int_0^1\int_0^1\frac{dsdt}{|s-t|^{1+2m\alpha-m}} &= \int_0^1 ds\Big\{\int_0^s\frac{dt}{(s-t)^{1-\varepsilon}}+\int_s^1\frac{dt}{(t-s)^{1-\varepsilon}}\Big\}\\
&= \int_0^1 ds\Big(\int_s^0\frac{-du}{u^{1-\varepsilon}}+\int_s^1\frac{du}{u^{1-\varepsilon}}\Big)\\
&= \int_0^1 ds\int_0^1\frac{du}{u^{1-\varepsilon}}\\
&= \varepsilon^{-1}\\
&= (m-2m\alpha)^{-1}\ . \qquad (7.57)
\end{aligned}
$$

Consequently, by virtue of (7.56) and (7.57), we get

$$
\|D^2\theta(x)\|_{H^{\otimes 2}} \leq 2\sqrt{2}d\cdot m(4m^2-8m+5)^{1/2}(m-2m\alpha)^{-1/m}\theta(x)^{(m-1)/m}\ .
$$

This completes the proof of (2).

Finally, we prove (3). Noting

$$
\begin{aligned}
L\theta(x) &= 2m(d+2m-2)\int_0^1\int_0^1\frac{|x(s)-x(t)|^{2(m-1)}}{|s-t|^{1+2m\alpha}}|s-t|dsdt\\
&\qquad -2m\int_0^1\int_0^1\frac{|x(s)-x(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\ . \qquad (7.58)
\end{aligned}
$$

By virtue of (7.57) and (7.58), we can conclude that

$$
\begin{aligned}
|L\theta(x)| \quad &\leq \quad 2m(d+2m-2)\Big(\int_0^1\int_0^1 \frac{|x(s)-x(t)|^{2(m-1)}}{|s-t|^{1+2m\alpha}}dsdt\Big)^{(m-1)/m} \\
&\qquad\qquad \times\Big(\int_0^1\int_0^1 \frac{dsdt}{|s-t|^{1+2m\alpha-m}}\Big)^{1/m}. \\
&\quad +2m\Big(\int_0^1\int_0^1 \frac{|x(s)-x(t)|^{2m}}{|s-t|^{1+2m\alpha}}dsdt\Big) \\
&= \quad 2m(d+2m-2)(m-2m\alpha)^{-1/m}\theta(x)^{(m-1)/m} \\
&\quad +2m\theta(x)\ .
\end{aligned}
$$

We complete the proof. □

*Proof of Theorem 7.2.* It is clear to get the upper estimate by using Theorem 2.10. Hence, we prove the lower estimate.

Let us explain our strategy to get the lower estimate. First, using the diffusion coeffient $a(\cdot)$, we define an approximate Dirichlet form $\mathcal{E}^\Phi$ whose Ricci curvature is bounded below. Next we compare $P_z$ with $P_z^\Phi$, where $P_z^\Phi$ is the transition probability associated with $\mathcal{E}^\Phi$.

We devide the proof into several steps.

**Step 1. (Construction of Dirichlet form $\mathcal{E}^\Phi$):** Let us take $C \subset B$ and $v \in H$ as in (5.29). Further we take a compact set $K \subset C$ with $\mu(K) > 0$ and fix a positive number $r > 0$. To control the growth of $\theta$, we need the following lemma.

**Lemma 7.4.** *For any $s > 0$, there exists a positive constant $C(s)$ such that*

$$\sup_{x \in K+U_H(s)} |\theta(x)| \leq C(s),$$

*where,* $U_H(s) := \{u \in H \mid \|u\|_H \leq s\}$.

*Proof.* For any $x \in K+U_H(r)$, there exist $y \in K$ and $h \in U_H(r)$ such that $x = y+h$ holds. By recalling $\|x\|_E = \theta(x)^{1/2m}$, the following inequality holds:

$$\theta(x)^{1/2m} \leq \theta(y)^{1/2m} + \theta(h)^{1/2m}.$$

Since $\theta(\cdot)$ is a continuous function on $E$ and $U_H(r)$ is a compact set in $E$, we can set the desired constant $C(r) < \infty$ as follows:

$$C(r) := \Big(\sup_{y \in K}|\theta(y)|^{1/2m} + \sup_{h \in U_H(r)}|\theta(h)|^{1/2m}\Big)^{2m}.$$

This completes the proof. □

Next we introduce a cut-off function $\Phi(\cdot) \in C_0^\infty(\mathbb{R}, [0,1])$ :

$$\Phi(x) = \begin{cases} 1 & |x| \le 2, \\ 0 & |x| \ge 3. \end{cases} \tag{7.59}$$

and define $(\mathcal{E}^\Phi, \mathcal{D}(\mathcal{E}^\Phi))$ as follows:

$$\mathcal{E}^\Phi(u,v) := \int_E (A_\Phi(x)Du(x), Dv(x))_H \mu(dx) \ , \tag{7.60}$$

where $\mathcal{D}(\mathcal{E}^\Phi) := \mathbb{D}_2^1(E,\mathbb{R})$ and $A_\Phi(x) := I_H + \Phi\Big(\dfrac{\theta(x)}{C(r)}\Big) \cdot a(x)$.

The reason for introducing this Dirichlet form is explained in the following Lemma.

**Lemma 7.5.** *The Ricci curvature of* $(\mathcal{E}^\Phi, \mathcal{D}(\mathcal{E}^\Phi))$ *is bounded.*

*Proof.* First, we recall that $L(E,H) \subset L_{(2)}(H,H)$ holds, i.e.,

$$\|T\|_{H^{\otimes 2}} \le \left(\int_E \|x\|_E^2 \mu(dx)\right)^{1/2} \|T\|_{L(E,H)} \quad \text{for} \quad T \in L(E,H).$$

For details see Kuo [18]. Also by the assumption on $a$, the norms

$$\|a(x)\|_{L(E,H)}, \|Da(x)\|_{L(E\times E,H)}, \|D^2a(x)\|_{L(E\times E\times E,H)}$$

are uniformly bounded with respect to $x \in E$. Therefore, by virtue of Lemma 6.1 and Lemma 7.3, it is sufficient to prove that there exists $\varphi(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\big\|La(x)\big\|_{L(E,H)} + \big\|D^*a(x)\big\|_H \le \ \varphi(\theta(x)) \ . \tag{7.61}$$

Next, we calculate $D^*a(x)$ as follows. $\mathcal{H} = \{h_i\}_{i=1}^\infty$ denotes a complete orthonormal basis of $H$.

$$\begin{aligned} D^*a(x) &= -\sum_{i=1}^\infty D_i a(x) \cdot h_i + a(x)x \\ &= -\int_E D_x a(x) \cdot x \ \mu(dx) + a(x)x \ . \end{aligned}$$

Hence there exists a positive constant $C$ which is independent of $x \in E$, such that

$$\begin{aligned} \big\|D^*a(x)\big\|_H &\le \big\|Da(x)\big\|_{L(E\times E,H)} \int_E \|x\|_E^2 \mu(dx) + \big\|a(x)\big\|_{L(E,H)} \cdot \|x\|_E \\ &\le C\Big(1 + \theta(x)^{1/2m}\Big) \ . \end{aligned} \tag{7.62}$$

By the same calculation for $La(x)$, we can get

$$\begin{aligned}\big\|La(x)\big\|_{L(E,H)} &\leq \big\|D^2a(x)\big\|_{L(E\times E\times E,H)} \int_E \|x\|_E^2 \mu(dx) \\ &\quad + \big\|Da(x)\big\|_{L(E\times E,H)} \cdot \|x\|_E \\ &\leq C\Big(1+\theta(x)^{1/2m}\Big). \end{aligned} \tag{7.63}$$

By combining (7.62) and (7.63), we conclude that (7.61) holds. □

**Step 2. (A Relation between $\mathcal{E}$ and $\mathcal{E}^{\Phi}$):** Here we will study the relation between $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ and $(\mathcal{E}^{\Phi},\mathcal{D}(\mathcal{E}^{\Phi}))$. First, we define an open set $G \subset E$ as follows.

$$G := \big\{x \in E \mid \theta(x) < 2C(r)\big\} .$$

Here $C(r)$ is a positive constant denoted in Lemma 7.4. Let us consider the part of the Dirichlet form $\mathcal{E}$ on $G$ which we denote by $\mathcal{E}_G$. By virtue of (7.60), we notice the following identity.

$$\begin{aligned}\mathcal{E}_G(u,v) &= \int_G (A(x)Du(x),Dv(x))_H \mu(dx) \\ &= \int_G (A_\Phi(x)Du(x),Dv(x))_H \mu(dx) = \mathcal{E}_G^{\Phi}(u,v) , \\ \mathcal{D}(\mathcal{E}_G) &= \big\{u \in \mathbb{D}_2^1(E,\mathbb{R}) \mid \tilde{u} = 0 \ for\ q.e.\ x \in E\setminus G\big\} \\ &= \mathcal{D}(\mathcal{E}_G^{\Phi}) .\end{aligned}$$

We denote by $X^{\Phi} = (X_t^{\Phi},P_z^{\Phi})$ the diffusion process corresponding to $(\mathcal{E}^{\Phi},\mathcal{D}(\mathcal{E}^{\Phi}))$. Then we can conclude that $X_G = X_G^{\Phi}$ holds by using the following proposition.

**Proposition 7.6 (Fukushima-Oshima-Takeda [11]).** *Let $\mathcal{E}$ be a regular Dirichlet form on $L^2(E,\mu)$ that is associated with $\mu$-symmetric Hunt process $\mathbf{M}$ on E. Let $G\subset E$ be a q.e. finely open set. Then the part $\mathcal{E}_G$ of $\mathcal{E}$ on $G$ is a Dirichlet form on $L^2(G,\mu)$. $\mathcal{E}_G$ is associated with $\mathbf{M}_G$ in the sense that the strongly continuous semigroup $\{T_t^G\}_{t>0}$ on $L^2(G,\mu)$ corresponding to $\mathcal{E}_G$ is determined by the transition function $\{p_t^G\}_{t>0}$ of $\mathbf{M}_G$.*

**Step 3. (Proof of the Lower Estimate):** First we note the following inclusion by virtue of Remark 2.6 and Lemma 7.4.

$$K_{M_2^{-1}} = \big\{x \in E \mid d_g(x,K) \leq M_2^{-1}\cdot r\big\} \subset K + U_H(r) \ \subset G .$$

Since $X_G = X_G^\Phi$, we have

$$\begin{aligned}
P_\mu(t,A,B) &= P_\mu\Big(\big\{X_0 \in A,\ X_t \in B\big\}\Big) \\
&\geq P_\mu\Big(\big\{X_0 \in K,\ X_t \in K+v\big\}\Big) \\
&\geq P_\mu\Big(\big\{X_0 \in K,\ X_t \in K+v, \\
&\qquad \sup_{0\le s\le t} d_g(X_s,K) \le M_2^{-1}\cdot r\big\}\Big) \\
&= P_\mu^\Phi\Big(\big\{X_0^\Phi \in K,\ X_t^\Phi \in K+v, \\
&\qquad \sup_{0\le s\le t} d_g(X_s^\Phi,K) \le M_2^{-1}\cdot r\big\}\Big) \\
&= P_\mu^\Phi\Big(\big\{X_0^\Phi \in K,\ X_t^\Phi \in K+v\big\}\Big) \\
&\quad -P_\mu^\Phi\Big(\big\{X_0^\Phi \in K,\ X_t^\Phi \in K+v, \\
&\qquad \sup_{0\le s\le t} d_g(X_s^\Phi,K) > M_2^{-1}\cdot r\big\}\Big) \\
&:= \mathrm{I} - \mathrm{II}. \qquad (7.64)
\end{aligned}$$

By using the proof of Theorem 2.10, we get

$$\begin{aligned}
\mathrm{II} &\le P_\mu^\Phi\Big(\big\{\sup_{0\le s\le t} d_g(X_s^\Phi,K) > M_2^{-1}\cdot r\big\}\Big) \\
&\le \frac{4}{\sqrt{2\pi}}\int_{M_2^{-1}\cdot r/\sqrt{t}}^{\infty} \exp(-\frac{s^2}{2})ds\ . \qquad (7.65)
\end{aligned}$$

By combining (7.64), (7.65), letting $r \to \infty$ and recalling the proof of Theorem 2.16, there exist a Borel measurable set $K' \subset K$ and a sequence $\{t_n\}_{n=1}^\infty \downarrow 0$ such that the following estimate holds :

$$\begin{aligned}
&4t_n \log P_\mu(t_n,A,B) \\
&\quad \ge 4t_n \log\left\{\left(\frac{1}{2}\right)^\alpha \cdot \mu(K')\cdot \exp\left(-\frac{\alpha(d_g(A,B)+\varepsilon)^2}{4(\alpha-1)}\cdot\frac{2K}{1-e^{-2Kt_n}}\right)\right\}.
\end{aligned}$$

Finally, we complete the proof of the lower estimate by letting $n \to \infty$, $\alpha \to \infty$ and $\varepsilon \downarrow 0$. □

**Remark 7.7.** In the finite dimensional case and if $\sigma(\cdot)$ is smooth, then $\Theta(z) = \|z\|_E^2$ satisfies (6.39) and (6.40) for suitable $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$. Contrary to finite dimensional cases, it is difficult to find the functions $\Theta(\cdot)$, $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ which satisfy (6.39) and (6.40) in infinite dimensional cases. In other words, it is difficult to find the Dirichlet form whose Ricci curvature is bounded from below.

# 8 Examples

In this section, we will discuss two examples. In Example 1, we consider the diffusion process whose diffusion coefficient is not smooth in the Fréchet sense. In Example 2, we consider the diffusion process arising from the statistical mechanics.

### Example 1. Discontinuous coefficient case

We shall consider an example involving multiple Wiener integrals. In general multiple Wiener integrals are not continuous in the Fréchet sense. We emphasize that our approach can handle such a case. First, we shall review the notion of multiple Wiener integrals. We define the set of symmetric multilinear forms of the Hilbert-Schmidt type as follows:

$$H^{(n)} := \{g \in H^{\otimes n} \mid g \text{ is a symmetric form}\} .$$

For each $f \in H^{(n)}$, its multiple Wiener integral $I_n(f)(\cdot) \in \mathbb{D}^{\infty}_{\infty-}(E, \mathbb{R})$ is defined in the following way:

$$I_n(f)(x) := \{(D^*)^n f\}(x).$$

For $1 \leq k < n$, we may identify $f \in H^{(n)}$ with a multiple form of the Hilbert-Schmidt type $f^{(k)} : H^{\otimes n-k} \to H^{\otimes k}$ given by

$$\left(f^{(k)}(h_1 \otimes \ldots \otimes h_{n-k}),\ h_{n-k+1} \otimes \ldots \otimes h_n\right)_{H^{\otimes k}} = f(h_1 \otimes \ldots \otimes h_n).$$

In particular, $I_k(f)$ denotes its $k$-tuple Wiener integral $I_k(f^{(n-k)})$ .

We shall recall the following fundamendal analyticity property of multiple Wiener integrals.

**Proposition 8.1 (Sugita-Taniguchi [30]).** *Let $n \in \mathbb{N}$ and $f \in H^{(n)}$. Then the multiple Wiener integrals $I_{n-k}(f)$, $k = 0, \ldots n$, admit $\mu$-versions $\hat{I}_{n-k}(f)$ such that $\hat{I}_{n-k}(f) \in C^{\omega,p}(E, H^{\otimes k})$ for any $p \in (1, \infty)$ and the following identity holds.*

$$\hat{I}_n(f)(x+h) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \left(\hat{I}_{n-k}(f)(x), h^{\otimes k}\right)_{H^{\otimes k}}, \tag{8.66}$$

*for any $x \in E$ and $h \in H$.*

Here we omit the definition of $C^{\omega,p}(E, H^{\otimes k})$. See Sugita-Taniguchi [30] for the details.

Next, we state our example. In the sequel, we denote $T := S^*S$ for $S \in L_{(2)}(H,H)$. For $f \in H^{(n)}$ and $S \in L_{(2)}(H,H)$, let

$$\sigma(x) := I_n(f)(x) \cdot \Phi(\Theta(x))^{1/2} \cdot S \ \in L_{(2)}(H,H), \tag{8.67}$$

where $\Phi(\cdot) \in C_0^\infty(\mathbb{R},\mathbb{R})$ is defined in (7.59) and $\Theta(\cdot) \in \mathbb{D}^\infty_{\infty-}(E,\mathbb{R})$ is defined as follows:

$$\begin{aligned} \Theta(x) \ &:= \ \sum_{k=0}^{n} \|D^k I_n(f)(x)\|^2_{H^{\otimes k}} + \sum_{k=0}^{2} \|D^k I_2(T)(x)\|^2_{H^{\otimes k}} \\ &:= \ \Theta_1(x) + \Theta_2(x) \ . \end{aligned} \tag{8.68}$$

Then $A(\cdot)$, the coefficient of the Dirichlet form in (2.3), is given as follows:

$$A(x) := I_H + a(x) = I_H + I_n(f)(x)^2 \cdot \Phi(\Theta(x)) \cdot T \ . \tag{8.69}$$

Below, we always assume that we are working with the modifications of the multiple Wiener integrals $I_n(f)(\cdot)$ and of $I_2(T)(\cdot)$ which satisfy the identity (8.66). By virtue of (7.59) and (8.68), we note that there exists $C > 1$ such that

$$\mathrm{esssup}_{x \in E} \|A(x)\|_{L(H,H)} \leq C$$

and for any $x \in E$, $A(x) - I_H$ is a positive symmetric operator.

By Proposition 8.1, $A(\cdot)$ is $H$-*continuous*. So we prove the assumption $(\mathrm{A}_4)$ to get the upper estimate. To this end, let us prove the $H$-*UC property* of the multiple Wiener integrals.

**Lemma 8.2.** (1) *The multiple Wiener integral $I_n(f)(\cdot)$ belongs to $H$-$UC(E,\mathbb{R})$.*

(2) *For $A(\cdot)$ which is denoted in* (8.69)*, $A(\cdot)^{-1}$ belongs to $H$-$UC(E,L(H,H))$.*

*Proof.* First, we construct $H$-$UC$ nest $\{K_m\}_{m=1}^\infty$. By applying Lusin's theorem to $\{I_{n-k}(f)(\cdot)\}_{k=0}^n$, $\{I_{2-k}(T)(\cdot)\}_{k=0}^2$, there exist a sequence of increasing compact sets $\{K_m\}_{m=1}^\infty$ in $E$ with $\mu(E \setminus K_m) \leq \frac{1}{m}$ such that

$$I_{n-k}(f) : K_m \to H^{\otimes k} \tag{8.70}$$

is continuous for any $0 \leq k \leq n$ and $m \in \mathbb{N}$ and

$$I_{2-k}(T) : K_m \to H^{\otimes k}$$

is continuous for any $0 \leq k \leq 2$ and $m \in \mathbb{N}$. We prove that $I_n(f)(\cdot)$ satisfies $H$-$UC$ property for each $K_m$. By Proposition 8.1 and (8.70), we notice the following estimate holds for any $x, y \in K_m$ and $R > 0$:

$$
\begin{aligned}
&\lim_{x \to y,\ x \in K_m} \Big( \sup_{\|h\|_H \le R} \big| I_n(f)(x+h) - I_n(f)(y+h) \big| \Big) \\
&\quad \le \lim_{x \to y,\ x \in K_m} \Bigg( \sup_{\|h\|_H \le R} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \\
&\qquad\qquad \times \Big\| I_{n-k}(f)(x) - I_{n-k}(f)(y) \Big\|_{H^{\otimes k}} \cdot \Big\| h^{\otimes k} \Big\|_{H^{\otimes k}} \Bigg) \\
&\quad \le \sum_{k=0}^{n} R^k \frac{n!}{k!(n-k)!} \cdot \Big( \lim_{x \to y,\ x \in K_m} \Big\| I_{n-k}(f)(x) - I_{n-k}(f)(y) \Big\|_{H^{\otimes k}} \Big) \\
&\quad = 0.
\end{aligned} \tag{8.71}
$$

This completes of the proof of (1). Next we prove that $A(\cdot)^{-1}$ satisfies $H$-$UC$ property for each $K_m$. By noticing (8.70), for any $x, y \in K_m$ and $h \in H$, we have

$$
\begin{aligned}
&\|A^{-1}(x+h) - A^{-1}(y+h)\|_{L(H,H)} \\
&\quad = \Big\| \Big( I_H + I_n(f)(x+h)^2 \Phi(\Theta(x+h)) T \Big)^{-1} \\
&\qquad - \Big( I_H + I_n(f)(y+h)^2 \Phi(\Theta(y+h)) T \Big)^{-1} \Big\|_{L(H,H)} \\
&\quad \le \Big\| \big( I_n(f)(x+h)^2 \Phi(\Theta(x+h)) T \big) \\
&\qquad - \big( I_n(f)(y+h)^2 \Phi(\Theta(y+h)) T \big) \Big\|_{L(H,H)} \\
&\quad \le \Big\| \Phi(\Theta(x+h)) \cdot \Big( I_n(f)(x+h)^2 - I_n(f)(y+h)^2 \Big) \cdot T \Big\|_{L(H,H)} \\
&\qquad + \Big\| I_n(f)(y+h)^2 \cdot \Big( \Phi(\Theta(x+h)) - \Phi(\Theta(y+h)) \Big) \cdot T \Big\|_{L(H,H)} \\
&\quad \le C \Big| I_n(f)(x+h)^2 - I_n(f)(y+h)^2 \Big| \\
&\qquad + C \Big| \Phi(\Theta(x+h)) - \Phi(\Theta(y+h)) \Big| ,
\end{aligned}
$$

where $C$ is a positive constant which is independent of $x, y \in K_m$. Thus (2) easily follows from (1). $\square$

To prove the lower estimate, we prove that the Ricci curvatue of Dirichlet form $\mathcal{E}$ is bounded from below. To this end, we prepare the following lemma.

**Lemma 8.3.** *For $\Theta(\cdot) \in \mathbb{D}^{\infty}_{\infty-}(E, \mathbb{R})$ which is defined in* (8.68)*, the following estimates hold for any $x \in E$.*

(1) $\|D\Theta(x)\|_H \le 2\ \Theta(x)$

(2) $\|D^2\Theta(x)\|_{H^{\otimes 2}} \le 4\sqrt{n+3}\ \Theta(x)$.

(3) $|L\Theta(x)| \le 2(n+4)\ \Theta(x)$.

*Proof.* First, we prove (1). By the definition of the multiple Wiener integrals, it is easy to see that

$$\begin{aligned}(D)^k I_n(f)(\cdot) &= 0 \qquad \text{for } k \geq n+1,\\ (D)^* I_n(f)(\cdot) &= 0. \end{aligned} \tag{8.72}$$

For $\Theta_1(x)$ which is defined in (8.68), the following estimate holds for any $x \in E$.

$$\begin{aligned}\|D\Theta_1(x)\|_H &\leq 2\sum_{k=0}^{n} \|D^k I_n(f)(x)\|_{H^{\otimes k}} \cdot \|D^{k+1} I_n(f)(x)\|_{H^{\otimes k+1}}\\ &\leq \sum_{k=0}^{n}\Big(\|D^k I_n(f)(x)\|^2_{H^{\otimes k}} + \|D^{k+1} I_n(f)(x)\|^2_{H^{\otimes k+1}}\Big)\\ &\leq 2\ \Theta_1(x)\ .\end{aligned}$$

We notice that the same estimate holds for $\Theta_2(x)$. We complete the proof of (1).

Next, we prove (2). First note that

$$\begin{aligned}&\Big(D_i D_j \|D^k I_n(f)(x)\|^2_{H^{\otimes k}}\Big)^2\\ &= \Big\{2\Big(D_i D_j D^k I_n(f)(x), D^k I_n(f)(x)\Big)_{H^{\otimes k}}\\ &\quad +2\Big(D_j D^k I_n(f)(x), D_i D^k I_n(f)(x)\Big)_{H^{\otimes k}}\Big\}^2\\ &\leq 8\big\|D_i D_j D^k I_n(f)(x)\big\|^2_{H^{\otimes k}} \cdot \big\|D^k I_n(f)(x)\big\|^2_{H^{\otimes k}}\\ &\quad +8\big\|D_i D^k I_n(f)(x)\big\|^2_{H^{\otimes k}} \cdot \big\|D_j D^k I_n(f)(x)\big\|^2_{H^{\otimes k}}\ .\end{aligned} \tag{8.73}$$

By using (8.73), $\Theta_1(x)$ can be estimated as follows:

$$\begin{aligned}\|D^2\Theta_1(x)\|^2_{H^{\otimes 2}} &= \sum_{i,j=1}^{\infty}\big(\sum_{k=0}^{n} D_i D_j \|D^k I_n(f)(x)\|^2_{H^{\otimes k}}\big)^2\\ &\leq 8(n+1)\Big(\sum_{k=0}^{n}\|D^{k+2} I_n(f)(x)\|^2_{H^{\otimes k+2}}\Big)\\ &\qquad \times\Big(\sum_{k=0}^{n}\|D^k I_n(f)(x)\|^2_{H^{\otimes k}}\Big)\\ &\quad +8(n+1)\Big(\sum_{k=0}^{n}\|D^{k+1} I_n(f)(x)\|^2_{H^{\otimes k+1}}\Big)^2\\ &\leq 16(n+1)\Theta_1(x)^2.\end{aligned} \tag{8.74}$$

By the same calculation, we can get the following estimate for $\Theta_2(x)$.

$$\|D^2\Theta_2(x)\|_{H^{\otimes 2}}^2 \le 48\Theta_2(x)^2 \; . \tag{8.75}$$

By combining (8.74), (8.75), we complete the proof of (2).

Finally, we shall prove (3). Recall the following commutation rule. We can find this in many places, for example, in Sugita-Tanigichi [30].

$$D^*D^p = D^pD^* - p\ D^{p-1} \qquad \text{for any } p \in \mathbb{N}.$$

It is easy to see

$$\begin{aligned} L\|D^kI_n(f)(x)\|_{H^{\otimes k}}^2 &= 2\|D^{k+1}I_n(f)(x)\|_{H^{\otimes k+1}}^2 \\ &\quad +2\Big(LD^kI_n(f)(x), D^kI_n(f)(x)\Big)_{H^{\otimes k}}. \end{aligned} \tag{8.76}$$

Using the commutation rule and (8.72), we have

$$\begin{aligned} LD^kI_n(f)(x) &= -D^*D^{k+1}(D^*)^n(f)(x) \\ &= -\big\{D^{k+1}D^* - (k+1)D^k\big\}(D^*)^n(f)(x) \\ &= (k+1)D^kI_n(f)(x). \end{aligned} \tag{8.77}$$

By combining (8.76) and (8.77), we can get

$$\begin{aligned} L\|D^kI_n(f)(x)\|_{H^{\otimes k}}^2 &\le 2\|D^{k+1}I_n(f)(x)\|_{H^{\otimes k+1}}^2 \\ &\quad +2(k+1)\|D^kI_n(f)(x)\|_{H^{\otimes k}}^2. \end{aligned} \tag{8.78}$$

Hence we have

$$\begin{aligned} \big|L\Theta_1(x)\big| &\le 2\sum_{k=0}^{n} \|D^{k+1}I_n(f)(x)\|_{H^{\otimes k+1}}^2 \\ &\qquad +2(n+1)\sum_{k=0}^{n} \|D^kI_n(f)(x)\|_{H^{\otimes k}}^2 \\ &\le 2(n+2)\ \Theta_1(x). \end{aligned} \tag{8.79}$$

On the other hand, we can get the following estimate for $\Theta_2(x)$ by using the same calculation:

$$\big|L\Theta_2(x)\big| \le 8\ \Theta_2(x). \tag{8.80}$$

Consequently, we complete the proof of (3) by combining (8.79) and (8.80). □

We are now ready to prove the boundedness of the Ricci curvature of the Dirichlet form $\mathcal{E}$ which is sufficient to prove the lower estimate of the short time asymptotics.

**Lemma 8.4.** *The Ricci curvature of Dirichlet form $\mathcal{E}$ is bounded.*

*Proof.* By the definition of $\Theta$,

$$\left\|I_n(f)(x)T\right\|_{H^{\otimes 2}} + \left\|DI_n(f)(x)T\right\|_{H^{\otimes 3}} + \left\|D^2 I_n(f)(x)T\right\|_{H^{\otimes 4}} \leq 3\Theta(x)^{1/2}.$$

Hence by Lemma 6.1 and Lemma 8.3, it suffices to prove that there exists $\varphi(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\left\|L(I_n(f)(x)^2 T)\right\|_{H^{\otimes 2}} + \left\|D^*(I_n(f)(x)^2 T)\right\|_H \leq \varphi(\Theta(x)).$$

By the property $LI_n(f)(x) = -nI_n(f)(x)$, we have

$$\begin{aligned}
&\left\|L(I_n(f)(x)^2 T)\right\|_{H^{\otimes 2}} \\
&\qquad = \left\|2LI_n(f)(x)I_n(f)(x)T + 2\|DI_n(f)(x)\|_H^2 T\right\|_{H^{\otimes 2}} \\
&\qquad \leq 2n\left|I_n(f)(x)\right|^2 \cdot \left\|T\right\|_{H^{\otimes 2}} + 2\left\|DI_n(f)(x)\right|_H^2 \cdot \left|T\right|_{H^{\otimes 2}} \\
&\qquad \leq (2n+2)\ \Theta_1(x) \cdot \big(\frac{1}{2}\Theta(x)^{1/2}\big) \\
&\qquad \leq (n+1)\ \Theta(x)^{3/2}\ .
\end{aligned} \tag{8.81}$$

Using the same calculation, we get

$$\begin{aligned}
\left\|D^*(I_n(f)(x)^2 T)\right\|_H &= \left\|2I_n(f)(x) \cdot D_{T\cdot} I_n(f)(x) + I_n(f)(x)^2 D^* T(x)\right\|_H \\
&\leq 2\left|I_n(f)(x)\right| \cdot \left\|DI_n(f)(x)\right\|_H \cdot \left\|T\right\|_{H^{\otimes 2}} \\
&\quad + \left|I_n(f)(x)\right|^2 \cdot \left\|D^*(T)(x)\right\|_H \\
&\leq \Theta_1(x) \cdot \Theta_2(x)^{1/2} + \frac{1}{2}\Theta_1(x) \cdot \Theta_2(x)^{1/2} \\
&\leq \frac{3}{2}\ \Theta(x)^{3/2}.
\end{aligned}$$

Therefore, we complete the proof. □

**Example 2. Reversible diffusion process with respect to Gibbs Measure**

We will discuss the diffusion process whose reversible measure is a finite volume Gibbs measure. See for example Funaki [12], Hariya and Osada [14] and Osada and Spohn [28].

For $\xi \in C(\mathbb{R}, \mathbb{R})$, let $W_{r;\xi} := C\big([-r, r] \to \mathbb{R}; w(-r) = \xi(-r), w(r) = \xi(r)\big)$ and denote by $P^W_{r,\xi}$ the pinned Brownian motion measure on $W_{r;\xi}$ and consider the probability measure $\mu_{r,\xi}$ on it:

$$d\mu_{r,\xi} = \frac{1}{Z_{r,\xi}} \exp\left(-H_r(\xi, w)\right) dP^W_{r,\xi}$$

where $Z_{r,\xi}$ is the normalization constant. $H_r(\xi, w)$ is given by

$$\begin{aligned} H_r(\xi, w) &= \int_{-r}^{r} V(w_t)\, dt + \frac{1}{2}\int_{-r}^{r}\int_{-r}^{r} W(t-u, w_t - w_u)\, dtdu \\ &\quad + \iint_{|t|\le r \le |u|} W(t-u, w_t - \xi_u)\, dtdu\ , \end{aligned} \tag{8.82}$$

where $V$ is a self interaction potential and $W$ is a two-body interaction potential. Here we assume that $V,\ W$ are $C^2$ functions. Below we denote the $L^2$ inner product for two functions $\phi_1, \phi_2$ by $\langle \phi_1, \phi_2\rangle = \int_{-r}^{r} \phi_1(t)\phi_2(t)\, dt$. Here we will consider the following Dirichlet form

$$\mathcal{E}(u,v) := \int_{W_{r;\xi}} \big\langle \nabla u, \nabla v\big\rangle d\mu_{r,\xi}, \tag{8.83}$$

where

$$\nabla u(w)_t := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma)\phi_i(t), \quad \text{for } u(w) = f\big(\langle w, \phi_1\rangle, \dots, \langle w, \phi_n\rangle\big).$$

Taking the domain which consists of the functions in (8) where $\phi_i \in C_0^\infty((-r,r),\mathbb{R})$, we get the closable Dirichlet form. Note that the "carré du champ" operator $\Gamma(u,v)$ can be written using the usual $H$-derivative $D$ on the pinned Wiener space as follows:

$$\Gamma(u,v) := \big\langle \nabla u(w), \nabla v(w)\big\rangle = \big(-A_0 Du(w), Dv(w)\big)_{H_0^1\left((-r,r),\mathbb{R}\right)}$$

where $A_0$ denotes the Laplacian $\dfrac{d^2}{dt^2}$ with the Dirichlet boundary condition. Note that in this case the Riemannian metric is $L^2$-metric. Zhang [35] proved the short time asymptotics for this diffusion process using the usual large deviation theory. We can apply our method which was developed in Section 4 and Theorem 2.16 by calculating the $\Gamma_2$ for this Dirichlet form. In fact we can prove the following lower bound of $\Gamma_2$:

$$\Gamma_2(u,u) \ge \big\langle A\nabla u, \nabla u\big\rangle,$$

where $A$ denotes the Schrödinger operator:

$$\begin{aligned} (A\phi)(t) &:= -(A_0\phi)(t) \\ &\quad + \left\{ V''(w(t)) + \int_{|u|\ge r} W''(t-u, w(t) - \xi(u))du \right\} \phi(t) \\ &\quad + \int_{-r}^{r} W''(t-u, w(t)-w(u))(\phi(t)-\phi(u))du \end{aligned}$$

Here $W''(t,x) = \dfrac{\partial^2 W}{\partial x^2}(t,x)$. Consequently, for example, under the assumptions that

$$\inf_{t,x\in\mathbb{R}} W''(t,x) \geq 0, \ \ \inf_{x\in\mathbb{R}} V''(x) \geq -C > -\infty, \ \ \int_0^\infty \sup_{x\in\mathbb{R}} W''(t,x)\ dt < \infty \tag{8.84}$$

we can derive the lower bound of the Ricci curvature of the Dirichlet form:

$$\Gamma_2(u,u) \geq -C\Gamma(u,u).$$

Note that the bound is independent of the volume of the space $[-r,r]$. Therefore we get the volume independent estimate

$$|T_t^{(r,\xi)}u(w)|^\alpha \leq T_t^{(r,\xi)}|u|^\alpha(w+v) \ \cdot \exp\left(\frac{\alpha\langle v,v\rangle}{4(\alpha-1)}\cdot\frac{2C}{1-e^{-2Ct}}\right). \tag{8.85}$$

where $T_t^{(r,\xi)}$ is the diffusion semigroup corresponding to the Dirichlet form (8.83). This gives the lower bound of the short time asymptotics for the diffusion in finite volume. Actually Osada and Spohn [28] proved the existence of the Gibbs measure itself associated with the interaction potential $V$ and $W$ under some assumptions which are weaker than (8.84). Hence using the estimate (8.85) which is independent of the size of the volume $2r$ and the boundary condition, we may establish (8.85) for the diffusion whose reversible measure is the Gibbs measure itself and we may get the lower bound of the short time asymptotics. These will be discussed in the forthcoming paper.

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## References

[1] S. Aida, Uniform positivity improving property, Sobolev inequality and spectral gaps, *J. Funct. Anal* **158** (1998), 152–185.

[2] S. Aida, On the irreducibility of Dirichlet forms on a domain in infinite dimensional spaces, preprint (1998). To appear in *Osaka. J. Math.*

[3] S. Aida and T. S. Zhang, On the small time asymptotics of diffusion processes on path groups, preprint (1999). To appear in *Potential Analysis.*

[4] D. Bakry and M. Emery, Diffusions hypercontractives, in: *Séminaire de probabilités* XIX, Lecture Notes in Math. **1123** (1985), Springer-Verlag, 177–206.

[5] D. Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups, in: *New Trends in Stochastic Analysis*, World Scientific Publishing, River Edge, NJ, (1997), 43–75.

[6] R. F. Bass, *Diffusions and Elliptic Operators*, Springer-Verlag, New York (1998).

[7] N. Bouleau and F. Hirsch, *Dirichlet Forms and Analysis on Wiener Space*, Walter de Gruyter (1991).

[8] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, U.K. (1989).

[9] S. Fang, On the Ornstein-Uhlenbeck process, *Stochastic and Stochastic Reports* **46** (1994), 141–159.

[10] S. Fang and T. S. Zhang, On the small time behavior of Ornstein-Uhlenbeck processes with unbounded linear drifts, *Probab. Th. Relat. Fields* **114** (1999), 487–504.

[11] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter (1994).

[12] T. Funaki, SPDE approach and log-Sobolev inequalities for continuum field with two-body interactions. Preprint, (1996).

[13] E. Getzler, Dirichlet forms on loop spaces, *Bull. Sci. Math.* **113**(2) (1989), 151–174.

[14] Y. Hariya and H. Osada, Diffusion processes on path space with interactions, preprint (1999).

[15] M. Hino, On short time asymptotic behavior of some symmetric diffusions on general state spaces, preprint (1999).

[16] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, Kodansha/North-Holland (1989).

[17] H. Kawabi, On the precise asymptotic formula for certain infinite dimensional diffusion processes, in preparation.

[18] H. H. Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Math. **463** (1975), Springer-Verlag.

[19] S. Kusuoka, Dirichlet forms and diffusion processes on Banach space, *J. Fac. Sci. Univ. Tokyo.* Sec. IA **29** (1982), 79–95.

[20] S. Kusuoka, The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity, *J. Fac. Sci.*, Univ. Tokyo, Sec. IA **29** (1982), 567–598.

[21] S. Kusuoka, Some remarks on Getzler's degree theorem, in Probability theory and mathematical statistics, Lecture Notes in Math. **1299** (1988), Springer-Verlag, 239–249.

[22] S. Kusuoka, Degree theorem in certain Wiener Riemannian manifolds, in Stochastic analysis, Lecture Notes in Math. **1322** (1988), Springer-Verlag, 93–108.

[23] S. Kusuoka, Analysis on Wiener spaces I, Nonlinear maps, *J. Funct. Anal.* **98** (1991), 122–168.

[24] S. Kusuoka, Analysis on Wiener spaces II, Differential forms, *J. Funct. Anal.* **103** (1992), 229–274.

[25] T. J. Lyons and W. A. Zheng, A crossing estimate for the canonical process on a Dirichlet space and a tightness result, *Astérisque* **157–8** (1988), 249–271.

[26] T. J. Lyons and T. S. Zhang, Decomposition of Dirichlet process and its applications, *Annals of Probab.* **22** (1994), 494–524.

[27] J. R. Norris, Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds, *Acta. Math* **179** (1997), 79–103.

[28] H. Osada and H. Spohn, Gibbs measure relative to Brownian motion, *Annals of Probab.* **27** (1999), 1183–1207.

[29] H. Sugita, Positive generalized Wiener functions and potential theory over abstract Wiener spaces, *Osaka. J. Math* **25** (1988), 665–696.

[30] H. Sugita and S. Taniguchi, Oscillatory integrals with quadratic phase function on a real abstract Wiener space, *J. Funct. Anal.* **155** (1998), 229–262.

[31] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math.* **20** (1967), 431–455.

[32] S. R. S. Varadhan, Diffusion processes in a small time interval, *Comm. Pure Appl. Math.* **20** (1967), 657–685.

[33] F-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, *Probab. Th. Relat. Fields* **109** (1997), 417–424.

[34] S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin's Calculus*, Tata Inst. of Fundamental Research, Springer-Verlag, (1984).

[35] T. S. Zhang, On the small time asymptotics of diffusion processes on Hilbert Spaces, preprint, (1998). To appear in *Annals of Probab.*

[36] T. S. Zhang, On the small time large deviations of diffusion processes on configuration spaces, preprint, (1999).

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