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Gibbs Measures and Nonlinear Dispersive Equations

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- **Aim**: Review recent progress in nonlinear dispersive equations and a probabilistic approach by the Gibbs measure.

  1. Introduction
  Time local well-posedness in low regularity space for the Cauchy problem of nonlinear dispersive equation
  2. Gibbs Measures and Global Solutions
  How to prove the global existence of solution by the Gibbs measure?

1 Introduction

- Nonlinear Schrödinger Equation (NLS)

$$i\partial_t u + \partial_x^2 u = \lambda |u|^5 u, \qquad (1)$$

$$t \in \mathbf{R},\ x \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$$

$$u(0,x) = u_0(x). \qquad (2)$$

Mass and Energy Conservation Laws

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \qquad (3)$$

$$E(u(t)) = E(u_0), \qquad (4)$$

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\lambda}{6}\|v\|_{L^6}^6.$$

- $\lambda = 1$ (defocusing) $\implies$ E is positive definite, no blowup solution, no nontrivial standing wave (soliton-like solution),
- $\lambda = -1$ (focusing) $\implies$ E is not bounded from below, blowup solutions, standing waves

We assume $\lambda = 1$ (defocusing) from now on. Let $X = H^s$ for any s with $1/2 > s > 1/3$.

Remark 1 We note that $X \supset \cap_{s<1/2} H^s$ and so X includes the paths of the Brownian motion.

Theorem 1 [Bourgain, 1996]

$$\forall u_0 \in X \implies \exists T > 0, \exists \text{unique solution}$$
$$u(t) \text{ to } (1) - (2) \text{ in } C([-T, T]; X),$$
$$\|u(t)\|_X \leq 2\|u_0\|_X \quad (|t| < T).$$

Furthermore, $T \geq \big(C\|u_0\|_X\big)^{-\alpha}$ for some $\alpha, C > 0$ and the continuous dependence in

X of solution on initial data holds.

Remark 2 The following three claims in Theorem 1 play an important role later.

$$T \geq \left(C\|u_0\|_X\right)^{-\alpha}, \tag{5}$$

$$\|u(t)\|_X \leq 2\|u_0\|_X \ (|t| < T), \tag{6}$$

$$u_{0,n} \to u_0 \ in \ X \implies$$
$$\|u_n(t) - u(t)\|_{C([-T,T];X)} \to 0. \tag{7}$$

- **Question**: For various nonlinear dispersive equations such as KdV and NLS, the time local existence of solutions has been proved within the framework of weak spaces including distributions by the Fourier restriction norm method. It is very interesting whether these solutions exist globally in time or not.

  How does the Gibbs measure work for the proof of the global existence?

2 Gibbs Measures and Global Solutions

$$P_N f = \sum_{|k| \le N} \hat{f}(k) e^{ikx}, \quad N \in \mathbf{N}.$$

Consider the Cauchy problem of the truncated equation:

$$i\partial_t u_N + \partial_x^2 u_N = P_N\left[|u_N|^2 u_N\right], \quad t \in \mathbf{R},\ x \in \mathbf{T} \tag{8}$$

$$u_N(0,x) = P_N u_0(x). \tag{9}$$

Remark 3 (i)

$$\forall u_0 \in X \subset \cap_{s<1/2} H^s \Longrightarrow$$
$$\exists \text{global solution of } (8) - (9)$$

(ii) The truncated NLS (8) is a finite dimensional Hamiltonian system.

- Finite Dimensional Gibbs Measure
  We can construct the Gibbs measure for the truncated NLS by the Liouville theorem,

because (8)-(9) is a finite dimensional problem.

$$
\begin{aligned}
d\mu_N &= Z_N^{-1} e^{-\frac{1}{6}\|\phi_N\|_{L^6}^6} d\rho_N, \qquad N \in \mathbf{N}, \\
d\rho_N &= \prod_{|k|\le N} e^{-\frac{1}{2}\sum |k|^2 |a_k|^2}\ d^2 a_k, \\
\phi_N &= \sum_{|k|\le N} a_k e^{ikx}, \qquad\qquad a_k \in \mathbf{C},
\end{aligned}
$$

where Z_N is the normalization constant.

Here, $d^2 a_k$ is regarded as the Lebesgue measure on $\mathbf{R}^2$ for two components of the real and the imaginary parts of a_k.

Lemma 1 $\forall T > 0$, $\forall \varepsilon > 0$, $\exists$ cylindrical set $\Omega_N \subset X$ such that

$$\mu_N\left(\Omega_N^c\right) < \varepsilon,$$

$$P_N u_0 \in \Omega_N \;\Rightarrow\; \|u_N(t)\|_X \leq C\left(\log \frac{T}{\varepsilon}\right)^{1/2},$$

$$|t| < T.$$

- Proof of Lemma 1

$\Phi_N(t)$; the solution map of (8),

$$\Omega_N = \bigcap_{|j| \leq [T/\delta]} \Phi_N^j(\delta)\big(\{\|P_N u_0\|_X \leq K\}\big).$$

Remark 4 The image $\Phi_N^j(\delta)$ is the set such that the value at $t = -j\delta$ of solution of (8) with its element as initial data is in the ball centered at the origin with radius K and so Ω_N is the set of all initial data such that if we choose its element as initial data, the solution u_N satisfies

$$\|u_N(t)\|_X \le 2K \quad (|t| < T).$$

(The last inequality follows (6) in Remark 2.)

If we choose $\delta \sim K^{-\alpha}$, we have by the invariance of μ_N

$$\mu_N\big(\Omega_N^c\big) \le C\frac{T}{\delta}\mu_N\big(\{\|P_N u_0\|_X > K\}\big)$$
$$\sim TK^{\alpha}e^{-cK^2} \sim Te^{-c_0K^2}.$$

Choose $K \sim \big(\log(T/\varepsilon)\big)^{1/2}$ and we have by (5) in Remark 2

$$\mu_N\big(\Omega_N^c\big) < \varepsilon.$$

Therefore,

$$\begin{aligned} \|u_N(t)\|_X &\leq 2K \\ &\sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}, \quad |t| < T \\ \Longrightarrow \ & \text{Lemma 1} \end{aligned}$$

Lemma 1 + weak convergence of $\{\mu_N\}$ to μ $\Longrightarrow \forall \varepsilon > 0$, $\exists \Omega_\varepsilon \subset X$ such that $\mu\big(\Omega_\varepsilon^c\big) < \varepsilon$ and for $u_0 \in \Omega_\varepsilon$, (1)-(2) has global solution u satisfying

$$\|u(t)\|_X \leq C\bigg(\log \frac{1+|t|}{\varepsilon}\bigg)^{1/2}, \quad t \in \mathbf{R}. \tag{10}$$

Estimate (10) + Borel-Cantelli Theorem $\Longrightarrow$ for a.s. $u_0 \in X$, $\exists$ global solution u of (1)-(2) satisfying (10) for some $\varepsilon > 0$.

Furhtermore, (7) in Remark 2 yields the

invariance of masure μ under the flow generated by equation (1).

Remark 5 (i) The crucial part of the above proof is based on the time local well-posedness in X of the Cauchy problem (1)-(2), while the probabilistic part of the above proof is standard. Especially, (5)-(7) are indispensable to the proof for the both parts of the global existence of solution and the invariance of measure μ.

(ii) In the global existence result proved above, “almost surely” means that the Gibbs measure zero set in X is exceptional. In fact, the space $H^s \setminus \cap_{\tau<1/2} H^\tau$ is measure zero with respect to the Gibbs measure. In other words, the above global existence result implies that for almost sure $u_0 \in \cap_{\tau<1/2} H^\tau$, (1)-(2) has a global solution. The probabilistic approach by the Gibbs measure is useful if one wishes to have the global existence result in the space corresponding to the support of the

Gibbs measure.

- **References**

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