On rough PDEs

Samy Tindel

University of Nancy

Rough Paths Analysis and Related Topics - Nagoya 2012

Joint work with: Aurélien Deya and Massimiliano Gubinelli

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim
2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim
2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim
2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

Equation under consideration

Equation:

- $dy_t = Ay_t dt + f(y_t) dx_t$, with $y_0 = \psi$
- $t \in [0, T]$, $y_t \in \mathcal{B}$, with $\mathcal{B}$ of the form L^p
- $A =$ Laplace operator, with semi-group $(S_t)_{t \geq 0}$
  $\hookrightarrow$ Generalization to divergence type operators
- f from $\mathcal{B}$ to an operator space
- x general noise, γ-Hölder continuous, $\gamma > 1/3$

Mild formulation:

$$y_t = S_t \psi + \int_0^t S_{ts} f(y_s) dx_s, \quad \text{with} \quad S_{ts} := S_{t-s}$$

Motivations: Natural generalization of SPDEs, continuity results

Methods of resolution (1)

- Brownian case
  Peszat-Zabczyk, Dalang (90s)
  Well understood by probabilistic methods
- Additive noise, x inf-dim fBm, for any H
  Tindel-Tudor-Viens ('03)
  Wiener integrals estimates for fBm
- Non-linear case, inf-dim fBm, $H > 1/2$
  Maslowski-Nualart ('03), smooth noise in space
  Fractional integrals
- Linear and nonlinear cases, x finite-dim
  Caruana-Friz ('08-'09), Diehl-Friz ('11)
  Abstract setting of rough paths, viscosity solutions

Methods of resolution (2)

- Non-linear case, x γ-Hölder path with $\gamma > 1/2$
  Non-smooth noise in space
  Young integration in infinite dimension
  Local solution, due to the fact that f is only locally Lipschitz
  Lejay-Gubinelli-Tindel ('06)
- Wave equation, $d = 1$, x space-time noise with $\gamma > 1/2$
  Young integration in the plane
  Quer-Tindel ('07)
- Genuine rough paths setting for SPDEs
  Applications: global solution for fBm, $H > 5/6$
  Brownian case, specific cases of f, infinite dimensional noise
  Gubinelli-Tindel ('10)

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim
2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

General aim

- Existence and uniqueness results in the mild sense for

$$dy_t = Ay_t dt + f(y_t) dx_t \tag{1}$$

- $y_t \in$ function in space, Hölder continuous in time
- A Laplace operator, f non-linear coefficient
- x_t finite dimensional, γ-Hölder continuous in time, $\gamma > 1/3$
- Application: $x \equiv$ fBm, with $H > 1/3$

Bibliographical note:

- Local solution obtained in Deya-Gubinelli-T. (PTRF, to appear)
- Global solution obtained by Deya (Elec. J. Probability)

Sketch of the talk

1 Introduction
- Pathwise type stochastic PDEs
- Main aim

2 Description of the results
- Abstract setting
- Applications
- Heuristic computations in the Young case

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim

2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

The rough paths black box for SDEs

Hypothesis: $x \in \mathcal{C}^{\gamma}(\mathbb{R}^d)$ with $\gamma > 1/3$

x allows to define a Levy area $\mathbf{x^2} \in \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \equiv \int dx \int dx$

Coefficient $\sigma \in C_b^3$

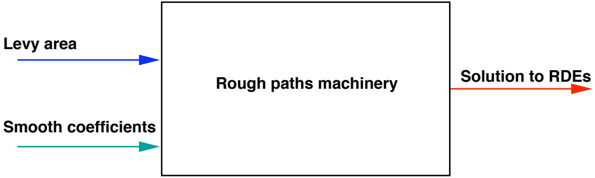
Main rough paths theorem: Let y be the solution to

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s.$$

Then (Lyons-Qian, Friz-Victoir, Gubinelli)

$$F : \mathbb{R}^n \times \mathcal{C}^{\gamma}(\mathbb{R}^d) \times \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \longrightarrow \mathcal{C}^{\gamma}(\mathbb{R}^n), \quad (a, x, \mathbf{x^2}) \mapsto y$$

is a continuous map



State space for the solution (SPDEs)

L^p space: $\mathcal{B}_p := L^p(\mathbb{R}^n)$

Fractional Sobolev spaces: for $\alpha \in [0, 1/2)$,

$$\mathcal{B}_{\alpha,p} := \mathcal{W}^{2\alpha,p}(\mathbb{R}^n) = [\mathrm{Id} - \Delta]^{-\alpha} \left(L^p(\mathbb{R}^n) \right)$$

Action of the heat semigroup:
Contraction: S_t contraction on $\mathcal{B}_{\alpha,p}$
Regularization: $\|S_t \varphi\|_{\mathcal{B}_{\alpha,p}} \leq c_\alpha t^{-\alpha} \|\varphi\|_{\mathcal{B}_p}$.

Remark: We work with $\mathcal{B}_{\alpha,p}$ with large p
$\hookrightarrow$ Because we want this space to be an algebra.

A (slightly) unusual formulation

- We solve equation (1) under the form:

$$y_t = S_t\psi + \sum_{i=1}^{N} \int_0^t S_{ts}\, dx_s^i\, f_i(y_s), \qquad (2)$$

- f_i function from $\mathcal{B}_p$ to $\mathcal{B}_p$
- x^i scalar noise
- Formulation (2) fits better to rough path type expansions

Example of nonlinear term: $[f(\varphi)](\xi) = \sigma(\xi, \varphi(\xi))$, where $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ rapidly decreasing function in ξ.

Operator valued Levy area

First order integrals: $a_{us} = S_{us} - \mathrm{Id}$

$$\begin{aligned} X^{x,i}_{ts}(\varphi) &= \int_s^t S_{tu}(\varphi)\, dx^i_u \\ X^{xa,i}_{ts}(\varphi,\psi) &= \int_s^t S_{tu}\left[a_{us}(\varphi)\cdot\psi\right]\, dx^i_u \end{aligned}$$

Second order integral: $\delta x_{us} = x_u - x_s$

$$X^{xx,ij}_{ts}(\varphi) = \int_s^t S_{tu}(\varphi)\, \delta x^j_{us}\, dx^i_u$$

More specifically: if $g :=$ heat kernel

$$\left[X^{xx,ij}_{ts}(\varphi)\right](\xi) = \int_s^t \left(\int_{\mathbb{R}^n} g_{t-u}(\xi-\eta)\varphi(\eta)\, d\eta\right)\, \delta x^j_{us}\, dx^i_u$$

General result (loose formulation)

Theorem

Assume

- x *allows to define* X^{x}, X^{xa} *and* X^{xx} *(operator valued on* $\mathcal{B}_{\gamma,p}$*)*
- $X^{x} \in \mathcal{C}^{\gamma}$ *and* $X^{xa}, X^{xx} \in \mathcal{C}^{2\gamma}$
- $\gamma > 1/3$
- f *and initial condition* ψ *regular enough (*$\psi \in \mathcal{B}_{\gamma,p}$*)*

Then the equation

$$y_t = S_t\psi + \int_0^t S_{ts} f(y_s)\, dx_s$$

has a unique global solution y *on* $[0, T]$

- y *is a continuous function of* X^{x}, X^{xa} *and* X^{xx}

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim

2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

Examples of application

Space-time dependence:
$\{\sigma^j;\, j \leq N\}$ collection of rapidly decreasing smooth functions
Generic form of the noise: $x_t = \sum_{j=1}^{N} \sigma^j \, B_t^j$

Application in the Young setting: $B^j \equiv$ fBm with $H > 1/2$

Application in the rough setting: $B^j \equiv$ fBm with $H > 1/3$

Equation which can be solved:

$$\partial_t y_t(\xi) = \Delta y_t(\xi) + \sum_{j=1}^{N} \sigma^j(\xi)\, f(y_t(\xi))\, dB_t^j$$

Sketch of the talk

1 Introduction
- Pathwise type stochastic PDEs
- Main aim

2 Description of the results
- Abstract setting
- Applications
- Heuristic computations in the Young case

Setting

Notational simplification:

- $x \in \mathcal{C}^\gamma$ with $\gamma > 1/2$, 1-dimensional
- $[f(y)](\xi) = \sigma(\xi, y(\xi))$.

Equation we wish to solve:

$$y_t = S_t \psi + \int_0^t S_{ts}\, dx_s\, f(y_s)$$

Hypothesis:
The solution y_t exists in a space $\mathcal{C}^\gamma([0, T]; \mathcal{B}_p)$, with $\gamma > 1/2$

Main step: define the integral $\int_0^t S_{ts}\, dx_s\, f(y_s)$
$\hookrightarrow$ fixed point argument

Formal computations

One way to catch the regularity of the solution y:

$$
\begin{aligned}
(\hat{\delta} y)_{ts} &:= y_t - S_{ts} y_s = \int_s^t S_{tu}\, dx_u\, f(y_u) \\
&= \left(\int_s^t S_{tu}\, dx_u \right) f(y_s) + \int_s^t S_{tu}\, dx_u\, \delta(f(y))_{us} \\
&= X^x_{ts}\, f(y_s) + y^\sharp_{ts}
\end{aligned}
$$

Definition of the terms:

- $X^x_{ts}\, f(y_s)$ well-defined as long as X^x is well-defined as an operator acting on $\mathcal{B}_p$
- $y^\sharp_{ts}$ defined as a Young integral if $\gamma > 1/2$ with Hölder regularity > 1

Integration result

Proposition

Assume that x allows to build X^x such that for $\gamma > 1/2$,

- *For any $\alpha \in [0, 1/2)$ s.t. $2\alpha p > 1$, $X^{x,i} \in \mathcal{C}^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))$*
- *The algebraic relation $\hat{\delta} X^{x,i} = 0$ is satisfied.*

Consider $z \in \mathcal{C}_1^0(\mathcal{B}_{\gamma,p}) \cap \mathcal{C}_1^\gamma(\mathcal{B}_p)$.

Then the element $\int_s^t S_{tu} dx_u z_u$

1. *Is well-defined as a Young integral*
2. *Defines an element of $\mathcal{C}^\gamma$, linearly bounded in terms of z*
3. *Is limit of $\sum_{(t_k)\in\Pi} S_{tt_{k+1}} X^{x,i}_{t_{k+1}t_k} z^i_{t_k}$ along partitions Π*

Perspectives

Some possible or ongoing projects:

1. Study of the solutions to rough SPDEs
   - Study of the law of y_t (first result with Deya)
   - Numerical schemes
   - Ergodic properties
2. Evolution equations via multiparametric integration
   $\hookrightarrow$ Walsh vs. Da Prato
3. Viscosity solutions: rough paths structures of sub or superjets