Higher order weak approximations of stochastic differential equations with and without jumps

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Outline

Introduction

Topics

1. We want to know the value of the expectation

$$E[f(X_T)]$$

where X_t is a solution of a stochastic differential equation (SDE).

2. Applications: finance, control, filtering, physics, etc.
3. However, we do <u>not</u> know the exact distribution of X_t in general. Therefore, we can not simulate X_t by MonteCarlo..
4. <u>Goal</u>: find a higher order approximation scheme $\overline{X}_t^{(n)}$ s.t.

$$E[f(X_T)] - E[f(\overline{X}_T^{(n)})] = O(n^{-k}).$$

(especially, $k \geq 2$)

What are merits of higher order methods ?

- Reducing computational cost

  (e.g. $k = 1, n = 1000 \Rightarrow k = 2, n =$several tens...)

- Quasi Monte Carlo: It (sometimes) holds that

$$E[f(\overline{X}_T^{(n)})] = \int_{\mathbf{R}^{\alpha(n)}} g(y)p(dy)$$

  - ${}^{\exists}g : \mathbf{R}^{\alpha(n)} \to \mathbf{R}$.
  - ${}^{\exists}p$: measure on $\mathbf{R}^{\alpha(n)}$.
  - If $\alpha(n)$ is not so large ($\sim$ 100, 1000?), QMC works well.

Approximations of SDEs without jumps

1. Introduction
2. **Approximations of SDEs without jumps**
   - Approximation of SDEs by random ODEs
   - (Extended) Cubature on Wiener space
   - Operator Splitting method as a Cubature
3. Approximations of SDEs with jumps I (finite jump intesity)
   - Operator Splitting
   - Jump-adapted approximation
4. Approximations of SDEs with jumps II (general Lévy process)
   - Overview
   - Small jump approximation
   - How to deal with compound Poisson term
5. Further Research

Setting

Consider a Stratonovich SDE ($\mathbf{R}^N$-valued)

$$X_t^x = x + \sum_{i=0}^{d} \int_0^t V_i(X_s^x) \circ dW^i(s) \tag{1}$$

- $(W^i)_{1 \le i \le d}$ is a d-dimensional Brownian motion.
- $W^0(s) = s$.
- $V_i \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$.

Our purpose: Compute

$$P_t f(x) := E[f(X_t^x)]$$

at time $t = T$, through time discretization methods.

Random Ordinary Differential Equation

Let $t > 0$ be a fixed time-scaling parameter. We want to construct an approximation process

$$\overline{X}_r^x = x + \sum_{i=0}^{d} \int_0^r V_i(\overline{X}_s^x) d\omega_t{}^i(s) \tag{2}$$

for $0 \leq r \leq t$.

- a driving random path ω has bounded variation paths.
- $\omega \approx W$ in some sense.
- ω depends on fixed (small) time t;

Approximation Operator

Approximation operator: For a scaling parameter t,

$$Q_t f(x) := E[f(\overline{X}_t^x)].$$

- Time partition: $\pi := \{0 = t_0 < t_1 < \cdots < t_n = T\}$
- Operator $Q_{t_1} Q_{t_2 - t_1} \cdots Q_{t_n - t_{n-1}}$

$\Leftrightarrow$ $^{\exists}$Markov chain whose transition follows from the random ODE (2) with scaling $t = t_{j+1} - t_j$.

$\Leftrightarrow$ $^{\exists}$Random ODE, defined in $[0, T]$, driven by (ω_π^i). Here $\omega_\pi^i(s)(0 \leq s \leq T)$ is defined for each time interval $[t_j, t_{j+1}]$ independently:

$$\overline{X}_r^{t_j, x} = x + \sum_{i=0}^{d} \int_{t_j}^{r} V_i(\overline{X}_s^{t_j, x}) d\omega_{t_{j+1} - t_j}^i(s), \quad r \in [t_j, t_{j+1}]$$

$((\omega_{t_{j+1} - t_j})_{j=0,\ldots,n-1}$ are independent.)

From local error to global error

(Rough sketch): $t_j - t_{j-1} \equiv T/n$

1. Assume $(P_t - Q_t)g = O(t^{k+1})$ for smooth g.
2. Then

$$\begin{aligned}(P_T - Q^n_{T/n})f &= (P^n_{T/n} - Q^n_{T/n})f \\ &= \sum_{j=0}^{n-1} Q^j_{t/n}(P_{T/n} - Q_{T/n})P_{(1-(j+1)/n)T}f \\ &= n \times O(n^{-(k+1)}) = O(n^{-k}).\end{aligned}$$

3. f: smooth & polynomial growth;
   see e.g. Talay & Tubaro(1990), Tanaka & Kohatsu-Higa (2009)
4. f: Lipschitz or more general, under Hörmander type condition;
   see Bally & Talay, Kohatsu-Higa, Kusuoka, etc.

Stochastic Taylor expansion I

- By using Itô formula, (V_i act as vector fields)

$$\begin{aligned} f(X_t^x) &= f(x) + \sum_{i=0}^{d} \int_0^t (V_i f)(X_s^x) \circ dW^i(s) \\ &= f(x) + \sum_{i=0}^{d} (V_i f)(x) \int_0^t \circ dW^i(s) \\ &\quad + \sum_{0 \le i,j \le d} \int_0^t \int_0^s (V_j V_i f)(X_r^x) \circ dW^j(r) \circ dW^i(s) \end{aligned}$$

- Notation for the order of convergence:
  For index $(i_1, \cdots, i_\ell)$, $i_j = 0, 1, \ldots, d$,

$$|(i_1, \cdots, i_\ell)| := \ell + \#\{j : i_j = 0\}.$$

Stochastic Taylor expansion I (continued)

- Stochastic Taylor expansion:

$$
\begin{aligned}
f(X_t^x) &= \sum_{|(i_1,\ldots,i_\ell)|\leq m} (V_{i_1}\cdots V_{i_\ell} f)(x) \\
&\quad \times \int_0^t \cdots \int_0^{s_2} \circ dW^{i_1}(s_1)\cdots \circ dW^{i_\ell}(s_\ell) \\
&\quad + R_m^X(t,f)
\end{aligned}
$$

  with

$$E[|R_m^X(t,f)|] \leq C(T,f)\, t^{(m+1)/2}.$$

- A similar expansion holds for $\overline{X}_t^x$.
  ($\circ dW$ is replaced by $d\omega$)

Cubature formulas (extended)

1. Definition: $\omega \in C_{0,\mathrm{bv}}([0,1];\mathbf{R}^{d+1})$ defines a cubature formula with degree m if for any index $(i_1,\dots,i_\ell)$ with $|(i_1,\dots,i_\ell)| \le m$,

$$E\Big[\int_0^1 \cdots \int_0^{s_2} \circ dW^{i_1}(s_1)\cdots \circ dW^{i_\ell}(s_\ell)\Big] = E\Big[\int_0^1 \cdots \int_0^{s_2} d\omega^{i_1}(s_1)\cdots d\omega^{i_\ell}(s_\ell)\Big],$$

and for $|(i_1,\dots,i_\ell)| = m+1$ or $m+2$,

$$\int_0^1 \cdots \int_0^{s_2} d|\omega^{i_1}|(s_1)\cdots d|\omega^{i_\ell}|(s_\ell) \in L^2(\Omega).$$

2. Lyons & Victoir (2004) assume in addition,
(We do not assume here)
   - $\omega^0(s) = s$.
   - ω has a discrete probability distribution on $C_{0,\mathrm{bv}}([0,1];\mathbf{R}^{d+1})$.

Time Scaling

1. Note that

$$\sqrt{t}^{|(i_1,\cdots,i_\ell)|}\int_0^1\cdots\int_0^{s_2}\circ dW^{i_1}(s_1)\cdots\circ dW^{i_\ell}(s_\ell)$$
$$\stackrel{\text{law}}{=}\int_0^t\cdots\int_0^{s_2}\circ dW^{i_1}(s_1)\cdots\circ dW^{i_\ell}(s_\ell).$$

2. Scaling for ω: For $0 \le s \le t$,

$$\begin{aligned}\omega_t^0(s) &:= t\omega^0(s/t)\\ \omega_t^i(s) &:= \sqrt{t}\omega_t^i(s/t),\ 1 \le i \le d.\end{aligned}$$

Error estimates

(Sketch)

- Assume that ω defines a cubature formula with degree m.
- $f \in C_b^\infty$.
- Then

$$\begin{aligned} E[f(X_t^x)] - E[f(\overline{X}_t^x)] &= E[R_m^X(t,f)] - E[R_m^{\overline{X}}(t,f)] \\ &= O(t^{(m+1)/2}). \end{aligned}$$

$\Rightarrow$ Order $k = (m-1)/2$ scheme !

$\Rightarrow$ For second order schemes, we need $\underline{m = 5}$.

Stochastic Taylor expansion II

Formal Taylor series of $t \mapsto e^{tL}$

- By taking expectations of stochastic Taylor expansions,

$$E[f(X_t^x)] = f(x) + tLf(x) + \frac{t^2}{2}L^2 f(x) + \frac{t^3}{3!}L^3 f(x) + \cdots$$

  where $L := \sum_{i=0}^{d} L_i$ ($\leftrightarrow$ the generator of P_t),
  - $L_0 := V_0$
  - $L_i = \frac{1}{2}V_i^2$, $1 \leq i \leq d$.

- Approximation of P_t $\Leftrightarrow$ Approximation of exponential map

Splitting of exponential maps on noncommutative algebra

1. Now we can not solve directly

$$e^{tL} = e^{t(L_0+L_1+\cdots+L_d)}.$$

2. However, each e^{tL_i} may be solvable ...

- Approximation of e^{tL} by $\{e^{tL_0}, \ldots, e^{tL_d}\}$
- $\{L_i\}$: noncommutative

Second order method via splitting

- $e^{t(L_i+L_j)} = I + t(L_i + L_j) + \frac{t^2}{2}(L_i + L_j)^2 + \cdots .$
- $e^{tL_i}e^{tL_j} = I + t(L_i + L_j) + \frac{t^2}{2}(L_i^2 + L_j^2 + 2L_iL_j) + \cdots .$

- Socond-order methods:

$$\begin{aligned} e^{t(L_i+L_j)} &= \frac{1}{2}e^{tL_i}e^{tL_j} + \frac{1}{2}e^{tL_j}e^{tL_i} + O(t^3) \\ e^{t(L_i+L_j)} &= e^{t/2L_i}e^{tL_j}e^{t/2L_i} + O(t^3) \end{aligned}$$

The idea can be found in the works of Strang (1960s).

ODE and coordinate SDE

What's e^{tL_i} ?

- Notation: $\exp(V)x$ is the solution of

$$\frac{dz_t(x)}{dt} = V(z_t(x)),\ z_0(x) = x$$

  at time $t = 1$.
- $\exp(W_t^i V_i)x$ is the solution of the SDE

$$dX_{i,t}^x = x + \int_0^t V_i(X_{i,s}^x) \circ dW^i(s) \in \mathbf{R}^N$$

- So, e^{tL_i} is solvable in the sense of

$$e^{tL_i} f(x) = E[f(X_{i,t}^x)] = E[f(\exp(W_t^i V_i)x)].$$

- $e^{tL_i} e^{tL_j} \leftrightarrow$ two step flow $\exp(W_t^j V_j) \exp(W_t^i V_i)x$

Ninomiya-Victoir scheme

Ninomiya-Victoir scheme

$$\overline{X}_t^x = \begin{cases} \exp(t/2V_0)\exp(\sqrt{t}Z^1V_1)\cdots\exp(\sqrt{t}Z^dV_d)\exp(t/2V_0)x, \\ \text{if } \Lambda = 1, \\ \exp(t/2V_0)\exp(\sqrt{t}Z^dV_d)\cdots\exp(\sqrt{t}Z^1V_1)\exp(t/2V_0)x, \\ \text{if } \Lambda = -1. \end{cases}$$

- $Z = (Z^i)_{1\le i\le d}$ and Λ are independent.
- $Z \sim N(0, I_d)$.
- $P(\Lambda = \pm 1) = 1/2$.
- If the ODE has no closed-form solution, then we can use Runge-Kutta methods.

Ninomiya-Victoir scheme as an extended cubature on Wiener space

Ninomiya-Victoir as cubature

$\omega^i(r) = \int_0^r \eta^i(s)ds$, $0 \le r \le 1$:

$$\eta^i(s) = \begin{cases} (d+1), & \text{if } i = 0, s \in [0, \frac{1}{2(d+1)}) \cup [\frac{2d+1}{2(d+1)}, 1), \\ (d+1)Z^i, & \text{if } i \ge 1, \Lambda = 1, s \in [\frac{2d-2i+1}{2(d+1)}, \frac{2d-2i+3}{2(d+1)}), \\ (d+1)Z^i, & \text{if } i \ge 1, \Lambda = -1, s \in [\frac{2i-1}{2(d+1)}, \frac{2i+1}{2(d+1)}), \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow \omega$: degree 5 formula. ($\omega^0(s) \neq s$.)

Theorem

Let $f \in C^6_{pol}(\mathbf{R}^N)$, and ω is defined by the above. Then

$$|P_T f(x) - Q^n_{T/n} f(x)| \le \frac{\text{const.}}{n^2}.$$

Remark : Cubature formula for Gaussian measure

- Cubature for Gaussian measure with degree *m*:
  A discrete-valued random variable Z such that

$$E[Z^\alpha] = \int_{\mathbf{R}^d} x^\alpha \frac{1}{(\sqrt{2\pi})^d} e^{-|x|^2/2} dx$$

  for $|\alpha| \leq m$.

- N-V + Gaussian cubature with degree 5

  $\Downarrow$

  Cubature on Wiener space with degree 5,
  and with finite number of paths (but $\omega^0(s) \neq s$).

  The number of paths is
  "2$\times$ (the number of points of Gaussian cubature)".

Approximations of SDEs with jumps I (finite jump intesity)

Setting: jump-type SDE

- Let $J(t)$ be a compound Poisson process.
- Consider the following SDE with jumps

$$X_t^x = x + \sum_{i=0}^{d} \int_0^t V_i(X_s^x) \circ dW^i(s) + \int_0^t h(X_{s-}^x) dJ(s)$$

- Question: How can we construct a higher order scheme ?
  - Point 1 : W and J are independent.
  - Point 2 : Number of jumps $=$ finite.

Approach (OS): Operator Splitting

1. Consider the equation

$$dX^x_{d+1,t} = x + \int_0^t h(X^x_{d+1,s-})dJ(s)$$

$\Rightarrow$ easy to simulate (if we can easily simulate jump size).

2. Generator:

$$L_{d+1}f(x) := \int_{\mathbf{R}^d} (f(x + h(x)y) - f(x))\nu(dy)$$

where ν is a finite Lévy measure.

Approach (OS): Operator Splitting (continued)

3. As in the continuous diffusion case, we can consider approximations of
$$e^{t\sum_{i=0}^{d+1} L_i}$$
by
$$\{e^{tL_0}, \ldots, e^{tL_d}, e^{tL_{d+1}}\}.$$

4. We can construct a second order scheme via
$$\frac{1}{2}e^{tL_0}\cdots e^{tL_{d+1}} + \frac{1}{2}e^{tL_{d+1}}\cdots e^{tL_0} = e^{t\sum_{i=0}^{d+1} L_i} + O(t^3)$$
(This is first considered by Fujiwara (2006, Master thesis))

Approach (JSAS): Jump (Size) Adapted Simulation

Ref. : Mordecki & Szepessy & Tempone & Zouraris (2008)

1. First, simulate $J(t)$:
   - Jump time: $0 < \tau_1 < \cdots < \tau_k < T$
2. Since $\tau_{j+1} - \tau_j$ may be large, we use another partition $\{\tilde{t}_j\}$ so that $\{t_i\} := \{\tilde{t}_j\} \cup \{\tau_j\}$ satisfies $t_{j+1} - t_j \leq^{\exists} \Delta$
3. (Jump adapted simulation)
   - Continuous term approximation:

$$\overline{X}_{t_j-} = \text{Euler or cubature scheme in time interval } [t_{j-1}, t_j).$$

     for continuous part $X_{t_j-}^{t_{j-1},x} = x + \sum_{i=0}^{d} \int_{t_{j-1}}^{t_j-} V_i(X_s^{t_{j-1},x}) \circ dW^i(s)$.
   - Jump simulation:

$$\overline{X}_{t_j} = \begin{cases} \overline{X}_{t_j-} + h(\overline{X}_{t_j-}) \times (\text{jump size}), & \text{if } t_j: \text{ jump time}, \\ \overline{X}_{t_j-}, & \textit{otherwise}. \end{cases}$$

Practical Problems

- Which is better, (OS) or (JSAS)?
- The best (adapted) choice of partition $t_1 < \cdots < t_n$
- How to simulate by QMC. (We need a restriction of the number of jumps.)

Approximations of SDEs with jumps II (general Lévy process)

Setting : Infinite Activity Lévy-driven SDE

- Genaral Lévy-driven SDE:

$$X_t^x = x + \sum_{i=0}^{d} \int_0^t V_i(X_s^x) \circ dW^i(s) + \int_0^t h(X_{s-}^x) dZ(s) \tag{3}$$

  where $Z(t)$ is a Lévy process without Brownian term, and $Z_t \in \cap_{p>1} L^p$.

- Generator:

$$\begin{aligned} L &= L_0 + \cdots + L_d + L_{d+1}, \\ L_{d+1} f(x) &:= \nabla f(x) h(x) b \\ &\quad + \int_{\mathbf{R}^d} (f(x + h(x)y) - f(x) - \nabla f(x) h(x) 1_{|y| \leq 1}) \nu(dy), \end{aligned}$$

  where $b \in \mathbf{R}^d$, ν is a Lévy measure. (Note: $\int_{|y|\leq 1} |y|^2 \nu(dy) < \infty$.)

Basic Strategy

- Extend the schemes (OS) & (JSAS) to the general case (3).
  1. (OS): Tanaka & Kohatsu(2009) $\rightarrow$ Kohatsu & Ngo(2011, submitted)
  2. (JSAS): Kohatsu & Tankov(2009) without Brownian term i.e. $V_i \equiv 0, 1 \leq i \leq d$.
- Approach (OS): Decompose $L_{d+1} = L_{d+1}^{(1)} + L_{d+1}^{(2)} + L_{d+1}^{(3)}$ where

$$
\begin{aligned}
L_{d+1}^{(1)} f(x) &= b_\epsilon (hf)(x), \\
L_{d+1}^{(2)} f(x) &= \int_{|y|\leq\epsilon} (f(x+h(x)y) - f(x) - \nabla f(x) h(x) y) \nu(dy), \\
L_{d+1}^{(3)} f(x) &= \int_{|y|>\epsilon} (f(x+h(x)y) - f(x)) \nu(dy),
\end{aligned}
$$

where $\epsilon < 1$, $b_\epsilon := b - \int_{\epsilon<|y|\leq 1} y\nu(dy)$.

Asmussen-Rosinski approximation for $L^{(2)}_{d+1}$

- $L^{(1)}_{d+1}$ and $L^{(3)}_{d+1}$ correspond to "drift" & "compound Poisson" term respectively.
- What is $L^{(2)}_{d+1}$? : Define $\Sigma_\epsilon := \left(\int_{|y|\le\epsilon} y_i y_j \nu(dy) \right)_{1\le i,j\le d}$

  $$L^{(2)}_{d+1} = h(x)\Sigma_\epsilon h^*(x) D^2 f(x) + O\Big(\int_{|y|\le\epsilon} |y|^3 \nu(dy) \Big).$$

  So we replace $L^{(2)}_{d+1}$ by a new "small diffusion"(Itô form) term

  $$\tilde{L}^{(2)}_{d+1} := h(x)\Sigma_\epsilon h^*(x) D^2 f(x)$$

  as an approximation. If necessary for simulation, we can modify $(L^{(1)}_{d+1}, \tilde{L}^{(2)}_{d+1})$ so that $\tilde{L}^{(2)}_{d+1}$ becomes the generator of the SDE of Stratonovich form.
- Control (ϵ, n) as $\int_{|y|\le\epsilon} |y|^3 \nu(dy) \approx O(n^{-k})$.

The case $\int_{|y|\le 1} |y|\nu(dy) < +\infty$

1. The number of jumps w.r.t. $L_{d+1}^{(3)}$ goes to $+\infty$, as $\epsilon \downarrow 0$. To avoid this, we need some restriction for jumps.
2. If we assume that $\int_{|y|\le 1} |y|\nu(dy) < +\infty$, then
   - Note: $|b_\epsilon| < +\infty$.
   - So we can use (OS) approximations for

$$\begin{aligned} (L_0 + L_{d+1}^{(1)}) &\quad : \quad \text{(drift)} \\ L_1, \dots, L_d, \tilde{L}_{d+1}^{(2)} &\quad : \quad \text{(2}d\text{-diffusion)} \\ L_{d+1}^{(3)} &\quad : \quad \text{(Jumps)} \end{aligned}$$

3. For $L_{d+1}^{(3)}$, we can construct an approximation process which has single or double jumps in (fixed) small time interval. See Tanaka & Kohatsu, or Kohatsu & Ngo.

The case $\int_{|y|\leq 1} |y|\nu(dy) = +\infty$

1. Kohatsu & Ngo discuss a case study where Z is a subordinated Brownian motion. They construct an algorithm under the case
   - $\int_{|y|\leq 1} |y|\nu(dy) = +\infty$, but $|\int_{|y|\leq 1} y\nu(dy)| < +\infty$
2. If we assume that $\int_{|y|\leq 1} |y|\nu(dy) = +\infty$, then in general,
   - Note: $|b_\epsilon| = +\infty$.
   - We should not decompose $(L_{d+1}^{(1)} + L_{d+1}^{(3)})$, since $L_{d+1}^{(1)}$ includes the truncation function for $L_{d+1}^{(3)}$.
   - Consider (OS) approximations for

$$\begin{aligned} L_1, \ldots, L_d, \tilde{L}_{d+1}^{(2)} &\quad : \quad (2d\text{-diffusion}) \\ L_0 + (L_{d+1}^{(1)} + L_{d+1}^{(3)}) &\quad : \quad (\text{ODE with Jumps}) \end{aligned}$$

What can we do for $(L_{d+1}^{(1)} + L_{d+1}^{(3)})$?

Further Research

Problems in time discretization

1. Find higher-order methods with degree more than 7 ($d \geq 2$). (We have known only "existence"!)

2. High order approximation of Lévy driven SDEs
   - How to deal with small jumps (which occurs infinitely many times).
   - How to simulate Lévy processes whose Lévy measure has high singularity ($\int_{|y|\leq 1} |y|\nu(dy) = +\infty$).

3. Problems in computing conditional expectations $E[\cdot|\mathcal{F}_t]$:
   - Pricing American/Bermudan options
   - Simulating forward-backward SDEs (FBSDEs)

   $\Rightarrow$ Recombination techniques: e.g. Chevance(1997), Lyons-Litterer(forthcoming, AAP), Tanaka(2011, submitted)

Thank you for your attention.