Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths

Hiroshi KAWABI

Department of Mathematics, Faculty of Science, Okayama University
3-1-1, Tsushima-Naka, Okayama 700-8530, Japan
kawabi@math.okayama-u.ac.jp

Summary. In this talk, we discuss asymptotic expansions for the Laplace approximations for the laws of solutions of formal Stratonovich type stochastic differential equations in Banach spaces. The rigorous meaning of the solutions is given by the rough path theory of T. Lyons. In our proof, the (stochastic) Taylor expansion in terms of the rough path theory plays a key role. The main example we have in mind is the loop group valued Brownian motion introduced by Malliavin.

This talk is based on recent jointworks [4], [5], [6] with Yuzuru Inahama (Tokyo Institute of Technology). Let (X, H, μ) be an abstract Wiener space, i.e., X is a real separable Banach space, H is the Cameron-Martin space and μ is the Wiener measure on X. Let Y be another real separable Banach space and $w = (w_t)_{0 \le t \le 1}$ be the X-valued Brownian motion. We denote by $L(X, Y)$ the space of bounded linear operators from X to Y. Let consider a class of Y-valued Wiener functionals $X^\varepsilon := (X^\varepsilon_t)_{0\le t\le 1}$ $(\varepsilon > 0)$ defined through the following <u>formal</u> Stratonovich type stochastic differential equation (SDE) on Y:

$$dX^\varepsilon_t = \sigma(X^\varepsilon_t) \circ \varepsilon dw_t + \sum_{i=1}^{N} a_i(\varepsilon) b_i(X^\varepsilon_t) dt \quad \text{ with } X^\varepsilon_0 = 0. \tag{1}$$

Here $\sigma \in C_b^\infty(Y, L(X, Y)), b_i \in C_b^\infty(Y, Y), i = 1, \ldots, N$, and $a = (a_1, \ldots, a_N) : [0, 1] \to \mathbb{R}^N$ is a smooth curve. In this talk, we discuss the Laplace type asymptotic expansion of the functional integral of the form $\mathbb{E}\big[G(X^\varepsilon) \exp(-F(X^\varepsilon/\varepsilon^2))\big]$ as $\varepsilon \searrow 0$. This is an application of the large deviation principle established in [4]. The Laplace method for the leading term $(= \alpha_0)$ was given in [5].

In order to give a precise meaning for our Wiener functional X^ε, we need to introduce some notations. For a real separable Banach space B, we set $P(B) := \{x \in C([0, 1], B) \mid x_0 = 0\}$, $BV(B) := \{\gamma \in P(B) \mid \|\gamma\|_1 < \infty\}$ and set by $G\Omega_p(B)$ $(2 < p < 3)$ the space of geometric rough paths over B. The law of εw on $P(X)$ is denoted by $\mathbb{P}'_\varepsilon$. $\mathcal{H}$ is the Cameron-Martin space for $(P(X), \mathcal{H}, \mathbb{P}'_1)$. In this talk, we assume that $(|\cdot|_{X\otimes X}, \mu)$ satisfies the <u>exactness condition</u> **(EX)** (cf. Ledoux-Lyons-Qian [7]), which implies the existence of Brownian rough path $\overline{w} = (1, \overline{w}_1, \overline{w}_2) \in G\Omega_p(X)$. The law of scaled Brownian rough path $\overline{\varepsilon w}$ on $G\Omega_p(X)$ is denoted by $\mathbb{P}_\varepsilon$. We set $\tilde{X} := X \oplus \mathbb{R}^N$ and define $\tilde{\sigma} \in C_b^\infty(Y, L(\tilde{X}, Y))$ by

$$\tilde{\sigma}(y)\big[(x, u)\big]_{\tilde{X}} := \sigma(y)x + \sum_{i=1}^{N} b_i(y)u_i, \quad y \in Y, x \in X, u = (u_1, \ldots, u_N) \in \mathbb{R}^N.$$

Now we consider the following ODE in the rough path sense:

$$dy_t = \tilde{\sigma}(y_t) d\tilde{x}_t \quad \text{ with } y_0 = 0. \tag{2}$$

For an input $\overline{\tilde{x}} \in G\Omega_p(\tilde{X})$, there is a unique solution $\overline{z} = (\overline{\tilde{x}}, \overline{y}) \in G\Omega_p(\tilde{X} \oplus Y)$ in the rough path sense. We denote it by $\overline{y} = \Phi(\overline{\tilde{x}})$ and also call it a solution of ODE (2). The Itô map $\Phi : G\Omega_p(\tilde{X}) \to G\Omega_p(Y)$ is locally Lipschitz continuous. For $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \mathrm{BV}(\mathbb{R}^N)$ and

$\overline{x} \in G\Omega_p(X)$, we define $\iota(\overline{x}, \lambda) \in G\Omega_p(\tilde{X})$ by $\iota(\overline{x}, \lambda)_1(s,t) = (\overline{x}_1(s,t), \lambda_t - \lambda_s)$,

$$\iota(\overline{x}, \lambda)_2(s,t) = \Big(\overline{x}_2(s,t), \int_s^t \overline{x}_1(s,u) \otimes d\lambda_u, \int_s^t (\lambda_u - \lambda_s) \otimes \overline{x}_1(s, du), \int_s^t (\lambda_u - \lambda_s) \otimes d\lambda_u\Big).$$

Note that the map $\iota : G\Omega_p(X) \times \mathrm{BV}(\mathbb{R}^N) \to G\Omega_p(\tilde{X})$ is also locally Lipschitz continuous.

For $\varepsilon \geq 0$, we set $\lambda^\varepsilon(t) := (a_1(\varepsilon)t, \dots, a_N(\varepsilon)t)$ and set $\Psi_\varepsilon : \mathrm{BV}(X) \to \mathrm{BV}(Y)$ by

$$\Psi_\varepsilon(h)_t := \Phi\big(\iota(\overline{h}, \lambda^\varepsilon)\big)_1(0,t), \qquad 0 \leq t \leq 1.$$

Then $y := \Psi_\varepsilon(h)$ is the unique solution of the following Y-valued usual ODE:

$$dy_t = \sigma(y_t)dh_t + \sum_{i=1}^N a_i(\varepsilon) b_i(X_t^\varepsilon)dt \quad \text{ with } y_0 = 0$$

With these observations in mind, we define our Wiener functional X^ε by

$$X_t^\varepsilon := \Phi\big(\iota(\overline{\varepsilon w}, \lambda^\varepsilon)\big)_1(0,t), \qquad 0 \leq t \leq 1.$$

By Wong-Zakai's approximation theorem, examples of X^ε include solutions of usual SDEs (1) on finite dimensional spaces and heat processes on loop spaces. (Heat processes are defined by a collection of finite dimensional SDEs. A process of this kind was first introduced by Malliavin [8] in the case of loop groups and then was generalized by many authors.)

In this paper, we impose the following conditions on the functions F and G. Below, we denote by D the Fréchet derivatives on $\mathrm{BV}(X), P(Y)$ and denote by ∇ that on Y.

(H1): F and G are real-valued bounded continuous functions defined on $P(Y)$.

(H2): The function $F_\Lambda := F \circ \Psi_0 + \| \cdot \|_{\mathcal{H}}^2/2$ defined on $\mathcal{H}$ attains its minimum at a unique point $\gamma \in \mathcal{H}$. For this γ, we write $\phi := \Psi_0(\gamma)$.

(H3): The functions F and G are $n+3$ and $n+1$ times Fréchet differentiable on a neighborhood $B(\phi)$ of $\phi \in P(Y)$, respectively. Moreover there exist positive constants $M_1, \dots, M_{n+3}$ such that

$$\begin{aligned} \big|D^k F(\eta)\big[y, \dots, y\big]\big| &\leq M_k \|y\|_{P(Y)}^k, \quad k = 1, \dots, n+3, \\ \big|D^k G(\eta)\big[y, \dots, y\big]\big| &\leq M_k \|y\|_{P(Y)}^k, \quad k = 1, \dots, n+1, \end{aligned}$$

hold for any $\eta \in B(\phi)$ and $y \in P(Y)$.

(H4): At the point $\gamma \in \mathcal{H}$, we consider the Hessian $A := D^2(F \circ \Psi_0)(\gamma)|_{\mathcal{H}\times\mathcal{H}}$. As a bounded self-adjoint operator on $\mathcal{H}$, the operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}}$ in the form sense. (By the min-max principle, it is equivalent to assume that all eigenvalues of A are strictly larger than -1.)

Theorem 1 *Under the conditions* **(EX)**, **(H1)**, **(H2)**, **(H3)** *and* **(H4)**, *we have the following asymptotic expansion:*

$$\begin{aligned} &\mathbb{E}\Big[G(X^\varepsilon)\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\Big] \\ &\quad= \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big)\exp\big(-c(\gamma)/\varepsilon\big)\cdot\big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_n\varepsilon^n + O(\varepsilon^{n+1})\big), \end{aligned} \tag{3}$$

where the constant $c(\gamma)$ in (3) *is given by $c(\gamma) := DF(\phi)[\Xi_1(\gamma)]$. Here $\Xi_j(\gamma) \in P(Y), j \in \mathbb{N}$, is the unique solution of the differential equation*

$$d\,\Xi_t - \nabla\sigma(\phi_t)[\Xi_t, d\gamma_t] - \sum_{i=1}^N a_i(0)\nabla b_i(\phi_t)[\Xi_t]dt = \sum_{i=1}^N a_i^{(j)}(0) b_i(\phi_t)dt \quad \textit{with} \quad \Xi_t = 0.$$

The coefficients $\{\alpha_m\}_{m=0}^n$ *are given by*

$$\begin{aligned}\alpha_0 &= G(\phi)\int_{G\Omega_p(X)} \exp\big(-J_F^{(2)}(\phi)(\overline{w})\big)\mathbb{P}_1(d\overline{w}),\\ \alpha_m &= \int_{G\Omega_p(X)} \Big\{J_G^{(m)}(\phi)(\overline{w}) + \sum_{j=0}^{m-1}\sum_{k=1}^{m}\frac{(-1)^k}{k!}J_G^{(j)}(\phi)(\overline{w})\\ &\qquad\times\sum_{\pi\in\mathcal{G}(j,k,m)}\prod_{i=1}^{k}J_F^{(\pi(i))}(\phi)(\overline{w})\Big\}\cdot\exp\big(-J_F^{(2)}(\phi)(\overline{w})\big)\mathbb{P}_1(d\overline{w}),\quad m=1,\ldots,n,\end{aligned}$$

where

$$J_F^n(\phi)(\overline{w}) := \sum_{k=1}^{n}\frac{1}{k!}\Big(\sum_{(i_1,\cdots,i_k)\in S_k^n} D^kF(\phi)\big[\phi^{i_1}(\overline{w})_1,\cdots,\phi^{i_k}(\overline{w})_1\big]\Big),$$

and

$$\mathcal{G}(j,k,m) := \Big\{\pi:\{1,\ldots,k\}\to\{3,\ldots,m+2\}\ \Big|\ j+\sum_{i=1}^{k}(\pi(i)-2)=m\Big\}.$$

Remark 2 *In the case* $a'(0)=0$*, we have* $c(\gamma)=0$ *and the explicit value of the coefficient* α_0 *can be written in terms of* $\det_2(\mathrm{Id}_{\mathcal{H}}+A)$*. See Theorem* 6.5 *in* [6] *for the detail.*

The key tool in our proof is the (stochastic) Taylor expansion with respect to the topology of $G\Omega_p(X)$ given as

$$\Phi(\iota(\overline{\gamma+\varepsilon w},\lambda^\varepsilon))_1 = \phi + \varepsilon\phi^1(\overline{w})_1 + \varepsilon^2\phi^2(\overline{w})_1 + \cdots + \varepsilon^n\phi^n(\overline{w})_1 + O(\varepsilon^{n+1}). \tag{4}$$

This expansion (4) is deterministic and it is natural to guess that the same method applies to asymptotic problems of other probability measures on $G\Omega_p(X)$. By combining (4) with the rough path version of the Cameron-Martin theorem and the Fernique theorem, we can proceed in the same way as in the proof of (scaled) Gaussian measures on Banach spaces.

We note that our method of the (stochastic) Taylor expansion is slightly different from Aida's method in [1]. He uses the derivative equation, whose coefficient is of course of linear growth. Since it is not known whether Lyons' continuity theorem holds or not for unbounded coefficients, he extends the continuity theorem for the case of the derivative equation. On the other hand, we employs the method in Azencott [2] and Ben Arous [3], and we only need the continuity theorem for the given equation (2) whose coefficient is bounded. The price we have to pay is that notations and proofs may seem slightly long. However, the strategy of this method is quite simple and straightforward.

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