# Large deviation principle of Freidlin-Wentzell type for pinned diffusion processes

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# 1 \Outline of the talk

- We will prove LDP of FW type for pinned diffusion measures.
- Main tools of our analysis are;
  - (1) Rough Path Theory
  - (2) Quasi sure analysis $\subset$ Malliavin calc.
- Probably new (?) But the case of pinned BM on complete Riem. mfd. was shown by E. P. Hsu (PTRF '90). Shown via estimates of heat kernel. No SDE in this paper. Not so "of FW type."

$w = (w_t)_{0 \le t \le 1}$: $d$-dim BM

$V_1, \ldots, V_d$: vector fields on $\mathbb{R}^n$

Consider the following (Stratonovich type) SDE;

$$dy_t = \sum_{i=1}^{d} V_i(y_t) \circ dw_t^i \qquad \text{with} \quad y_0 = a \in \mathbb{R}^n.$$

The correspondence $w \mapsto y = (y_t)_{0 \le t \le 1}$ is called Itô map. NOT continuous. Write $y = \Phi(w)$.

- $y$ is a diffusion proc. corresponding to

$$\mathcal{L} = (1/2) \sum_{i=1}^{d} V_i^2$$

Let $\varepsilon > 0$ be a small parameter. Consider

$$dy_t^\varepsilon = \sum_{i=1}^d V_i(y_t^\varepsilon) \circ \varepsilon dw_t^i \quad \text{with} \quad y_0^\varepsilon = a \in \mathbb{R}^n.$$

- $y^\varepsilon$ is a diffusion proc. corresponding to

$$\mathcal{L}^\varepsilon = (\varepsilon^2/2) \sum_{i=1}^d V_i^2$$

- Formally, $y^\varepsilon = \Psi(\varepsilon w)$

## [Fact: FW's LDP]

**The law of $y^{\varepsilon}$ satisfies LDP as $\varepsilon \searrow 0$.** $\blacksquare$

**- The law of $(\varepsilon w_t)$ (=scaled Wiener measure) satisfies LDP of Schilder type as $\varepsilon \searrow 0$.**

**- From this and contraction principle for LDP, FW's LDP is immediate if $\Phi$ were continuous.**

**- In the usual stochastic analysis, the proof is done by using "exponentially good approximation."**

- In Rough Path Theory, we consider not only $w$ itself, but its iterated path integrals:

$$\boldsymbol{W}^{1}_{s,t} = \boldsymbol{w}_t - \boldsymbol{w}_s, \quad \boldsymbol{W}^{2}_{s,t} = \int_s^t (\boldsymbol{w}_u - \boldsymbol{w}_s) \otimes \circ d\boldsymbol{w}_u$$

for $0 \leq s \leq t \leq 1$.

- We call $\boldsymbol{W} = (\boldsymbol{W}^1, \boldsymbol{W}^2)$ Brownian RP.
- T. Lyons developed a thoery of line integrals and ODE for RPs, in which Lyons-Itô map $\boldsymbol{\Phi}$ is continous w.r.t. RP topology. (Note: those are deterministic)

**Roughly speaking, $\Phi(\varepsilon W) = y^{\varepsilon}$. (i.e., sol. of Stratonovich SDE is obtained via RP theory)**

**A new proof by Ledoux-Qian-Zhang ('02)**
**The law of $\varepsilon W = (\varepsilon W^1, \varepsilon^2 W^2)$ satisfies LDP of Schilder type on RP space. From this, FW's LDP is immediate** $\blacksquare$

**A nice idea! Followed by many results on LDP for various Gaussian RPs w.r.t. various Banach norms.**

## 2 \A natural question arises:

Can one prove a similar LDP for pinned diffusion measure $\mathbb{Q}^{\varepsilon}_{a,a'}$ ??

Here, $\mathbb{Q}^{\varepsilon}_{a,a'}$ is the pinned diff. meas. associated with $\mathcal{L}^{\varepsilon}$, which starts/ends at $a/a'$, resp.

Method: quasi-sure analysis for BRP. Recall that original motivation for QSA was to analyze (the pullback of) a pinned diffusion measure.

# 3 \Setting and Assumptions

$V_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$: vector field, $C_b^\infty$, $1 \le i \le d$
$V_0 : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$: $\varepsilon$-dep. vector field, $C_b^\infty$

$$dy_t^\varepsilon = \sum_{i=1}^d V_i(y_t^\varepsilon) \circ \varepsilon dw_t^i + V_0(\varepsilon, y_t^\varepsilon)dt, \quad y_0^\varepsilon = a.$$

Generator; $\mathcal{L}^\varepsilon = (\varepsilon^2/2)\sum_{i=1}^d V_i^2 + V_0(\varepsilon, \cdot\,)$

Examples; $V_0(\varepsilon, \cdot\,) = \hat{V}(\,\cdot\,)$ or $\varepsilon^2\hat{V}(\,\cdot\,)$

We assume "everywhere ellipticity"

**(H1)** For all $a \in \mathbb{R}^n$, the set of vectors $\{V_1(a), \ldots, V_d(a)\}$ linearly spans $\mathbb{R}^n$.

Pinned diff. meas. $\mathbb{Q}^{\varepsilon}_{a,a'}$ is well-defined and sits on

$$C^{\alpha-H}_{a,a'}([0,1], \mathbf{R}^n)$$
$$= \{x \in C_{a,a'}([0,1], \mathbf{R}^n) \mid \alpha\text{-Hölder conti } \}$$

for any $\alpha \in (1/3, 1/2)$.

Heuristically, $\mathbb{Q}^{\varepsilon}_{a,a'}$ is the law of map $w \mapsto y^{\varepsilon}$ under conditional measure $\mathbb{P}(\,\cdot\,|y^{\varepsilon}_1 = a')$.

$\mathcal{H}$: Cameron-Martin space of BM

For $h \in \mathcal{H}$, $\phi^0 = \phi^0(h)$ is a unique solution of;

$$d\phi_t^0 = \sum_{i=1}^{d} V_i(\phi_t^0)dh_t^i + V_0(0, \phi_t^0)dt, \qquad \phi_0^0 = a.$$

Set $\mathcal{K}^{a,a'} := \{h \in \mathcal{H} \mid \phi^0(h)_1 = a'\} \neq \emptyset$.

Define a good rate function $J : C^{\alpha-H}_{a,a'}([0,1],\mathbb{R}^n) \longrightarrow [0,\infty]$ by

$$J(y) = \inf\{\frac{\|h\|^2_{\mathcal{H}}}{2} \mid h \in \mathcal{K}^{a,a'} \text{ with } y = \phi^0(h)\} - \min\{\frac{\|h\|^2_{\mathcal{H}}}{2} \mid h \in \mathcal{K}^{a,a'}\}$$

if $y = \phi^0(h)$ for some $h \in \mathcal{K}^{a,a'}$ and $J(y) = \infty$ if no such $h \in \mathcal{K}^{a,a'}$ exists.

# 4 \Main result (LDP of FW-type)

**Theorem:** Let $1/3 < \alpha < 1/2$ and assume **(H1)**. Then, $\{\mathbb{Q}^{\varepsilon}_{a,a'}\}_{\varepsilon>0}$ satisfies an LDP on $C^{\alpha-H}_{a,a'}([0,1],\mathbb{R}^d)$ as $\varepsilon \searrow 0$ with a good rate function $J$. $\blacksquare$

# 5 \RP space with Besov norm

$\triangle := \{(s,t) \mid 0 \le s \le t \le 1\}$. For $Y \in C(\triangle, \mathbb{R}^d)$,

$$\|Y\|^m_{\alpha,m-B} := \iint_\triangle \frac{|Y_{s,t}|^m}{|t-s|^{1+m\alpha}} dsdt$$

(Besov norm, $\quad m \ge 1,\ 0 < \alpha \le 1$)

$T^2(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2}$: truncated $\otimes$-alg.

$X = (1, X^1, X^2) \in C(\triangle \to T^2(\mathbb{R}^d))$ is called multiplicative if $\quad X^1_{s,t} = X^1_{s,u} + X^1_{u,t}$,
$X^2_{s,t} = X^2_{s,u} + X^2_{u,t} + X^1_{s,u} \otimes X^1_{u,t}$ (Chen's id.)

♠ $m \geq 2$, $1/3 < \alpha < 1/2$, $\alpha - m^{-1} > 1/3$.

$X = (1, X^1, X^2)$ is said to be $(\alpha, m)$-Besov RP if it is multiplicative and
$\|X^1\|_{\alpha,m-B} < \infty$, $\|X^2\|_{2\alpha,m/2-B} < \infty$.
(We will write $(X^1, X^2)$ by omitting "1")

**Example:** $x \in C_0([0,1], \mathbb{R}^d)$, Lipschitz conti.,

$$X^1_{s,t} = x_t - x_s, \quad X^2_{s,t} = \int_s^t (x_u - x_s) \otimes dx_u$$

(the smooth RP above $x$, or the lift of $x$)

**A geometric RP is a RP obtained as a limit of a sequence of smooth RPs w.r.t. $(\alpha, m)$-besov top.**

**$G\Omega^B_{\alpha,m}(\mathbb{R}^d)$: totality of $(\alpha, m)$-geometric RPs**

**♡ Relation to Hölder RP spaces.**

$$G\Omega^H_{\alpha'}(\mathbb{R}^d) \hookrightarrow G\Omega^B_{\alpha,m}(\mathbb{R}^d) \hookrightarrow G\Omega^H_{\alpha-m^{-1}}(\mathbb{R}^d)$$

**for $m \geq 2$, $1/3 < \alpha < \alpha' < 1/2$, $\alpha - m^{-1} > 1/3$.**

**♠ Interpretation of Besov indices;**

**$\alpha \approx$ Hölder index**

**m: supplementary index. (basically, very large)**

♠ **Remark:** (i) Large deviation estimate for the law of Brownian RP. (weight of [large ball]$^c$)
(ii) LDP of Schilder type for the laws of scaled Brownian RP.

These are known on for the Hölder case.
Because of the continuous embedding, they hold true on $G\Omega^B_{\alpha,m}(\mathbb{R}^d)$, too.

# 6 \Quasi sure existence of Brownian RP

Roughly, BRP $W := \lim_{n \to \infty} W(n)$. Here $W(n)$ is the lift of $w(n)$, polygonal approx associated with $\{k2^{-n} | 0 \le k \le 2^n\}$. (a.s. convergence)

Actually, $W(n)$ converges quasi surely.

- Higuchi (Master thesis, '06) / Aida (JFA '11)
- I. (IDAQP '06)
- Watanabe (Proc. Abel Symp. '07)

Higuchi/Aida seems best. We will follow them.

$\mathcal{W} = C_0([0,1], \mathbb{R}^d)$: Wiener sp. with sup-norm

$\mathcal{Z}_{\alpha,4m} := \{w \in \mathcal{W} | \ \{W(k)\}_{k=1}^{\infty}$ Cauchy in $G\Omega^B_{\alpha,4m}\}$

- We assume (H2) $\quad m \in \mathbb{N}, \quad 1/3 < \alpha < 1/2,$
$\alpha - (4m)^{-1} > 1/3, \quad 4m(1/2 - \alpha) > 1.$

- They proved that under (H2), $\mathcal{Z}_{\alpha,4m}$ is slim, i.e., $\mathrm{cap}_{q,r}(\mathcal{Z}_{\alpha,4m}) = 0$ for any $1 < q < \infty, r \geq 0$.

- We will fix this version of BRP $W$. Then, $w \mapsto W$ is $\infty$-quasi conti. (Aida '11)

# 7 2 quick remarks

Two nice theorems in QSA became obvious.

(J. Ren, Bull Sci Math '90) Quasi sure refinement of Wong-Zakai approx. thm.

Note that we only need $C_b^3$-condition for coefficients.

(Malliavin-Nualart, JFA '94) The solution of an SDE, as a path space-valued functional, admits an $\infty$-quasi conti. mdf.

-Note that we only need $C_b^3$-condition for coefficients.

- We don't know this results via RP theory is better or not, (since we don't know the UMD Banach norm in MN is weaker than Hölder-type norms.)

- But, al least in spirit, this proof seems better than the one in MN.

# 8 \Pinned process via Lyons-Itô map

From now on, we assume drift term in SDE $\equiv 0$.

$\Phi$: Lyons–Itô map associated with $V_1, \ldots, V_d$.

Then, $y^\varepsilon = \Phi(\varepsilon W)^1$ q.e. $w \in \mathcal{W}$.

$y_1^\varepsilon$: non-deg. in Malliavin's sense.

$\Longrightarrow$ $\delta_{a'}(y_1^\varepsilon)$: (positive) Watanabe dist.

$\Longrightarrow$ $p_1^\varepsilon(a, a') = \mathbb{E}[\delta_{a'}(y_1^\varepsilon)]$

By Sugita's theorem, $\delta_{a'}(y_1^\varepsilon)$ is actually a finite Borel measure on $\mathcal{W}$ ($=: \theta_{a,a'}^\varepsilon(dw)$)

**Its lift is well-defined. We set**

$$\mu^{\varepsilon}_{a,a'} = [\varepsilon \cdot \mathrm{Lift}]_*(\theta^{\varepsilon}_{a,a'}) = [w \mapsto \varepsilon W]_*(\theta^{\varepsilon}_{a,a'})$$

**a measure on geom. RP sp.**

$\hat{\mu}^{\varepsilon}_{a,a'}, \hat{\theta}^{\varepsilon}_{a,a'}$: **normalized ones (probabilities).**

♡ **Fact** $\mathbb{Q}^{\varepsilon}_{a,a'} = \Phi_*(\hat{\mu}^{\varepsilon}_{a,a'})$

**- First pointed out in I. ('06)**

**- But, almost immediate from the quasi-conti of** $w \mapsto W$**.**

We have obtained the pinned diffusion as an image of continuous Lyons-Itô map

Thus, our main results is immediate from
LDP for $\mu^{\varepsilon}_{a,a'}$ on $G\Omega^{B}_{\alpha,4m}(\mathbb{R}^d)$ as $\varepsilon \searrow 0$
(Note that the whole set is both open and closed)

A sketch of proof of this LDP will be given.

# 9 \Key points of proof

## (I) large deviation estimate for capacities

$1 < \forall q < \infty$, $\forall r \geq 0$, $\exists C = C(q, r, \alpha, m)$ s.t,

$$\mathbf{cap}_{q,r}(\{w \mid \|W^i\|^{1/i}_{i\alpha, 4m/i-B} \geq R\}) \leq e^{-CR^2}$$

as $R \nearrow \infty$. $(i = 1, 2)$

♠ This kind of estimate is well-known for the law of Gaussian RP. (e.g., Friz-Oberhauser '10). But, we are dealing with not measures, but capacities.

## (II) Integration by parts for Watanabe distributions

$F = (F^1, \ldots, F^n) \in \mathbb{D}_\infty(\mathbb{R}^n)$: non-deg

$G \in \mathbb{D}_\infty$ and $T \in \mathcal{S}'(\mathbb{R}^n, \mathbb{R})$.

$\Longrightarrow \mathbb{E}\big[(\partial_i T \circ F) \cdot G\big] = \mathbb{E}\big[(T \circ F) \cdot \Psi_i(\,\cdot\,; G)\big]$

Here $\Psi_i(\,\cdot\,; G)$ is given by

$$- \sum_{j=1}^{d} \Big\{ - \sum_{k,l=1}^{d} G \cdot \gamma^{ik} \cdot \gamma^{jl} \langle D\tau^{kl}, DF^j \rangle_{\mathcal{H}} + \gamma^{ij} \langle DG, DF^j \rangle_{\mathcal{H}} + \gamma^{ij} \cdot G \cdot LF^j \Big\}.$$

where $\tau$ is Malliavin cov. matrix and $\gamma = \tau^{-1}$

**how to use it.** **In upper estimate, we must treat**

$$\mathbb{E}\Big[\{\text{something like indicator of a set}\} \cdot \delta_{a'}(y_1^{\varepsilon})\Big]$$

**But,** $\delta_{a'}(\xi) = (\partial_1^2 \cdots \partial_n^2) A(\xi), \qquad (\xi \in \mathbb{R}^n)$

**with** $A(\xi) := \prod_{j=1}^{n} |\xi_j - a'_j|_+$

**So we can apply IBP and make Schwartz distribution on RHS vanish. After that we may use the usual Scilder-type LDP for BRP**

# (III) Taylor expansion of Itô map around $h \in \mathcal{H}$

Watanabe's asymptotic expansion of

$\tilde{y}_1^\varepsilon = \Phi(\varepsilon W + h)$ in $\mathbb{D}_\infty$-topology (for given $h$)

Here, SDE for $\tilde{y}_t^\varepsilon$ is given by

$$d\tilde{y}_t^\varepsilon = \sum_{i=1}^d V_i(\tilde{y}_t^\varepsilon) \circ [\varepsilon dw_t^i + dh_t^i]$$

Then, as $\varepsilon \searrow 0$, $\tilde{y}_1^\varepsilon \sim \phi^0(h)_1 + \varepsilon \phi^1(h, w)_1 + \cdots$

Notice taht if $h \in \mathcal{K}^{a,a'} \iff \phi^0(h)_1 = a'$.

$\phi^1(h, w)_1$ is a non-deg. $\mathbb{R}^n$-val. Gaussian r.v.

**Watanabe's asymptotic theory further claims;**

$$\delta_0\Big(\frac{\tilde{y}_1^{\varepsilon} - a'}{\varepsilon}\Big) \to \delta_0(\phi^1(h,\,\cdot\,)_1)$$

**in $\mathbb{D}_{q,-r}$-topology ($1 < \exists q < \infty, \exists r \geq 0$).**

**RHS defines a non-trivial meas. by Sugita's thm**

**In lower estimate, we use this. Just a sketch.**

**$B_\rho(X)$: small ball around RP X $\quad (\rho > 0)$**

**Let $h \in \mathcal{K}^{a,a'}$ and let $H$ be its lift.**

**Want to estimate the weight of the ball from below**

Roughly, we calculate as follows;

$$\mathbb{E}[1_{B_\rho(H)}(\varepsilon W)\cdot\delta_{a'}(y_1^\varepsilon)]$$
$$=\mathbb{E}[\exp(-\frac{\|h\|^2}{2\varepsilon^2}-\langle\frac{h}{\varepsilon},w\rangle)\,1_{B_{\rho/\varepsilon}(0)}(W)\,\delta_{a'}(\tilde{y}_1^\varepsilon)]$$
$$=e^{-\frac{\|h\|^2}{2\varepsilon^2}}\mathbb{E}[e^{-\langle\frac{h}{\varepsilon},w\rangle}1_{B_{\rho/\varepsilon}(0)}(W)\frac{1}{\varepsilon^n}\delta_0(\frac{\tilde{y}_1^\varepsilon-a'}{\varepsilon})]$$

If $\langle h,\cdot\rangle\in\mathcal{W}^*$, $\varepsilon^2\log$ of RHS is

$$\geq-\frac{\|h\|^2}{2}-\text{const}\cdot\rho+o(1)$$

Since $\rho>0$ is arbitrary, the lower estimate is done