On a global estimate for Fourier transforms of smooth rough paths

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(based on a joint work with T.Lyons)

What is Smooth Rough Path?

- A "smooth rough path" is actually smooth.
- But it is actually a rough path in the sense of Rough Path Theory.
- It means the smooth path that satisfies the rough path estimates.

The control function $\omega(s, t)$

You know our famous control function $\omega(s, t)$ of rough path theory.
So, we skip the definition!
(But, remember the important property:

$$\omega(s, u) + \omega(u, t) \leq \omega(s, t)$$

for $s \leq u \leq t$.)

Minimal definition of Smooth rough paths

Let $p \geq 1$ be a real number. Let $F : \mathbb{R} \to \mathbb{R}^d$ be an infinitely differentiable function and $F^i_{s,t}$ be the ith iterated integral:

$$F^i_{s,t} = \int \cdots \int_{s<u_1<\cdots<u_i<t} dF_{u_1} \otimes \cdots \otimes dF_{u_i} \quad (i = 1, 2, 3, \dots),$$

and let $\|\cdot\| = \|\cdot\|_i$ be the norm in the ith tensor product space of $\mathbb{R}^d$.

If there exists a control function ω with a constant M such that

$$\|F^i_{s,t}\|^{p/i} \leq \omega(s,t) < M < \infty \quad (i = 1, 2, \dots, \lfloor p \rfloor)$$

for any $-\infty < s < t < \infty$, we call F a (p-)smooth rough path with a global control ω.

(That is all. We need not assume the Chen's identity because it always holds by Chen's theorem for smooth paths.)

What for?

- A toy model of the (serious) rough path theory.
  - Easy to define. (Need not functional analytic setting, etc.)
  - Easy to study. (You can freely use any analytic technique. After that, consider what the essential points are as rough paths.)
- Interesting as itself.
  - Study the global oscillation of a path.
  - Another possibility of the power of 'rough path theory'

A secret motivation (related to Nonlinear Fourier transform)

- It seems that we need the deep estimate of Fourier transforms to study the Nonlinear Fourier transform.
- Especially the estimate of the global oscillation seems important.
- We might be able to define and study the NLFT with rough path theory!
- (But, it must be very hard task to get the deeper estimate for the Fourier transform enough to study NLFT....)

What is Nonlinear Fourier transform?

A rough sketch of NLFT according to T. Tao and C. Thiele.

We like to define the Nonlinear Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ under some reasonable conditions.

Let the matrix-valued Dirac operator $L = L(f)$ be

$$L = \begin{pmatrix} D & -f \\ f & -D \end{pmatrix},$$

where $D = d/dt$ is the differentiation operator. Consider the following eigenfunction problem.

$$L \begin{pmatrix} \phi(k;t) \\ \psi(k;t) \end{pmatrix} = ik \begin{pmatrix} \phi(k;t) \\ \psi(k;t) \end{pmatrix}.$$

(Note that k is a real number because the operator L is anti-selfadjoint.)

We suppose that the eigenfunctions have some regularity and the boundary conditions $\phi(k;t) \sim e^{ikt}, \psi(k;t) \sim 0$ as $t \to +\infty$.
It is natural to set

$$\phi(k;t) = a(k;t)e^{ikt} \quad \text{and} \quad \psi(k;t) = b(k;t)e^{-ikt},$$

considering the free case $f \equiv 0$. Then the equation becomes

$$\begin{aligned} \frac{d}{dt}a(k;t) &= b(k;t)\overline{f(t)}e^{-2ikt}, \\ \frac{d}{dt}b(k;t) &= a(k;t)f(t)e^{2ikt} \end{aligned}$$

with the boundary conditions $a(k;+\infty) = 1$ and $b(k;+\infty) = 0$.
If the solutions exist, we define the nonlinear Fourier transform of f as

$$\overbrace{f}[k] = \begin{pmatrix} a(k;-\infty) \\ b(k;-\infty) \end{pmatrix}.$$

A smooth rough path and NLFT?

Let us rewrite those equations as follows:

$$d\begin{pmatrix} \overline{a(k;t)} \\ b(k;t) \end{pmatrix} = \overline{\begin{pmatrix} \dfrac{b(k;t)}{a(k;t)} \end{pmatrix}} e^{ikt} dF(t), \quad (\, F'(t) = f(t)\,).$$

Or, the equivalent form in the real Euclidean space is

$$\begin{pmatrix} a_1 \\ -a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ -b_2 & b_1 \\ a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} d\begin{pmatrix} \widetilde{F_1}(t) \\ \widetilde{F_2}(t) \end{pmatrix},$$

where

$$d\begin{pmatrix} \widetilde{F_1}(t) \\ \widetilde{F_2}(t) \end{pmatrix} = \begin{pmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{pmatrix} d\begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}$$

and $F(t) = F_1(t) + iF_2(t)$ with real valued functions F_1 and F_2.

A secret (ambitious) plan

- This form shows a possibility that we might be able to study the NLFT through the input "rough signal" $d\widetilde{F} = e^{ikt}dF$ with the rough path theory.
- For example, use the fundamental theorem like "the solution is also a rough path if the input signal is so"?
- establish the theory of smooth rough paths (or global rough path on non-compact interval), and check the rough path estimate of the input signals?
- The first step is to study the rough path estimate of $d\widetilde{F} = e^{ikt}dF$.
- It is already non-trivial...

Today's Problem (The first level estimate)

Let $1 < p < 2$ and $F : \mathbf{R} \to \mathbf{R}$ be a smooth p-rough path, i.e., it has globally finite p-variation controlled by a control function ω_F.

Let us define the Fourier type integral $G^\theta(t)$ by $dG^\theta = e^{i\theta t} dF_t$, i.e.,

$$G^\theta(t) - G^\theta(s) = \int_s^t e^{i\theta t} dF_t \quad \text{for } s < t.$$

Then, is it a smooth rough path? Or, does it has a globally finite p'-variation with a control $\omega_{G^\theta,(p',\theta)}$ for some p'?

If we remember the point-wise convergence of Fourier transform, this problem is far from obvious. We will show the following answer as a partial result:

An answer

Suppose that $1 \leq p < 2$ and F is a smooth p-rough path, or that it has the (globally) finite p-variation controlled by ω_F. If $p < p' < 2$, then $\omega_{G^\theta,(p',\theta)}$ of G^θ is (globally finite) for almost every θ.

The rough sketch of the proof

Our proof has the three components.

- Estimate of the expectation of the integral with
  - Lyons' basic lemma (Part 1)
  - Young's idea in 2 dimension (Part 2)
- Apply Hambly-Lyons' dyadic argument to the estimate with a little trick (a time change according to a control $\omega(s,t)$). (Part 3).

Part 1 and 2

Get the following estimate of the second moment of the integral $\int_S^T e^{i\theta t} dF_t$ with Gaussian θ:

$$\begin{aligned} \mathbb{E}\left[\left|\int_S^T e^{i\theta t} dF_t\right|^2\right] &= \int_S^T \int_S^T e^{-(t-s)^2/2} dF_s dF_t \\ &\leq C\omega_F(S,T)^{2/p}. \end{aligned}$$

This is not innocent as it looks. Actually we will carefully use Lyons' Lemma (Research Letters Paper, 1981) and Young's argument in two dimension.

Part 3

We use the estimate above to apply Hambly-Lyons' dyadic argument (Hambly-Lyons, 1998).

Lemma (B.Hambly and T.J.Lyons (1998))

Suppose that $(X^k_{s,t})$ is a continuous multiplicative functional on $\Delta(0,1)$. Then there exists a constant $C(p)$ such that $(X^k_{s,t})$ will have finite p-variation on $[0,1]$ if

$$\sum_{n=0}^{\infty} n^{C(p)} \sum_{k=1}^{2^n} \max_{m \leq p} \left| X^m\left(\frac{k}{2^n}\right) - X^m\left(\frac{k+1}{2^n}\right) \right|^{p/m} < \infty.$$

Roughly saying, this argument is to paste the estimates on dyadic intervals like $[k/2^n, (k+1)/2^n] (k = 0, \ldots, 2^n - 1)$ in a clever way to get an estimate on $[0,1]$.

We need to modify this argument a little because our integral is defined on the whole real line $\mathbb{R}$.
For this, we will take a time change $\rho : [0,1] \to \mathbb{R}$ suitably (according to ω_F) and we already checked in Part 1 that our estimate for

$$\mathbb{E}\left[\left|\int_{\rho\left(\frac{k}{2^n}\right)}^{\rho\left(\frac{k+1}{2^n}\right)} e^{i\theta t} dF_t\right|^2\right]$$

is sharp enough for the dyadic argument to work.

Putting them together

$$
\begin{aligned}
& \mathbb{E}\left[\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_t\right|^{p'}\right] \\
\leq \quad & \mathbb{E}\left[\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t} dF_t\right|^{2}\right]^{p'/2} \\
\leq \quad & (\mathrm{const}) \cdot \omega_F\left(\rho\left(\frac{k+1}{2^n}\right), \rho\left(\frac{k}{2^n}\right)\right)^{(2/p)\cdot(p'/2)} \\
\leq \quad & (\mathrm{const}) \cdot \omega_F\left(\rho\left(\frac{k+1}{2^n}\right), \rho\left(\frac{k}{2^n}\right)\right)^{p'/p} .
\end{aligned}
$$

Now using the fact that we properly chose the time change ρ according to the control ω_F and that $p < p' < 2$, we will finally get the result thanks to Hambly-Lyons' dyadic argument. Actually, we have

$$\mathbb{E}^{\theta}\left[\sum_{n=0}^{\infty} n^{C(p')} \sum_{k=1}^{2^n}\left|G^{\theta}\left(\rho\left(\frac{k}{2^n}\right)\right) - G^{\theta}\left(\rho\left(\frac{k+1}{2^n}\right)\right)\right|^{p'}\right]$$

$$\leq D(p)\omega_F(-\infty,\infty)^{p'/p}\sum_{n=0}^{\infty} n^{C(p')}2^{n(1-(p'/p))} < \infty$$

by Beppo-Levi.

We have checked the assumption of Hambly-Lyons' lemma for the almost surely paths. Therefore, the time-changed path $G^{\theta}(\rho(\cdot))$ has uniformly finite p'-variation by Hambly-Lyons' dyadic argument. Since the variation does not depend on any time-change, we reach the conclusion.

The detail of the proof: Part 1

Choose the Gaussian for θ (mean 0, variance σ^2).
By Fubini's theorem and simple computations, we have

$$\begin{aligned} \mathbb{E}^\theta \left[\left| \int_S^T e^{i\theta s} dF_s \right|^2 \right] &= \mathbb{E}^\theta \left[\int_S^T e^{i\theta s} dF_s \int_S^T e^{-i\theta t} dF_t \right] \\ &= \int_S^T \int_S^T \mathbb{E}^\theta \left[e^{i\theta(s-t)} \right] dF_s dF_t \\ &= \int_S^T \int_S^T e^{-(t-s)^2\sigma^2/2} dF_s dF_t . \end{aligned}$$

Now we want to estimate the Gauss type integral $\int_S^T \int_S^T e^{-(t-s)^2} dF_t dF_s$. Though it might seem innocent, this procedure is not easy. (The diagonal $\{(t,s) : t = s\}$ seems dangerous!)

Proposition

Suppose that F and G are p-rough paths controlled by ω_F and ω_G respectably, which are both globally finite, i.e., $\omega_F(-\infty,\infty), \omega_G(-\infty,\infty) < M < \infty$ for a constant M. Suppose further that $\omega_F(0,\infty) = \omega_G(-\infty,0) = 0$ (, which means that F is constant on the right half line and G is constant on the left half line). Then, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \leq C \cdot \omega_F(-\infty,\infty)^{1/p} \omega_G(-\infty,\infty)^{1/p}$$

for a constant $C > 0$.

Since the integral above does not contain the diagonal $\{s = t\}$, the result is a direct consequence of the following lemma by T.J.Lyons (Research Letters Paper, 1981).

Lemma (T.J.Lyons)

Let $1/p + 1/q > 1$. If f is of finite q-variation controlled by ω_f and g is of finite p-variation controlled by ω_g, then the integral $z_t = \int_0^t f(s)dg(s)$ has its p-variation controlled by

$$(\|f\|_\infty + C_{1/p+1/q}\,\omega_f(0,t)^{1/q})^p \omega_g(0,t).$$

Therefore, in particular if f has bounded 1-variation and we take $q = 1$, the variation is at most $(\|f\|_\infty + C_{1/p+1}\,\omega_f(0,t))^p \omega_g(0,t)$.

Since F_s has bounded variation, it has a limit as $s \to -\infty$. Therefore, we can apply a suitable time change $s \mapsto -s$ to make it run backwards. Let us denote the backward function by $\tilde{F}_s$. Note that this time change does not change the shape of the path, so it is a p-rough path again. Then, we have

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \\
= & \int_{t>0} \int_{s<0} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \\
= & \int_{t>0} \int_{s>0} e^{-(t+s)^2\sigma^2/2} d\tilde{F}_s dG_t \\
\leq & \; C \cdot \omega_{\tilde{F}}(-\infty, \infty)^{1/p} \omega_G(-\infty, \infty)^{1/p} \\
= & \; C \cdot \omega_F(-\infty, \infty)^{1/p} \omega_G(-\infty, \infty)^{1/p},
\end{aligned}$$

if we can show the inequality above in the middle.

Therefore, our task is to estimate

$$\int_0^\infty \left(\int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s \right) dY_t$$

for a p-rough paths X, Y controlled by global bounded control functions ω_X, ω_Y respectively defined on $[0, \infty]$. According to Lyons' Lemma (1981), It is enough to control the uniform norm and 1-variation of

$$Z(t) = \int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s.$$

First we check the uniform bound. Certainly the integral above makes sense since the integrand $s \mapsto \exp(-(t+s)^2\sigma^2/2)$ is bounded and has globally bounded 1-variation thanks to Lemma above.

More precisely, we have the Young integral bound at $t > 0$ and taking the supremum over $t > 0$, we get

$$\begin{aligned} |Z(t)| &= \left| \int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s \right| \\ &\leq \omega_X(0,\infty)^{1/p}(e^{-t^2\sigma^2/2} + C_{1/p+1} e^{-t^2\sigma^2/2}) \\ &\leq \omega_X(0,\infty)^{1/p}(1 + C_{1/p+1})\, e^{-t^2\sigma^2/2} \\ &\leq \omega_X(0,\infty)^{1/p}(1 + C_{1/p+1}), \end{aligned}$$

because the supremum norm and the 1-variational norm of $s \mapsto e^{-(s+t)^2\sigma^2/2}$ are less than or equal to $e^{-(t+0)^2\sigma^2/2}$. So we have got the uniform bound of $|Z(t)|$.

Next, to estimates the 1-variational norm for $Z(t)$, we study the derivative of $Z(t)$:

$$Z(t)' = \int_0^\infty -(t+s)\sigma^2\, e^{-(t+s)^2\sigma^2/2} dX_s.$$

Let us denote the integrand by $-z(t)$, i.e.,

$$z(t) = (t+s)\sigma^2\, e^{-(t+s)^2\sigma^2/2} \quad (t > 0),$$

to see the behaviour closely. Simple calculation shows

$$z'(t) = \sigma^2 e^{-(t+s)^2\sigma^2/2} \left(1 - \sigma^2(t+s)^2\right).$$

Therefore, we have two cases. If $0 \leq s \leq 1/\sigma$, the maximum of $z(t)$ occurs exactly once when $(t+s)^2\sigma^2 = 1$ and the maximal value is $z(1/\sigma - s) = 1/\sigma \cdot \sigma^2 e^{-1/2} = \sigma e^{-1/2}$. On the other hand, if $s > 1/\sigma$, the function $z(t)$ is monotone and the maximal value is $z(0) = s\sigma^2 e^{-s^2\sigma^2/2}$.

This observation with Lyons' Lemma allows us to estimate of the integral $Z'(t)$ as follows.

$$\left|Z'(t)\right| = \left|\int_0^\infty -(t+s)\sigma^2 e^{-(t+s)^2\sigma^2/2} dX_s\right|$$

$$\leq \begin{cases} \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})\sigma e^{-1/2}, & \text{if } t \leq 1/\sigma, \\ \omega_X(0,\infty)^{1/p}(1+C_{1/p+1})t\sigma^2 e^{-t^2\sigma^2/2}, & \text{if } t > 1/\sigma. \end{cases}$$

Since $|Z(v)-Z(u)| = \left|\int_u^v Z'(t)dt\right| \leq \int_u^v |Z'(t)|dt$, we can get the bound of the 1-variation of $Z(t)$ with the estimate above for $|Z'(t)|$ as follows.

For any sequence $0 \leq t_0 < t_1 < t_2 < \cdots < \infty$, we have

$$\begin{aligned}
& \sum_j |Z(t_{j+1}) - Z(t_j)| \\
\leq \; & \sum_{j<k} |Z(t_{j+1}) - Z(t_j)| + |Z(1/\sigma) - Z(t_k)| \\
& + |Z(t_{k+1}) - Z(1/\sigma)| + \sum_{j \geq k+1} |Z(t_{j+1}) - Z(t_j)| \\
\leq \; & \omega_X(0,\infty)^{1/p}(1 + C_{1/p+1})\sigma e^{-1/2}\frac{1}{\sigma} \\
& + \omega_X(0,\infty)^{1/p}(1 + C_{1/p+1})\sigma^2 \int_{1/\sigma}^{\infty} te^{-t^2\sigma^2/2} dt \\
= \; & 2\omega_X(0,\infty)^{1/p}(1 + C_{1/p+1})e^{-1/2}.
\end{aligned}$$

Therefore, the 1-variation of $Z(t) = \int_0^\infty \exp(-(t+s)^2\sigma^2/2) dX_s$ is at most

$$2\omega_X(0,\infty)^{1/p}(1 + C_{1/p+1})e^{-1/2}.$$

Therefore, we have the estimates for the supremum norm and 1-variation norm of $Z(t)$.
Finally applying Lemma to $\int_0^\infty Z(t)dY_t$ again, we get

$$
\begin{aligned}
\left|\int_0^\infty \int_0^\infty e^{-(t+s)^2\sigma^2/2} dX_s dY_t\right| &= \left|\int_0^\infty Z(t)dY_t\right| \\
&\leq \omega_Y^{1/p}(0,\infty)\Big((1+C_{1/p+1})\omega_X^{1/p}(0,\infty) \\
&+ C_{1/p+1}(1+C_{1/p+1})2e^{-1/2}\omega_X^{1/p}(0,\infty)\Big) \\
&\leq C\omega_X^{1/p}(0,\infty)\omega_Y^{1/p}(0,\infty)
\end{aligned}
$$

for some constant C.
Therefore, we have finished the 1st part.

The detail of the proof: Part 2

To show the following uniform bound

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t\right| \leq C\omega_F(-\infty,\infty)^{1/p}\omega_G(-\infty,\infty)^{1/p},$$

we show the next lemma with Young's trick and the key estimate in Part 1.

Definition

Let $D = \{s_0 = -\infty < s_1 < s_2 < \cdots < s_r = +\infty\}$ be a finite partition of $\mathbb{R}$ and consider the domain $\Delta(S,T) = \{(s,t) : S \leq s < t \leq T\}$, and the domain $\Delta^D = \Delta(-\infty,\infty) \setminus \cup_{i=1,\dots,r}\Delta(s_{i-1},s_i)$. We define the D-approximate integral

$$I^D = \int\int_{\Delta^D} e^{-(t-s)^2\sigma^2/2} dF_s dG_t.$$

We have the following estimate for the approximate integral I^D with the partition D.

Lemma

Suppose that $1 \leq p < 2$. Then, the following estimate holds.

$$|I^D| \leq C\omega_F(-\infty, \infty)^{1/p}\omega_G(-\infty, \infty)^{1/p},$$

for some constant C.

Without loss of generality, we can rescale F and G such that $\omega_F(-\infty,\infty) = \omega_G(-\infty,\infty)$. Let $\omega = \omega_F + \omega_G$. (So ω controls the both of F and G.) If $r = 1$, i.e., D is $(-\infty,\infty)$, then Δ^D is empty , $|I^D| = 0$, and no problem. Otherwise, there is an i such that

$$\omega(s_{i-1}, s_{i+1}) \leq \begin{cases} 2\frac{\omega(-\infty,\infty)}{r-2} & \text{if } r > 2, \\ \omega(-\infty,\infty) & \text{if } r = 2. \end{cases}$$

Let $D' = D \setminus \{s_i\}$. Then

$$\begin{aligned} I^D - I^{D'} &= \int\int_{\Delta^D \setminus \Delta^{D'}} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \\ &= 2\int\int_{[s_{i-1},s_i]\times[s_i,s_{i+1}]} e^{-(t-s)^2\sigma^2/2} dF_s dG_t. \end{aligned}$$

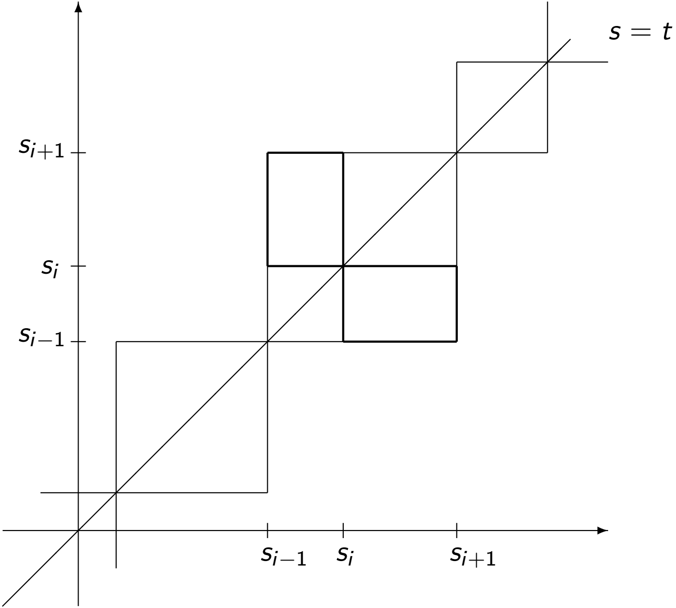

Figure: Young's trick in 2 dimension

Now we apply our key estimate to this integral on the off-diagonal box and deduce that

$$
\begin{aligned}
|I^D - I^{D'}| &\leq \left| \int 2 \int_{[s_{i-1},s_i]\times[s_i,s_{i+1}]} e^{-(t-s)^2\sigma^2/2} dF_s dG_t \right| \\
&\leq C\omega_F(s_{i-1}, s_i)^{1/p} \omega_G(s_i, s_{i+1})^{1/p} \\
&\leq C' \left(\frac{\omega(-\infty,\infty)}{r-2} \right)^{2/p}.
\end{aligned}
$$

Since $p < 2$, we can sum up these estimates to get

$$
\begin{aligned}
|I^D| \leq\ & C' 2^{2/p} \left(1 + \sum_{r=2}^{\infty} \left(\frac{2}{r} \right)^{2/p} \right) \\
& \omega_F(-\infty,\infty)^{1/p} \omega_G(-\infty,\infty)^{1/p},
\end{aligned}
$$

uniformly in D.
So, we have finished the second part.

The detail of the proof: Part 3

Let us recall the Hambly-Lyons' dyadic argument.

Lemma (B.Hambly and T.J.Lyons (1998))

Suppose that $(X^k_{s,t})$ is a continuous multiplicative functional on $\Delta(0,1)$. Then there exists a constant $C(p)$ such that $(X^k_{s,t})$ will have finite p-variation on $[0,1]$ if

$$\sum_{n=0}^{\infty} n^{C(p)} \sum_{k=1}^{2^n} \max_{l \leq p} \left| X^l\left(\frac{k}{2^n}\right) - X^l\left(\frac{k+1}{2^n}\right) \right|^{p/l} < \infty.$$

But our object G^θ is defined on $\mathbb{R}$ instead of $[0,1]$. Therefore we need a little work to adjust the situation.

To adjust our functions to Hambly-Lyons' lemma, we prepare a suitable time change. Our time change should be not only a map from $[0,1]$ to $\mathbb{R}$, but also go well with the control function.

Lemma

If ω is a control function where $\omega(-\infty,\infty) < M < \infty$ for a constant M, then there exists a continuous strictly increasing function $\rho : [0,1] \to [-\infty,\infty]$ with the property that

$$\omega(\rho(u),\rho(v)) \leq M|u-v|.$$

Proof.

Consider $\tau(t) = \omega(-\infty,t)$. (Basically) the inverse of τ is $\rho : [0,1] \to [-\infty,\infty]$. (You need to check it works well.) $\square$

The detail of the proof: Putting all together

Now we can put all together to get the result.

$$
\begin{aligned}
&\mathbb{E}^{\theta}\left(\left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{i\theta t}\, dF_u\right|^2\right)^{p'/2} \\
&= \left|\int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} \int_{\rho(k/2^n)}^{\rho((k+1)/2^n)} e^{-(t-s)^2\sigma^2/2}\, dF_s\, dF_t\right|^{p'/2} \\
&\leq D(p)\left(\omega_F\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)^{(2/p)(p'/2)} \\
&\leq D(p)\left(\frac{\omega_F(-\infty,\infty)}{2^n}\right)^{(2/p)(p'/2)} \\
&= D(p)\omega_F(-\infty,\infty)^{p'/p}\cdot 2^{-n(p'/p)},
\end{aligned}
$$

where we used the super-additivity of ω_F in the second inequality. Here it is very crucial to get the order $2^{-n(p'/p)}$ for $p' > p$.

This order assures us to satisfy the assumption of Hambly-Lyons' lemma in Section 3 as follows:

$$\mathbb{E}^{\theta}\left[\sum_{n=0}^{\infty} n^{C(p')} \sum_{k=1}^{2^n} \left| G^{\theta}\left(\rho\left(\frac{k}{2^n}\right)\right) - G^{\theta}\left(\rho\left(\frac{k+1}{2^n}\right)\right)\right|^{p'}\right]$$

$$\leq D(p)\omega_F(-\infty,\infty)^{p'/p} \sum_{n=0}^{\infty} n^{C(p')} 2^{n(1-(p'/p))} < \infty$$

by Beppo-Levi.

We have checked the assumption of Hambly-Lyons' lemma for the almost surely paths. Therefore, the time-changed path $G^{\theta}(\rho(\cdot))$ has uniformly finite p'-variation by Hambly-Lyons' dyadic argument. Since the variation does not depend on any time-change, we reach the conclusion.